Symmetry, Integrability and Geometry: Methods and Applications SIGMA **8** (2012), 097, 27 pages

Construction of a Lax Pair for the $E_6^{(1)}$ q-Painlevé System

Nicholas S. WITTE † and *Christopher M. ORMEROD* ‡

† *Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia*
E-mail: *nsw@ms.unimelb.edu.au*
URL: http://www.ms.unimelb.edu.au/~nsw/

‡ *Department of Mathematics and Statistics, La Trobe University, Bundoora VIC 3086, Australia*
E-mail: *C.Ormerod@latrobe.edu.au*

Received September 05, 2012, in final form November 29, 2012; Published online December 11, 2012
http://dx.doi.org/10.3842/SIGMA.2012.097

Abstract. We construct a Lax pair for the $E_6^{(1)}$ q-Painlevé system from first principles by employing the general theory of semi-classical orthogonal polynomial systems characterised by divided-difference operators on discrete, quadratic lattices [arXiv:1204.2328]. Our study treats one special case of such lattices – the q-linear lattice – through a natural generalisation of the big q-Jacobi weight. As a by-product of our construction we derive the coupled first-order q-difference equations for the $E_6^{(1)}$ q-Painlevé system, thus verifying our identification. Finally we establish the correspondences of our result with the Lax pairs given earlier and separately by Sakai and Yamada, through explicit transformations.

Key words: non-uniform lattices; divided-difference operators; orthogonal polynomials; semi-classical weights; isomonodromic deformations; Askey table

2010 Mathematics Subject Classification: 39A05; 42C05; 34M55; 34M56; 33C45; 37K35

1 Background and motivation

Since the recent discoveries of q-analogues of the Painlevé equations, see for example [4] and [13] which are of relevance to the present study, and their classification (of these and others) according to the theory of rational surfaces by Sakai [16] interest has grown in finding Lax pairs for these systems. This problem also has the independent interest as a search for discrete and q-analogues to the isomonodromic systems of the continuous Painlevé equations, and an appropriate analogue to the concept of monodromy. Such interest, in fact, goes back to the period when the discrete analogues of the Painlevé equations were first discussed, as one can see in [12].

In this work we illustrate a general method for constructing Lax pairs for all the systems in the Sakai scheme, as given in the study [17], with the particular case of the $E_6^{(1)}$ system. In this method all aspects of the Lax pairs are constructed, and in the end we verify the identification with the $E_6^{(1)}$ system by deriving the appropriate coupled first-order q-difference equations. We will utilise the form of the $E_6^{(1)}$ q-Painlevé system as given in [6] and [5] in terms of the variables f, g under the mapping

$$(t, f, g) \mapsto \big(qt, f(qt) \equiv \hat{f}, g(qt) \equiv \hat{g}\big),$$

and $f(q^{-1}t) \equiv \check{f}$, etc. In these variables the coupled first-order q-difference equations are

$$(g\check{f} - 1)(gf - 1) = t^2 \frac{(b_1 g - 1)(b_2 g - 1)(b_3 g - 1)(b_4 g - 1)}{(g - b_6 t)(g - b_6^{-1} t)}, \tag{1.1}$$

$$(f\hat{g}-1)(fg-1) = qt^2\frac{(f-b_1)(f-b_2)(f-b_3)(f-b_4)}{(f-b_5qt)(f-b_5^{-1}t)}, \tag{1.2}$$

with five independent parameters $b_1,\dots,b_6$ subject to the constraint $b_1b_2b_3b_4 = 1$.

Our approach is to construct a sequence of τ-functions starting with a deformation of a specific weight in the Askey table of hypergeometric orthogonal polynomial systems [7]. However for the purposes of the present work we will not explicitly exhibit these τ-functions although one could do so easily. The weight that we will take is the big q-Jacobi weight[1] given by equation (14.5.2) of [7]

$$w(x) = \frac{\left(a^{-1}x, c^{-1}x; q\right)_\infty}{\left(x, bc^{-1}x; q\right)_\infty}. \tag{1.3}$$

The essential property of this weight, and the others in the Askey table, that we will utilise is that they possess the q-analogue of the *semi-classical* property with respect to x, namely that it satisfies the linear, first-order homogeneous q-difference equation

$$\frac{w(qx)}{w(x)} = \frac{a(1-x)(c-bx)}{(a-x)(c-x)},$$

where the right-hand side is manifestly rational in x. Another feature of this weight is that the discrete lattice forming the support for the orthogonal polynomial system is the q-linear lattice, one of four discrete quadratic lattices. Consequently the perspective provided by our theoretical approach, then indicates that this case is the *master case* for the q-linear lattices (as opposed to the $D_5^{(1)}$ system, for example) and all systems with such support will be degenerations of it. The weight (1.3) has to be generalised, or *deformed*, in order to become relevant to q-Painlevé systems, and such a generalisation turns out to introduce a new variable t and associated parameter so that it retains the semi-classical character with respect to this variable. Using such a sequence of τ-functions one employs arguments to construct three systems of linear divided-difference equations which in turn characterise these. One of these is the three-term recurrence relation of the polynomials orthogonal with respect to the deformed weight, which in the Painlevé theory context is a distinguished Schlesinger transformation, while the two others are our Lax pairs with respect to the spectral variable x and the deformation variable t. Our method constructs a specific sequence of classical solutions to the $E_6^{(1)}$ system and thus is technically valid for integer values of a particular parameter, however we can simply analytically continue our results to the general case.

Lax pairs have been found for the $E_6^{(1)}$ system system using completely different techniques. In [15] Sakai used a particular degeneration of a two-variable Garnier extension to the Lax pairs for the $D_5^{(1)}$ q-Painlevé system[2] (see [14] for details on the multi-variable Garnier extension). More recently Yamada [18] has reported Lax pairs for the $E_6^{(1)}$ system by employing a degeneration starting from a Lax pair for the $E_8^{(1)}$ q-Painlevé equation through a sequence of limits $E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)}$.

The plan of our study is as follows. In Section 2 we recount the notations, definitions and basic facts of the general theory [17] in a self-contained manner omitting proofs. We draw heavily upon this theory in Section 3 where we apply it to the q-linear lattice and a natural extension or deformation of the big q-Jacobi weight. Again, using techniques first expounded in [17], we find explicit forms for the Lax pairs and verify the identification with the $E_6^{(1)}$ q-Painlevé system. At the conclusion of our study, in Section 4, we relate our Lax pairs with those of both Sakai and Yamada.

[1] However we will employ a different parameterisation of the big q-Jacobi weight from that of the conventional form (1.3) in order that our results conform to the the $E_6^{(1)}$ q-Painlevé system as given by (1.1), (1.2); see (3.1).

[2] This later system is also known as the q-$\mathrm{P_{VI}}$ system and its Lax pairs were constructed in [4].

2 Deformed semi-classical OPS on quadratic lattices

We begin by summarising the key results of [17], in particular Sections 2, 3, 4 and 6 of that work, which relate to semi-classical orthogonal polynomial systems with support on discrete, quadratic lattices.

Let $\Pi_n[x]$ denote the linear space of polynomials in x over $\mathbb{C}$ with degree at most $n \in \mathbb{Z}_{\geq 0}$. We define the *divided-difference operator* (DDO) $\mathbb{D}_x$ by

$$\mathbb{D}_x f(x) := \frac{f(\iota_+(x)) - f(\iota_-(x))}{\iota_+(x) - \iota_-(x)}, \tag{2.1}$$

and impose the condition that $\mathbb{D}_x : \Pi_n[x] \to \Pi_{n-1}[x]$ for all $n \in \mathbb{N}$. In consequence we deduce that $\iota_\pm(x)$ are the two y-roots of the quadratic equation

$$\mathcal{A}y^2 + 2\mathcal{B}xy + \mathcal{C}x^2 + 2\mathcal{D}y + 2\mathcal{E}x + \mathcal{F} = 0. \tag{2.2}$$

Assuming $\mathcal{A} \neq 0$ the two y-roots $y_\pm := \iota_\pm(x)$ for a given x-value satisfy

$$\iota_+(x) + \iota_-(x) = -2\frac{\mathcal{B}x + \mathcal{D}}{\mathcal{A}}, \qquad \iota_+(x)\iota_-(x) = \frac{\mathcal{C}x^2 + 2\mathcal{E}x + \mathcal{F}}{\mathcal{A}},$$

and their inverse functions $\iota_\pm^{-1}$ are defined by $\iota_\pm^{-1}(\iota_\pm(x)) = x$. For a given y-value the quadratic (2.2) also defines two x-roots, if $\mathcal{C} \neq 0$, which are consecutive points on the x-*lattice*, x_s, x_{s+1} parameterised by the variable $s \in \mathbb{Z}$ and therefore defines a map $x_s \mapsto x_{s+1}$. Thus we have the sequence of x-values $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ given by $\ldots$, $\iota_-(x_0) = \iota_+(x_{-1})$, $\iota_-(x_1) = \iota_+(x_0)$, $\ldots$ which we denote as the *lattice* or the *direct lattice* G, and the sequence of y-values $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ given by $\ldots$, $y_0 = \iota_-(x_0)$, $y_1 = \iota_-(x_1)$, $y_2 = \iota_-(x_2)$, $\ldots$ as the *dual lattice* $\tilde{G}$ (and distinct from the former in general). A companion operator to the divided-difference operator $\mathbb{D}_x$ is the mean or average operator $\mathbb{M}_x$ defined by

$$\mathbb{M}_x f(x) = \tfrac{1}{2}\left[f(\iota_+(x)) + f(\iota_-(x))\right],$$

so that the property $\mathbb{M}_x : \Pi_n[x] \to \Pi_n[x]$ is ensured by the condition we imposed upon $\mathbb{D}_x$. The difference between consecutive points on the dual lattice is given a distinguished notation through the definition $\Delta y(x) := \iota_+(x) - \iota_-(x)$.

We will also employ an operator notation for the mappings from points on the direct lattice to the dual lattice $E_x^\pm f(x) := f(\iota_\pm(x))$ so that (2.1) can be written

$$\mathbb{D}_x f(x) = \frac{E_x^+ f - E_x^- f}{E_x^+ x - E_x^- x},$$

for arbitrary functions $f(x)$. The inverse functions $\iota_\pm^{-1}(x)$ define operators $(E_x^\pm)^{-1}$ which map points on the dual lattice to the direct lattice and also an adjoint to the divided-difference operator $\mathbb{D}_x$

$$\mathbb{D}_x^* f(x) := \frac{f\big(\iota_+^{-1}(x)\big) - f\big(\iota_-^{-1}(x)\big)}{\iota_+^{-1}(x) - \iota_-^{-1}(x)} = \frac{(E_x^+)^{-1} f - (E_x^-)^{-1} f}{(E_x^+)^{-1} x - (E_x^-)^{-1} x}.$$

The composite operators $E_x := (E_x^-)^{-1} E_x^+$ and $E_x^{-1} = (E_x^+)^{-1} E_x^-$ map between consecutive points on the direct lattice[3].

[3]However in the situation of a symmetric quadratic $\mathcal{A} = \mathcal{C}$ and $\mathcal{D} = \mathcal{E}$, which entails no loss of generality, then we have $(E_x^+)^{-1} = E_x^-$ and $(E_x^-)^{-1} = E_x^+$ and consequently there is no distinction between the divided-difference operator and its adjoint.

Assuming $\mathcal{AC} \neq 0$ one can classify these non-uniform quadratic lattices (or *SNUL*, special non-uniform lattices) according to two parameters – the discriminant $\mathcal{B}^2 - \mathcal{AC}$ and

$$\Theta = \det \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{D} \\ \mathcal{B} & \mathcal{C} & \mathcal{E} \\ \mathcal{D} & \mathcal{E} & \mathcal{F} \end{pmatrix},$$

or $\mathcal{A}\Theta = (\mathcal{B}^2 - \mathcal{AC})(\mathcal{D}^2 - \mathcal{AF}) - (\mathcal{BD} - \mathcal{AE})^2$. The quadratic lattices are classified into four sub-cases [9, 10]: q-quadratic ($\mathcal{B}^2 - \mathcal{AC} \neq 0$ and $\Theta < 0$), quadratic ($\mathcal{B}^2 - \mathcal{AC} = 0$ and $\Theta < 0$), q-linear ($\mathcal{B}^2 - \mathcal{AC} \neq 0$ and $\Theta = 0$) and linear ($\mathcal{B}^2 - \mathcal{AC} = 0$ and $\Theta = 0$), as the conic sections are divided into the elliptic/hyperbolic, parabolic, intersecting straight lines and parallel straight lines respectively. The q-quadratic lattice, in its general non-symmetrical form, is the most general case and the other lattices can be found from this by limiting processes. For the quadratic class of lattices the parameterisation on s can be made explicit using trigonometric/hyperbolic functions or their degenerations so we can employ a parameterisation such that $y_s = \iota_-(x_s) = x_{s-1/2}$ and $y_{s+1} = \iota_+(x_s) = x_{s+1/2}$. We denote the totality of lattice points by $G[x_0] := \{x_s : s \in \mathbb{Z}\}$ with the point x_0 as the *basal point*, and of the dual lattice by $\tilde{G}[x_0] := \{x_s : s \in \mathbb{Z}+\frac{1}{2}\}$.

We define the $\mathbb{D}$-*integral* of a function defined on the x-lattice $f : G[x] \to \mathbb{C}$ with basal point x_0 by the Riemann sum over the lattice points

$$I[f](x_0) = \int_G \mathbb{D}x\, f(x) := \sum_{s\in\mathbb{Z}} \Delta y(x_s) f(x_s),$$

where the sum is either a finite subset of $\mathbb{Z}$, namely $\{0, \ldots, \mathfrak{N}\}$, $\mathbb{Z}_{\geq 0}$, or $\mathbb{Z}$. This definition reduces to the usual definition of the difference integral and the Thomae–Jackson q-integrals in the canonical forms of the linear and q-linear lattices respectively. Amongst a number of properties that flow from this definition we have an analog of the fundamental theorem of calculus

$$\int_{x_0 \leq x_s \leq x_{\mathfrak{N}}} \mathbb{D}x\, \mathbb{D}_x f(x) = f(E_x^+ x_{\mathfrak{N}}) - f(E_x^- x_0). \tag{2.3}$$

Central to our study are orthogonal polynomial systems (OPS) defined on G, and a general reference for a background on these and other considerations is the monograph by Ismail [3]. Our OPS is defined via orthogonality relations with support on G

$$\int_G \mathbb{D}x\; w(x) p_n(x)\, l_m(x) = \begin{cases} 0, & 0 \leq m < n, \\ h_n, & m = n, \end{cases} \qquad n \geq 0, \qquad h_n \neq 0,$$

where $\{l_m(x)\}_{m=0}^{\infty}$ is any system of polynomial bases with exact $\deg_x(l_m) = m$. Such relations define a sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ under suitable conditions (see [3]). An immediate consequence of orthogonality is that the orthogonal polynomials satisfy a three term recurrence relation of the form

$$\begin{aligned} &a_{n+1} p_{n+1}(x) = (x - b_n) p_n(x) - a_n p_{n-1}(x), \qquad n \geq 0, \\ &a_n \neq 0, \qquad p_{-1} = 0, \qquad p_0 = \gamma_0. \end{aligned} \tag{2.4}$$

However we require non-polynomial solutions to this linear second-order difference equation, which are linearly independent of the polynomial solutions. To this end we define the *Stieltjes function*

$$f(x) \equiv \int_G \mathbb{D}y\, \frac{w(y)}{x - y}, \qquad x \notin G.$$

A set of non-polynomial solutions to (2.4), termed *associated functions* or *functions of the second kind*, and which generalise the Stieltjes function, are given by

$$q_n(x) \equiv \int_G \mathbb{D}y\, w(y) \frac{p_n(y)}{x-y}, \qquad n \geq 0, \qquad x \notin G.$$

The associated function solutions differ from the orthogonal polynomial solutions in that they have the initial conditions $q_{-1} = 1/a_0\gamma_0$, $q_0 = \gamma_0 f$. The utility and importance of the Stieltjes function lies in the fact that that it connects p_n and q_n whereby the difference $fp_n - q_n$ is exactly a polynomial of degree $n-1$ which itself satisfies (2.4) in place of p_n. This relation is crucial for the arguments adopted in [17]. With the polynomial and non-polynomial solutions we form the 2×2 matrix variable, which occupies a primary position in our theory:

$$Y_n(x) = \begin{pmatrix} p_n(x) & \dfrac{q_n(x)}{w(x)} \\ p_{n-1}(x) & \dfrac{q_{n-1}(x)}{w(x)} \end{pmatrix}.$$

In this matrix variable the three-term recurrence relation takes the form

$$Y_{n+1} = K_n Y_n, \qquad K_n(x) = \frac{1}{a_{n+1}} \begin{pmatrix} x - b_n & -a_n \\ a_{n+1} & 0 \end{pmatrix}, \qquad \det K_n = \frac{a_n}{a_{n+1}}. \tag{2.5}$$

A key result required in the analysis of OPS are the expansions of polynomial solutions about the fixed singularity at $x = \infty$

$$p_n(x) = \gamma_n \left[x^n - \left(\sum_{i=0}^{n-1} b_i \right) x^{n-1} + \left(\sum_{0 \leq i < j < n} b_i b_j - \sum_{i=1}^{n-1} a_i^2 \right) x^{n-2} + \mathrm{O}\big(x^{n-3}\big) \right], \tag{2.6}$$

valid for $n \geq 1$, while for the associated functions the expansions read

$$q_n(x) = \gamma_n^{-1} \left[x^{-n-1} + \left(\sum_{i=0}^{n} b_i \right) x^{-n-2} + \left(\sum_{0 \leq i \leq j \leq n} b_i b_j + \sum_{i=1}^{n+1} a_i^2 \right) x^{-n-3} + \mathrm{O}\big(x^{-n-4}\big) \right], \tag{2.7}$$

valid for $n \geq 0$.

In order to proceed any further we need to impose some structure on the weight characterising our OPS – in particular its spectral characteristics – and this takes the form of the definition of a $\mathbb{D}$-*semi-classical weight* [9]. Such a weight satisfies a first-order homogeneous divided-difference equation

$$\frac{w(y_+)}{w(y_-)} = \frac{W + \Delta y V}{W - \Delta y V}(x), \tag{2.8}$$

where $W(x)$, $V(x)$ are irreducible polynomials in the spectral variable x, which we call *spectral polynomials*. As a consequence of this, under reasonable assumptions on the parameters of the weight, the Stieltjes function satisfies an inhomogeneous form of (2.8)

$$W \mathbb{D}_x f = 2V \mathbb{M}_x f + U, \tag{2.9}$$

where in addition $U(x)$ is a polynomial of x. A *generic* or *regular* $\mathbb{D}$-*semi-classical weight* has two properties:

(i) strict inequalities in the degrees of the spectral data polynomials, i.e., $\deg_x W = M$, $\deg_x V = M - 1$ and $\deg_x U = M - 2$, and

(ii) the lattice generated by any zero of $(W^2 - \Delta y^2 V^2)(x)$, say $\tilde{x}_1$, does not coincide with another zero, $\tilde{x}_2$, i.e. if $(W^2 - \Delta y^2 V^2)(\tilde{x}_2) = 0$ then $\tilde{x}_2 \notin \iota_{\pm}^{2\mathbb{Z}} \tilde{x}_1$.

Further consequences of semi-classical assumptions are a system of spectral divided-difference equations for the matrix variable Y_n, i.e., the *spectral divided-difference* equation

$$\begin{aligned} \mathbb{D}_x Y_n(x) &:= A_n \mathbb{M}_x Y_n(x) \\ &= \frac{1}{W_n(x)} \begin{pmatrix} \Omega_n(x) & -a_n\Theta_n(x) \\ a_n\Theta_{n-1}(x) & -\Omega_n(x) - 2V(x) \end{pmatrix} \mathbb{M}_x Y_n(x), \qquad n \geq 0, \end{aligned} \tag{2.10}$$

with A_n termed the *spectral matrix*. For the $\mathbb{D}$-semi-classical class of weights the coefficients appearing in the spectral matrix, W_n, Ω_n, Θ_n, are polynomials in x, with fixed degrees independent of the index n. These *spectral coefficients* have terminating expansions about $x = \infty$ with the leading order terms

$$\begin{aligned} W_n(x) &= \tfrac{1}{2}W + \tfrac{1}{4}[W + \Delta y V]\left(\frac{y_+}{y_-}\right)^n \\ &\quad + \tfrac{1}{4}[W - \Delta y V]\left(\frac{y_-}{y_+}\right)^n + \mathrm{O}\big(x^{M-1}\big), \qquad n \geq 0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \Theta_n(x) &= \frac{1}{y_-\Delta y}[W + \Delta y V]\left(\frac{y_+}{y_-}\right)^n \\ &\quad - \frac{1}{y_+\Delta y}[W - \Delta y V]\left(\frac{y_-}{y_+}\right)^n + \mathrm{O}\big(x^{M-3}\big), \qquad n \geq 0, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \Omega_n(x) + V(x) &= \frac{1}{2\Delta y}[W + \Delta y V]\left(\frac{y_+}{y_-}\right)^n \\ &\quad - \frac{1}{2\Delta y}[W - \Delta y V]\left(\frac{y_-}{y_+}\right)^n + \mathrm{O}\big(x^{M-2}\big), \qquad n \geq 0, \end{aligned} \tag{2.13}$$

where $M = \deg_x(W_n)$.

Compatibility of the spectral divided-difference equations (2.10) and recurrence relations (2.5) imply that the spectral matrix and the recurrence matrix satisfy

$$\begin{aligned} &K_n(y_+)\left(1 - \tfrac{1}{2}\Delta y A_n\right)^{-1}\left(1 + \tfrac{1}{2}\Delta y A_n\right) \\ &\quad = \left(1 - \tfrac{1}{2}\Delta y A_{n+1}\right)^{-1}\left(1 + \tfrac{1}{2}\Delta y A_{n+1}\right) K_n(y_-), \qquad n \geq 0. \end{aligned} \tag{2.14}$$

These relations can be rewritten in terms of the spectral coefficients arising in (2.10) as recurrence relations in n,

$$W_{n+1} = W_n + \tfrac{1}{4}\Delta y^2 \Theta_n, \qquad n \geq 0, \tag{2.15}$$

$$\Omega_{n+1} + \Omega_n + 2V = (\mathbb{M}_x x - b_n)\Theta_n, \qquad n \geq 0, \tag{2.16}$$

$$\begin{aligned} &(W_n\Omega_{n+1} - W_{n+1}\Omega_n)(\mathbb{M}_x x - b_n) \\ &\quad = -\tfrac{1}{4}\Delta y^2 \Omega_{n+1}\Omega_n + W_n W_{n+1} + a_{n+1}^2 W_n \Theta_{n+1} - a_n^2 W_{n+1}\Theta_{n-1}, \qquad n \geq 0. \end{aligned} \tag{2.17}$$

Another important deduction from these relations is that the spectral coefficients satisfy a bilinear relation

$$W_n(W_n - W) = -\tfrac{1}{4}\Delta y^2 \det \begin{pmatrix} \Omega_n & -a_n\Theta_n \\ a_n\Theta_{n-1} & -\Omega_n - 2V \end{pmatrix}, \qquad n \geq 0. \tag{2.18}$$

The matrix product appearing in (2.14), and recurring subsequently, is called the *Cayley transform* of A_n and it has the evaluation

$$\left(1 - \tfrac{1}{2}\Delta y A_n\right)^{-1}\left(1 + \tfrac{1}{2}\Delta y A_n\right) \tag{2.19}$$

$$= \frac{1}{W+\Delta y V}\begin{pmatrix} 2W_n - W + \Delta y(\Omega_n + V) & -\Delta y a_n \Theta_n \\ \Delta y a_n \Theta_{n-1} & 2W_n - W - \Delta y(\Omega_n + V)\end{pmatrix}, \qquad n \ge 0.$$

This result motivates the following definitions

$$\mathfrak{W}_\pm := 2W_n - W \pm \Delta y(\Omega_n + V), \qquad \mathfrak{T}_+ := \Delta y a_n \Theta_n,$$
$$\mathfrak{T}_- := \Delta y a_n \Theta_{n-1}, \qquad n \ge 1, \tag{2.20}$$

whilst for $n = 0$ we have $\mathfrak{W}_\pm(n=0) := W \pm \Delta y V$, $\mathfrak{T}_+(n=0) := -\Delta y a_0 \gamma_0^2 U$, and $\mathfrak{T}_-(n=0) := 0$. Thus we define

$$A_n^* := \begin{pmatrix} \mathfrak{W}_+ & -\mathfrak{T}_+ \\ \mathfrak{T}_- & \mathfrak{W}_- \end{pmatrix}. \tag{2.21}$$

In a scalar formulation of the matrix linear divided-difference equation (2.10) one of the components, p_n say, satisfies a linear second-order divided-difference equation of the form

$$E_x^+\left(\frac{W+\Delta y V}{\Delta y \Theta_n}\right)(E_x^+)^2 p_n + E_x^-\left(\frac{W-\Delta y V}{\Delta y \Theta_n}\right)(E_x^-)^2 p_n$$
$$- \left\{ E_x^+\left(\frac{\mathfrak{W}_+}{\Delta y \Theta_n}\right) + E_x^-\left(\frac{\mathfrak{W}_-}{\Delta y \Theta_n}\right)\right\} E_x^+ E_x^- p_n = 0. \tag{2.22}$$

Thus far our theoretical construction can only account for the OPS occurring in the Askey table – the hypergeometric and basic hypergeometric orthogonal polynomial systems [7]. To step beyond these, and in particular to make contact with the discrete Painlevé systems, one has to introduce pairs of deformation variables and parameters into the OPS. We denote such a single deformation variable by t, defined on a quadratic lattice (and possibly distinct from that of the spectral variable), with advanced and retarded nodes at $\iota_\pm(t) = u_\pm$, $\Delta u = \iota_+(t) - \iota_-(t)$. We introduce such deformations with imposed structures that are analogous to those of the spectral variable. Thus, corresponding to the definition (2.8), we deem that a *deformed* $\mathbb{D}$*-semi-classical weight* $w(x;t)$ satisfies the additional first-order homogeneous divided-difference equation

$$\frac{w(x;u_+)}{w(x;u_-)} = \frac{R+\Delta u S}{R-\Delta u S}(x;t), \tag{2.23}$$

where the deformation data polynomials, $R(x;t)$, $S(x;t)$, are irreducible polynomials in x. The spectral data polynomials, $W(x;t)$, $V(x;t)$, and the deformation data polynomials, $R(x;t)$, $S(x;t)$, now must satisfy the compatibility relation

$$\frac{W+\Delta y V}{W-\Delta y V}(x;u_+)\frac{R+\Delta u S}{R-\Delta u S}(y_-;t) = \frac{W+\Delta y V}{W-\Delta y V}(x;u_-)\frac{R+\Delta u S}{R-\Delta u S}(y_+;t). \tag{2.24}$$

The deformed $\mathbb{D}$-semi-classical deformation condition that corresponds to (2.9) is that the Stieltjes transform satisfies an inhomogeneous version of (2.23)

$$R\mathbb{D}_t f = 2S\mathbb{M}_t f + T,$$

with $T(x;t)$ being an irreducible polynomial in x with respect to R and S. Compatibility of spectral and deformation divided-difference equations for f implies the following identity on U and T

$$\Delta y\left[\frac{(W+\Delta y V)(x;u_+)}{(W+\Delta y V)(x;u_-)}(R+\Delta u S)(y_-;t)U(x;u_-) - (R-\Delta u S)(y_-;t)U(x;u_+)\right]$$
$$= \Delta u\left[(W+\Delta y V)(x;u_+)T(y_-;t) - (W-\Delta y V)(x;u_+)\frac{(R-\Delta u S)(y_-;t)}{(R-\Delta u S)(y_+;t)}T(y_+;t)\right].$$

Corresponding to the (2.10) the deformed $\mathbb{D}$-semi-classical OPS satisfies the *deformation divided-difference* equation

$$\mathbb{D}_t Y_n := B_n \mathbb{M}_t Y_n = \frac{1}{R_n} \begin{pmatrix} \Gamma_n & \Phi_n \\ \Psi_n & \Xi_n \end{pmatrix} \mathbb{M}_t Y_n, \qquad n \geq 0. \tag{2.25}$$

The deformation coefficients appearing in matrix B_n above satisfy a linear identity

$$\Psi_n = -\frac{a_n}{a_{n-1}} \Phi_{n-1}, \qquad n \geq 1, \tag{2.26}$$

and a trace identity

$$\Delta u (\Gamma_n + \Xi_n) = 2H_n \left[\frac{R + \Delta u S}{a_n(u_-)} - \frac{R - \Delta u S}{a_n(u_+)} \right], \qquad n \geq 0,$$

which means that only three of these are independent. Here H_n is a constant with respect to x and arises as a decoupling constant which will be set subsequently in applications to a convenient value. The deformation coefficients are all polynomials in x, with fixed degrees independent of the index n but with non-trivial t dependence. Let $L = \max(\deg_x R, \deg_x S)$. As $x \to \infty$ we have the leading orders of the terminating expansions of the following deformation coefficients

$$\frac{2}{H_n} R_n = -(\gamma_n(u_+) + \gamma_n(u_-)) \left[\frac{R - \Delta u S}{\gamma_{n-1}(u_+)} + \frac{R + \Delta u S}{\gamma_{n-1}(u_-)} \right] + \mathrm{O}\big(x^{L-1}\big) \qquad n \geq 0, \tag{2.27}$$

$$\frac{\Delta u}{2H_n} \Phi_n = \left[(R + \Delta u S) \frac{\gamma_n(u_+)}{\gamma_n(u_-)} - (R - \Delta u S) \frac{\gamma_n(u_-)}{\gamma_n(u_+)} \right] x^{-1} + \mathrm{O}\big(x^{L-2}\big), \qquad n \geq 0, \tag{2.28}$$

and

$$\frac{\Delta u}{H_n} \Gamma_n = (\gamma_n(u_-) - \gamma_n(u_+)) \left[\frac{R + \Delta u S}{\gamma_{n-1}(u_-)} + \frac{R - \Delta u S}{\gamma_{n-1}(u_+)} \right] + \mathrm{O}\big(x^{L-1}\big), \qquad n \geq 0. \tag{2.29}$$

Compatibility of the deformation divided-difference equation (2.25) and the recurrence relation (2.5) implies the relation

$$\begin{aligned} &K_n(;u_+) \left(1 - \tfrac{1}{2}\Delta u B_n\right)^{-1} \left(1 + \tfrac{1}{2}\Delta u B_n\right) \\ &\quad = \left(1 - \tfrac{1}{2}\Delta u B_{n+1}\right)^{-1} \left(1 + \tfrac{1}{2}\Delta u B_{n+1}\right) K_n(;u_-), \qquad n \geq 0. \end{aligned} \tag{2.30}$$

From this we can deduce that the deformation coefficients, R_n, Γ_n, Φ_n, satisfy recurrence relations in n in parallel to those of (2.15), (2.16)

$$\begin{aligned} &\frac{a_{n+1}(u_-)}{H_{n+1}}(-2R_{n+1} + \Delta u \Gamma_{n+1}) + \frac{a_n(u_-)}{H_n}(2R_n + \Delta u \Gamma_n) \\ &\quad = -[x - b_n(u_-)] \frac{\Delta u}{H_n} \Phi_n + 2a_n(u_-) \left(\frac{R + \Delta u S}{a_n(u_-)} - \frac{R - \Delta u S}{a_n(u_+)} \right), \qquad n \geq 0, \\ &\frac{a_{n+1}(u_+)}{H_{n+1}}(2R_{n+1} + \Delta u \Gamma_{n+1}) + \frac{a_n(u_+)}{H_n}(-2R_n + \Delta u \Gamma_n) \\ &\quad = -[x - b_n(u_+)] \frac{\Delta u}{H_n} \Phi_n + 2a_n(u_+) \left(\frac{R + \Delta u S}{a_n(u_-)} - \frac{R - \Delta u S}{a_n(u_+)} \right), \qquad n \geq 0. \end{aligned}$$

The deformation coefficients satisfy the bilinear or determinantal identity

$$R_n^2 + \tfrac{1}{4}\Delta u^2 \left[\Gamma_n \Xi_n - \Phi_n \Psi_n\right] = -H_n R_n \left[\frac{R + \Delta u S}{a_n(u_-)} + \frac{R - \Delta u S}{a_n(u_+)} \right], \qquad n \geq 0,$$

which is the analogue of (2.18). The matrix product given in (2.30) has the evaluation

$$\left(1-\tfrac{1}{2}\Delta uB_n\right)^{-1}\left(1+\tfrac{1}{2}\Delta uB_n\right)=\frac{a_n(u_-)}{2H_n(R+\Delta uS)}\\ \times\begin{pmatrix}2R_n+2H_n\dfrac{R-\Delta uS}{a_n(u_+)}+\Delta u\Gamma_n & \Delta u\Phi_n\\ \Delta u\Psi_n & 2R_n+2H_n\dfrac{R+\Delta uS}{a_n(u_-)}-\Delta u\Gamma_n.\end{pmatrix},\qquad n\geq 0.$$

This again motivates the definitions

$$\mathfrak{R}_\pm:=2R_n+2H_n\frac{R\mp\Delta uS}{a_n(u_\pm)}\pm\Delta u\Gamma_n,\\ \mathfrak{P}_+:=-\Delta u\Phi_n,\qquad \mathfrak{P}_-:=\Delta u\Psi_n,\qquad n\geq 1, \tag{2.31}$$

together with

$$B_n^*:=\begin{pmatrix}\mathfrak{R}_+ & -\mathfrak{P}_+\\ \mathfrak{P}_- & \mathfrak{R}_-\end{pmatrix}.$$

Our final relation expresses the compatibility of the spectral and deformation divided-difference equations. The spectral matrix $A_n(x;t)$ and the deformation matrix $B_n(x;t)$ satisfy the $\mathbb{D}$-*Schlesinger equation*

$$\left(1-\tfrac{1}{2}\Delta yA_n(;u_+)\right)^{-1}\left(1+\tfrac{1}{2}\Delta yA_n(;u_+)\right)\left(1-\tfrac{1}{2}\Delta uB_n(y_-;)\right)^{-1}\left(1+\tfrac{1}{2}\Delta uB_n(y_-;)\right) \tag{2.32}\\ =\left(1-\tfrac{1}{2}\Delta uB_n(y_+;)\right)^{-1}\left(1+\tfrac{1}{2}\Delta uB_n(y_+;)\right)\left(1-\tfrac{1}{2}\Delta yA_n(;u_-)\right)^{-1}\left(1+\tfrac{1}{2}\Delta yA_n(;u_-)\right).$$

Let us define the quotient

$$\chi\equiv\frac{(W+\Delta yV)(x;u_+)}{(W+\Delta yV)(x;u_-)}\frac{(R+\Delta uS)(y_-;t)}{(R+\Delta uS)(y_+;t)}=\frac{(W-\Delta yV)(x;u_+)}{(W-\Delta yV)(x;u_-)}\frac{(R-\Delta uS)(y_-;t)}{(R-\Delta uS)(y_+;t)}.$$

The compatibility relation (2.32) can be rewritten as the matrix equation

$$\chi B_n^*(y_+;t)A_n^*(x;u_-)=A_n^*(x;u_+)B_n^*(y_-;t), \tag{2.33}$$

or component-wise with the new variables in the more practical form as

$$\chi\left[\mathfrak{W}_+(x;u_-)\mathfrak{R}_+(y_+;t)-\mathfrak{T}_-(x;u_-)\mathfrak{P}_+(y_+;t)\right]\\ \qquad=\mathfrak{W}_+(x;u_+)\mathfrak{R}_+(y_-;t)-\mathfrak{T}_+(x;u_+)\mathfrak{P}_-(y_-;t), \tag{2.34}$$

$$\chi\left[\mathfrak{T}_+(x;u_-)\mathfrak{R}_+(y_+;t)+\mathfrak{W}_-(x;u_-)\mathfrak{P}_+(y_+;t)\right]\\ \qquad=\mathfrak{T}_+(x;u_+)\mathfrak{R}_-(y_-;t)+\mathfrak{W}_+(x;u_+)\mathfrak{P}_+(y_-;t), \tag{2.35}$$

$$\chi\left[\mathfrak{T}_-(x;u_-)\mathfrak{R}_-(y_+;t)+\mathfrak{W}_+(x;u_-)\mathfrak{P}_-(y_+;t)\right]\\ \qquad=\mathfrak{T}_-(x;u_+)\mathfrak{R}_+(y_-;t)+\mathfrak{W}_-(x;u_+)\mathfrak{P}_-(y_-;t), \tag{2.36}$$

$$\chi\left[\mathfrak{W}_-(x;u_-)\mathfrak{R}_-(y_+;t)-\mathfrak{T}_+(x;u_-)\mathfrak{P}_-(y_+;t)\right]\\ \qquad=\mathfrak{W}_-(x;u_+)\mathfrak{R}_-(y_-;t)-\mathfrak{T}_-(x;u_+)\mathfrak{P}_+(y_-;t). \tag{2.37}$$

For a general quadratic lattice there exists two fixed points defined by $\iota_+(x)=\iota_-(x)$, and let us denote these two points of the x-lattice by x_L and x_R. By analogy with the linear lattices we conjecture the existence of fundamental solutions to the spectral divided-difference equation about $x=x_L,x_R$ which we denote by Y_L, Y_R respectively. Furthermore let us define the *connection matrix*

$$P(x;t):=Y_R(x;t)^{-1}Y_L(x;t).$$

From the spectral divided-difference equation (2.10) it is clear that P is a $\mathbb{D}$-constant function with respect to x, that is to say

$$P(y_+; t) = P(y_-; t).$$

In addition it is clear from the deformation divided-difference equation (2.25) that this type of deformation is also a *connection preserving deformation* in the sense that

$$P(x; u_+) = P(x; u_-).$$

This is our analogue of the monodromy matrix and generalises the connection matrix of Birkhoff and his school [1, 2], although we emphasise that we have made an empirical observation of this fact and not provided any rigorous statement of it.

3 Big q-Jacobi OPS

As our central reference on the Askey table of basic hypergeometric orthogonal polynomial systems we employ [8], or its modern version [7]. We consider a sub-case of the quadratic lattices, in particular the q-linear lattice in both the spectral and deformation variables x and t in its standardised form, so that $\iota_+(x) = qx$, $\iota_-(x) = x$, $\Delta y(x) = (q-1)x$ and $\iota_+(t) = qt$, $\iota_-(t) = t$, $\Delta u(t) = (q-1)t$. In [7] the big q-Jacobi weight given by equation (14.5.2) is

$$w(x) = \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty},$$

subject to $0 < aq, bq < 1$, $c < 0$ with respect to the Thomae–Jackson q-integral

$$\int_{bq}^{aq} d_q x\, f(x).$$

The q-shifted factorials have the standard definition

$$(a; q)_\infty = \prod_{j=0}^{\infty} \left(1 - aq^j\right), \qquad |q| < 1, \qquad (a_1, \ldots, a_n; q)_\infty = (a_1; q)_\infty \cdots (a_n; q)_\infty.$$

We deform this weight by introducing an extra q-shifted factorial into the numerator and denominator containing the deformation variable and parameter, and relabeling the big q-Jacobi parameters. We propose the following weight

$$w(x; t) = \frac{\left(b_2 x, b_3 x, b_6^{-1} x t^{-1}; q\right)_\infty}{\left(b_1 x, b_4 x, b_6 x t^{-1}; q\right)_\infty}. \tag{3.1}$$

A condition $b_1 b_2 b_3 b_4 = 1$ will apply, so we have four free parameters. We do not need to specify the support for this weight for the purposes of our work, but suffice it to say that any Thomae–Jackson q-integral with terminals coinciding with any pair of zeros and poles of the weight would be suitable.

The spectral data polynomials are computed to be

$$\begin{aligned} W + \Delta y V &= b_6 \left(1 - b_1 x\right) \left(1 - b_4 x\right) \left(t - b_6 x\right), \\ W - \Delta y V &= \left(1 - b_2 x\right) \left(1 - b_3 x\right) \left(b_6 t - x\right). \end{aligned} \tag{3.2}$$

Clearly the regular $M = 3$ case is applicable and we seek solutions to the spectral coefficients with $\deg_x W_n = 3$, $\deg_x \Omega_n = 2$, $\deg_x \Theta_n = 1$. Our procedure is to employ the following algorithm, as detailed in [17]. Firstly we parameterise the spectral matrix in a minimal way; secondly we relate the parameterisation of the deformation matrix to that of the spectral matrix and thus close the system of unknowns; and finally utilise these parameterisations in the system of over-determined equations to derive evolution equations for our primary variables. What constitutes the primary variables will emerge from the calculations themselves.

Proposition 1. *Let us define a new parameter b_5 replacing q^n by*

$$q^n = \frac{b_5}{b_1 b_4 b_6}, \qquad n \in \mathbb{Z}_{\geq 0}.$$

Let the parameters satisfy the conditions $q \neq 1$, $b_5 \neq q^{-1/2}, \pm 1, q^{1/2}$, $b_1 b_4 \neq 0, \infty$ and $b_2 b_3 \neq 0, \infty$. Given the degrees of the spectral coefficients we parameterise these by

$$\begin{aligned}
&2W_n - W = w_3 x^3 + w_2 x^2 + w_1 x + w_0,\\
&\Omega_n + V = v_2 x^2 + v_1 x + v_0,\\
&\Theta_n = u_1 (x - \lambda_n).
\end{aligned}$$

Let λ_n be the unique zero of the $(1,2)$ component of A_n^*, i.e., $\Theta_n(x)$ and define the further variables $\nu_n = (2W_n - W)(\lambda_n, t)$ and $\mu_n = (\Omega_n + V)(\lambda_n, t)$. Then the spectral coefficients are given by

$$\begin{aligned}
2W_n - W = {} & x^2 \frac{\nu_n}{\lambda_n^2} + \tfrac{1}{2}(x - \lambda_n)\\
& \times \left[-b_6 \left(b_5 + b_5^{-1}\right) x^2 + \frac{1 + (b_1 + b_2 + b_3 + b_4)\, b_6 t + b_6^2}{\lambda_n} x - 2 t b_6 \frac{x + \lambda_n}{\lambda_n^2} \right], \quad (3.3)\\
\Omega_n + V = {} & \mu_n + \frac{b_6}{2 b_5 (1-q)\left(1 - b_5^2\right) \lambda_n^2} (x - \lambda_n) \Big\{ -\left(1 - b_5^2\right)^2 \lambda_n^2 x\\
& - 2 b_5^2 \left[b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1} + \left(b_6 + b_6^{-1}\right) t - 2\lambda_n \right] \lambda_n^2\\
& + b_5 b_6^{-1} \left(1 + b_5^2\right) \left[\left(1 + (b_1 + b_2 + b_3 + b_4) b_6 t + b_6^2\right) \lambda_n + 2\nu_n - 2 t b_6 \right] \Big\}, \quad (3.4)
\end{aligned}$$

and

$$\Theta_n = -\frac{b_6 \left(1 - q b_5^2\right)}{q(1-q) b_5} (x - \lambda_n). \tag{3.5}$$

We note that λ_n, μ_n, ν_n satisfy the quadratic relation

$$\nu_n^2 = (1-q)^2 \lambda_n^2 \mu_n^2 + b_6 (b_1 \lambda_n - 1)(b_2 \lambda_n - 1)(b_3 \lambda_n - 1)(b_4 \lambda_n - 1)(\lambda_n - t b_6)(b_6 \lambda_n - t). \tag{3.6}$$

Proof. Consistent with the known data, i.e., the degrees, from (2.11), (2.12), (2.13) we compute the leading coefficients to be

$$u_1 = -\frac{b_6 \left(1 - q b_5^2\right)}{q(1-q) b_5}, \qquad v_2 = -\frac{b_6 \left(1 - b_5^2\right)}{2(1-q) b_5}, \qquad w_3 = -\frac{b_6 \left(1 + b_5^2\right)}{2 b_5},$$

confirming the relation given by the coefficient of $[x^6]$ in (2.18), $w_3^2 = (q-1)^2 v_2^2 + b_6^2$. In addition we identify the diagonal elements of the $[x^3]$ coefficient of A_n^*

$$\kappa_+ \equiv w_3 + (q-1) v_2 = -b_5 b_6, \qquad \kappa_- \equiv w_3 - (q-1) v_2 = -\frac{b_6}{b_5}.$$

From the coefficient of $[x^0]$ in (2.18) we deduce (modulo a sign ambiguity)

$$w_0 = b_6 t,$$

and from the coefficient of $[x^1]$ in (2.18) we similarly find

$$w_1 = -\tfrac{1}{2}\left[1 + t(b_1 + b_2 + b_3 + b_4)b_6 + b_6^2\right].$$

Now utilising the condition $\nu_n = (2W_n - W)(\lambda_n, t)$ we invert this to compute

$$w_2 = \frac{1 + t(b_1 + b_2 + b_3 + b_4)b_6 + b_6^2}{2\lambda_n} + \tfrac{1}{2} b_6\left(b_5 + b_5^{-1}\right)\lambda_n + \frac{\nu_n - tb_6}{\lambda_n^2}.$$

Proceeding further we infer from the coefficient of $[x^5]$ in (2.18) that

$$v_1 = \frac{\left(1 + b_5^2\right)w_2 - \left(b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1}\right)b_5 b_6 - b_5\left(1 + b_6^2\right)t}{(1-q)\left(1 - b_5^2\right)},$$

and employing the previous result for w_2 we derive

$$\begin{aligned}(1-q)\frac{\left(1 - b_5^2\right)}{\left(1 + b_5^2\right)} v_1 = &-\frac{b_5 b_6\left[b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1} + \left(b_6 + b_6^{-1}\right)t\right]}{1 + b_5^2} \\ &+ \frac{1 + t(b_1 + b_2 + b_3 + b_4)b_6 + b_6^2}{2\lambda_n} + \frac{b_6\left(1 + b_5^2\right)}{2b_5}\lambda_n + \frac{\nu_n - tb_6}{\lambda_n^2}.\end{aligned}$$

This leaves v_0 to be determined. Imposing the relation $\mu_n = (\Omega_n + V)(\lambda_n, t)$ we can invert this and find

$$\begin{aligned}(1-q)\left(1 - b_5^2\right)v_0 = {}&(1-q)\left(1 - b_5^2\right)\mu_n + b_5 b_6\left[b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1} + \left(b_6 + b_6^{-1}\right)t\right]\lambda_n \\ &- 2b_5 b_6 \lambda_n^2 - \tfrac{1}{2}\left(1 + b_5^2\right)\left[1 + t(b_1 + b_2 + b_3 + b_4)b_6 + b_6^2\right] + \frac{\left(1 + b_5^2\right)}{\lambda_n}(b_6 t - \nu_n).\end{aligned}$$

This concludes our proof. ■

Remark 1. We observe that the appearance of the quantity $q^n b_1 b_4 b_6$ with $n \in \mathbb{Z}_{\geq 0}$ and its replacement by the new parameter b_5 constitutes a special condition. This condition is one of the necessary conditions for a member of our particular sequence of classical solutions to the $E_6^{(1)}$ q-Painlevé equations, and is built-in by our construction. The other condition derives from the initial conditions $n = 0$ in our construction, see (2.4) and following (2.20).

From our deformed weight (3.1) we compute the deformation data polynomials to be

$$R + \Delta u S = \frac{1}{b_6}(b_6 q t - x), \qquad R - \Delta u S = (qt - b_6 x). \tag{3.7}$$

We can verify that the compatibility relation (2.24) is identically satisfied by our spectral and deformation data polynomials. We see that this places us in the class $L = 1$. We will employ an abbreviation for the dependent variables evaluated at advanced or retarded times, e.g.,

$$\lambda_n(t) = \lambda_n, \qquad \lambda_n(qt) = \hat{\lambda}_n, \qquad \lambda_n\left(q^{-1}t\right) = \check{\lambda}_n.$$

In the second stage of our algorithm we parameterise the Cayley transform of the deformation matrix

$$B_n^* = \begin{pmatrix} \mathfrak{R}_+ & -\mathfrak{P}_+ \\ \mathfrak{P}_- & \mathfrak{R}_- \end{pmatrix}, \qquad n \geq 0,$$

consistent with known degrees, i.e., $\deg_x \mathfrak{R}_\pm = 1$, $\deg_x \mathfrak{P}_\pm = 0$, so that

$$\mathfrak{R}_\pm = r_{1,\pm}x + r_{0,\pm}, \qquad \mathfrak{P}_\pm = p_\pm.$$

Lemma 1. *Let us assume $b_6 \neq 0$ and $b_5 \neq q^{-1/2}, q^{1/2}$. Then the off-diagonal components of the deformation matrix are given by*

$$p_+ = -\hat{a}_n r_{1,-} + a_n r_{1,+}, \tag{3.8}$$
$$p_- = -a_n r_{1,-} + \hat{a}_n r_{1,+}. \tag{3.9}$$

Proof. We resolve the A-B compatibility relation (2.33) into monomials of x. Examining the x^7 coefficient of the $(1,2)$ and $(2,1)$ components yields (3.8) and (3.9) respectively. ■

Lemma 2. *Let us assume $b_5 \neq q^{1/2}$, $a_n, \hat{a}_n \neq 0$ and $\lambda_n \neq b_6 t, b_6^{-1} t$. Then the spectral and deformation matrices satisfy the following residue formulae*

$$\mathfrak{R}_-(b_6 qt, t) + \frac{\mathfrak{W}_+(b_6 qt, qt)}{\mathfrak{T}_+(b_6 qt, qt)} \mathfrak{P}_+(b_6 qt, t) = 0, \tag{3.10}$$
$$\mathfrak{R}_-(b_6^{-1} qt, t) + \frac{\mathfrak{W}_+(b_6^{-1} qt, qt)}{\mathfrak{T}_+(b_6^{-1} qt, qt)} \mathfrak{P}_+\big(b_6^{-1} qt, t\big) = 0, \tag{3.11}$$

and

$$\mathfrak{R}_+(b_6 qt, t) + \frac{\mathfrak{W}_-(b_6 t, t)}{\mathfrak{T}_+(b_6 t, t)} \mathfrak{P}_+(b_6 qt, t) = 0, \tag{3.12}$$
$$\mathfrak{R}_+(b_6^{-1} qt, t) + \frac{\mathfrak{W}_-(b_6^{-1} t, t)}{\mathfrak{T}_+(b_6^{-1} t, t)} \mathfrak{P}_+\big(b_6^{-1} qt, t\big) = 0. \tag{3.13}$$

Proof. In this step we compute the residues of the A-B compatibility relation, with respect to x, at the zeros and poles of

$$\chi(x,t) = \frac{(x - qb_6 t)(b_6 x - qt)}{q(x - b_6 t)(b_6 x - t)}. \tag{3.14}$$

From the residue of (2.34) at the zero $x = b_6 qt$ we deduce (3.10), and from the same equation at the zero $x = b_6^{-1} qt$ we deduce (3.11). From the residue of (2.37) at the pole $x = b_6 t$ we deduce (3.12), and from the same equation at the pole $x = b_6^{-1} t$ we deduce (3.13). ■

Remark 2. Although the above proof appealed to the vanishing of the right-hand side of one of the compatibility conditions, namely (2.34), at either of the two zeros of χ, in fact under these conditions the right-hand sides of all the other compatibility conditions, i.e. (2.35), (2.36), and (2.37), also vanish. This is because $\chi = 0$ implies $\big(R^2 - \Delta u^2 S^2\big)\big(b_6^{\pm 1} qt; t\big) = 0$ and $\big(W^2 - \Delta y^2 V^2\big)\big(b_6^{\pm 1} qt; qt\big) = 0$, and furthermore the spectral and deformation matrices satisfy the determinantal identities

$$\det A_n^* = \mathfrak{W}_+ \mathfrak{W}_- + \mathfrak{T}_+ \mathfrak{T}_- = W^2 - \Delta y^2 V^2,$$
$$\det B_n^* = \mathfrak{R}_+ \mathfrak{R}_- + \mathfrak{P}_+ \mathfrak{P}_- = \frac{a_n}{\hat{a}_n}\big(R^2 - \Delta u^2 S^2\big).$$

Therefore under the specialisations $x = b_6^{\pm 1} qt$ the right-hand sides of (2.35), (2.36), and (2.37) are proportional to the right-hand side of (2.34), and the vanishing of the latter implies the vanishing of the former. In this way we ensure that all components of the A-B compatibility vanish under the single condition. A similar observation applies to the left-hand sides of the compatibility relations at the zeros of χ^{-1}, i.e. $x = b_6^{\pm 1} t$.

We introduce our first change of variables, $\mu_n, \nu_n \mapsto z_\pm$, via the relations

$$\nu_n = \frac{1}{2}\lambda_n[\kappa_+ z_+ + \kappa_- z_-], \tag{3.15}$$

$$\mu_n = \frac{1}{2(q-1)}[\kappa_+ z_+ - \kappa_- z_-]. \tag{3.16}$$

The new variables satisfy an identity corresponding to (3.6) which reads

$$\kappa_+\kappa_- z_+ z_- = \frac{1}{\lambda_n^2}\left[W^2 - \Delta y^2 V^2\right](\lambda_n, t).$$

Next we subtract (3.11) from (3.10), in order to eliminate both z_- and $r_{0,-}$. This yields

$$\frac{qp_+}{\left(1-qb_5^2\right)\hat{a}_n}\left[-\frac{b_5\left(b_5qt\hat{\lambda}_n - 1\right)}{b_6\hat{\lambda}_n} + \frac{qb_5^2t}{\left(\hat{\lambda}_n - b_6qt\right)\left(b_6\hat{\lambda}_n - qt\right)}\hat{z}_+\right] + qt\frac{1}{b_6}r_{1,-} = 0. \tag{3.17}$$

This result motivates the definition of the new variable $\mathcal{Z}$

$$\mathcal{Z} = -\frac{b_5b_6qt}{(\hat{\lambda}_n - b_6qt)(b_6\hat{\lambda}_n - qt)}\hat{z}_+ + \left.\frac{b_5qt\hat{\lambda}_n - 1}{\hat{\lambda}_n}\right|_{t\to q^{-1}t}.$$

Definition 1. In terms of this new variable $\mathcal{Z}$ we have

$$z_+ = \frac{1}{b_5b_6t}\frac{(\lambda_n - tb_6)(b_6\lambda_n - t)[(b_5t - \mathcal{Z})\lambda_n - 1]}{\lambda_n}, \tag{3.18}$$

$$z_- = b_5t\frac{(b_1\lambda_n - 1)(b_2\lambda_n - 1)(b_3\lambda_n - 1)(b_4\lambda_n - 1)}{\lambda_n[(b_5t - \mathcal{Z})\lambda_n - 1]}. \tag{3.19}$$

Our final rewrite of the dependent variables is

$$\lambda_n(t) \to g(t), \tag{3.20}$$

$$\mathcal{Z}(t) \to b_5t - f(q^{-1}t). \tag{3.21}$$

We are now in a position to undertake the third stage of our derivation. The first of the evolution equations is given in the following result.

Proposition 2. *Let us assume that* $q \neq 0$, $b_5 \neq q^{-1/2}$, $t \neq 0$, $g \neq 0, b_6t, b_6^{-1}t$ *and* $a_n \neq 0$. *The variables* f, g *satisfy the first-order* q*-difference equation*

$$(g\check{f} - 1)(gf - 1) = t^2\frac{\left(g - b_1^{-1}\right)\left(g - b_2^{-1}\right)\left(g - b_3^{-1}\right)\left(g - b_4^{-1}\right)}{(g - b_6t)\left(g - b_6^{-1}t\right)}. \tag{3.22}$$

This evolution equation is identical to the second equation of equation (4.15) *of Kajiwara et al.* [6] *and to the second equation of equation* (3.23) *of Kajiwara et al.* [5].

Proof. Subtract (3.13) from (3.12) in order to eliminate both z_+ and $r_{0,+}$. This yields the relation

$$\frac{qp_+}{\left(-1+qb_5^2\right)a_n}\left[\frac{t}{(b_6t - \lambda_n)(-t + b_6\lambda_n)}z_- - \frac{(b_5 - t\lambda_n)}{b_6\lambda_n}\right] + qt\frac{1}{b_6}r_{1,+} = 0. \tag{3.23}$$

Thus we have two different ways of computing the ratio of $r_{1,+}$ to $r_{1,-}$; on the one hand we have from (3.17)

$$r_{1,+} = -\frac{[b_5(qb_5t - \hat{\mathcal{Z}}) - t]\hat{a}_n}{b_5\hat{\mathcal{Z}}a_n}r_{1,-}, \tag{3.24}$$

whereas using (3.23) we have

$$\begin{aligned}\frac{b_5 a_n}{\hat{a}_n}\frac{r_{1,+}}{r_{1,-}} = \Big[&b_5 b_6 t^2 (b_1\lambda_n - 1)(b_2\lambda_n - 1)(b_3\lambda_n - 1)(b_4\lambda_n - 1)\\ &+ (b_5 - t\lambda_n)(\lambda_n - tb_6)(b_6\lambda_n - t)[\lambda_n(b_5 t - \mathcal{Z}) - 1]\Big]\\ \div \Big[&t^2 b_6 (b_1\lambda_n - 1)(b_2\lambda_n - 1)(b_3\lambda_n - 1)(b_4\lambda_n - 1)\\ &- (qb_5 t\lambda_n - 1)(\lambda_n - tb_6)(b_6\lambda_n - t)[\lambda_n(b_5 t - \mathcal{Z}) - 1]\Big].\end{aligned}$$

Equating these two forms gives (3.22). ■

The second evolution equation, to be paired with the first (3.22) as a coupled system, is given next.

Proposition 3. *Let us make the following assumptions: $t \neq 0$, $b_5 \neq 1, q^{-1/2}, q^{-1}$, $f \neq 0$, $b_5 f \neq t$, $g \neq 0, \hat{g} \neq 0$ and $a_n \neq 0$. In addition let us assume that the condition*

$$\hat{g} \neq \frac{1 - qb_5 tg - qb_5^2 + qb_5^2 fg}{f - b_5 qt},$$

holds. The variables f, g satisfy the first-order q-difference equation

$$(f\hat{g} - 1)(fg - 1) = qt^2 \frac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - b_5 qt)\big(f - b_5^{-1} t\big)}. \tag{3.25}$$

This evolution equation is the same as the first equation of equation (4.15) *in Kajiwara et al.* [6] *and the first equation of equation* (3.23) *in Kajiwara et al.* [5]*, both subject to typographical corrections.*

Proof. Cross multiplying the relations (3.10), (3.11), (3.12), (3.13) we can eliminate all reference to the deformation matrix and deduce the identity

$$\frac{\mathfrak{W}_+(b_6 qt, qt)}{\mathfrak{T}_+(b_6 qt, qt)}\frac{\mathfrak{W}_-(b_6 t, t)}{\mathfrak{T}_+(b_6 t, t)} = \frac{\mathfrak{W}_+\big(qb_6^{-1} t, qt\big)}{\mathfrak{T}_+\big(qb_6^{-1} t, qt\big)}\frac{\mathfrak{W}_-\big(b_6^{-1} t, t\big)}{\mathfrak{T}_+\big(b_6^{-1} t, t\big)}. \tag{3.26}$$

Into this identity we employ the following evaluations for the advanced and retarded values of $z_\pm$

$$\begin{aligned}\hat{z}_+ &= \frac{q^{-1}}{b_5 b_6 t}\frac{(f\hat{g} - 1)(\hat{g} - b_6 qt)(\hat{g} b_6 - qt)}{\hat{g}},\\ \hat{z}_- &= qb_5 t \frac{(\hat{g} b_1 - 1)(\hat{g} b_2 - 1)(\hat{g} b_3 - 1)(\hat{g} b_4 - 1)}{\hat{g}(f\hat{g} - 1)},\\ z_+ &= \frac{t}{b_5}\frac{(gb_1 - 1)(gb_2 - 1)(gb_3 - 1)(gb_4 - 1t)}{g(fg - 1)},\\ z_- &= \frac{b_5}{b_6 t}\frac{(fg - 1)(g - b_6 t)(gb_6 - t)}{g}.\end{aligned}$$

We find that this relation factorises into two non-trivial factors, the first of which is proportional to

$$\hat{g} - \frac{1 - qb_5 tg - qb_5^2 + qb_5^2 fg}{f - qb_5 t}.$$

Assuming this is non-zero our evolution equation is then the remaining factor of (3.26)

$$(f\hat{g} - 1)(fg - 1) = qb_5 t^2 \frac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - qb_5 t)(b_5 f - t)},$$

or alternatively (3.25). ■

Lastly we have an auxiliary evolution equation which controls the normalisation of the orthogonal polynomial system.

Proposition 4. *Let us assume $b_6 \neq 0$, $b_5 \neq q^{-1/2}$, $f \neq b_5 qt, b_5^{-1} t$ and $\gamma_n \neq 0$ for $n \geq 0$. The leading coefficient of the polynomials or second-kind solutions (see (2.6), (2.7)) satisfy the first-order q-difference equation*

$$\left(\frac{\hat{\gamma}_n}{b_6 \gamma_n}\right)^2 = \frac{f - b_5^{-1} t}{f - b_5 qt}. \tag{3.27}$$

Proof. Using the leading order, i.e., the $[x]$ terms, in the expansions (2.27), (2.29) with definitions (2.31) we can compute $r_{1,+}$. However by using these same expansions to compute $r_{1,-}$ and the equation (3.24), which relates these two quantities, we have an alternative expression for $r_{1,+}$. Equating these expressions then gives (3.27). ■

We conclude our discussion by summarising our results for the spectral and deformation matrices in terms of the f_n, g_n variables. Henceforth we will restore the index n on all our variables. The form of the spectral matrix is given in the following proposition.

Proposition 5. *Assume that $|q| \neq 1$, $b_6 \neq 0$, $b_5^2 \neq q^{-1}, 1, q$ and $a_n \neq 0$. The spectral matrix elements (2.19), (2.20), (2.21) are given by*

$$\mathfrak{T}_{n,+} = \frac{b_6}{b_5}\left(q^{-1} - b_5^2\right) a_n x (x - g_n), \qquad \mathfrak{T}_{n,-} = \frac{b_6}{b_5}\left(1 - q^{-1} b_5^2\right) a_n x (x - g_{n-1}),$$

and

$$\begin{aligned} \frac{\mathfrak{W}_{n,+}}{x(x-g_n)} &= -x b_5 b_6 + \frac{b_6}{\left(1-b_5^2\right)}\left[-\frac{b_5^2}{t} + b_5\left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}\right)\right] \\ &\quad - \frac{b_6 (b_5 f_n - t)}{(1-b_5^2) t}\left[\frac{g_n (t - f_n b_5)}{f_n} + \frac{t b_5 \left(1 + b_6^2\right)}{b_6}\right] \\ &\quad + \frac{b_6 t}{(1-b_5^2)}\left[\frac{b_5^2}{g_n^2} - \frac{1-b_5^2}{x g_n} - (b_1 + b_2 + b_3 + b_4) b_5^2 \frac{1}{g_n} + b_5^2 \frac{f_n}{g_n} - \frac{g_n}{f_n} \right. \\ &\quad \left. + \frac{(1 - g_n b_1)(1 - g_n b_2)(1 - g_n b_3)(1 - g_n b_4)\left(g_n - x b_5^2\right)}{(1 - f_n g_n) g_n^2 (x - g_n)}\right], \end{aligned} \tag{3.28}$$

or

$$\begin{aligned} \frac{\mathfrak{W}_{n,+}}{x(x-g_n)} &= -\frac{b_6 t}{\left(1-b_5^2\right)} \frac{(f_n - b_1)(f_n - b_2)(f_n - b_3)(f_n - b_4)\left(1 - b_5^2 f_n x\right)}{f_n^2 (1 - f_n g_n)(1 - f_n x)} \\ &\quad + \frac{b_6 t (1 - x b_1)(1 - x b_2)(1 - x b_3)(1 - x b_4)}{x (1 - x f_n)(x - g_n)} - \frac{b_6 (b_5 f_n - t)}{f_n} x \\ &\quad - \frac{b_6 (b_5 f_n - t)}{b_5 \left(1 - b_5^2\right) t}\left\{\frac{b_5^2 \left(1 + b_6^2\right) t}{b_6} + \frac{b_5 g_n}{f_n}(t - b_5 f_n) \right. \\ &\quad \left. + \frac{b_5}{f_n^2}\left[t + b_5 f_n - t f_n \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4}\right)\right]\right\}, \end{aligned}$$

and

$$\frac{t \mathfrak{W}_{n,-}}{x(x-g_n)} = \frac{(1 - x f_n)(x - b_6 t)(x b_6 - t)}{x (x - g_n)} - \frac{b_6}{1 - b_5^2} \frac{(t - b_1 b_5)(t - b_2 b_5)(t - b_3 b_5)(t - b_4 b_5)}{b_5 (t g_n - b_5)}$$

$$+\frac{b_6(b_5f_n-t)}{b_5\left(1-b_5^2\right)}\left[\left(1-b_5^2\right)x-\frac{t^2}{g_n}-g_nb_5^2+\frac{b_5^2\left(1+b_6^2\right)t}{b_6}\right.$$
$$\left.-\frac{b_5t^2(1-g_nb_1)(1-g_nb_2)(1-g_nb_3)(1-g_nb_4)}{g_n(1-f_ng_n)(tg_n-b_5)}\right], \tag{3.29}$$

or

$$\frac{t\mathfrak{W}_{n,-}}{x(x-g_n)}=\frac{(1-xf_n)(x-b_6t)(b_6x-t)}{x(x-g_n)}+\frac{b_6t^2}{\left(1-b_5^2\right)}\frac{(f_n-b_1)(f_n-b_2)(f_n-b_3)(f_n-b_4)}{f_n^2(1-f_ng_n)}$$
$$+\frac{b_6(b_5f_n-t)}{b_5\left(1-b_5^2\right)}\left\{\left(1-b_5^2\right)x+\frac{b_5^2\left(1+b_6^2\right)t}{b_6}+\frac{b_5g_n}{f_n}(t-b_5f_n)\right.$$
$$\left.+\frac{b_5}{f_n^2}\left[t+b_5f_n-tf_n\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}\right)\right]\right\}.$$

Proof. This follows from applying the transformations (3.15), (3.16), (3.18), (3.19) and (3.20), (3.21) successively to (3.5), (3.3) and (3.4). The alternative forms arise from applying partial fraction expansions with respect to either of f_n or g_n. ∎

The deformation matrix is summarised in the next result.

Proposition 6. *Assume that $|q|\neq 1$, $b_6\neq 0$, $b_5^2\neq q^{-1},1,q$. The deformation matrix elements are given by*

$$\mathfrak{R}_{n,+}=\frac{\hat{\gamma}_n}{b_6\gamma_n}\left\{x+\frac{1}{1-b_5^2}\left[-\frac{q\left(t^2b_6+g_n^2b_5^2b_6-tg_nb_5^2\left(1+b_6^2\right)\right)}{g_nb_6}\right.\right.$$
$$+\frac{qt^2b_5(1-b_1g_n)(1-b_2g_n)(1-b_3g_n)(1-b_4g_n)}{g_n(f_ng_n-1)(tg_n-b_5)}$$
$$\left.\left.-\frac{q(t-b_1b_5)(t-b_2b_5)(t-b_3b_5)(t-b_4b_5)}{(tg_n-b_5)(f_nb_5-t)}\right]\right\}, \tag{3.30}$$

and

$$\mathfrak{R}_{n,-}=\frac{b_6\gamma_{n-1}}{\hat{\gamma}_{n-1}}\left\{x+\frac{1}{1-b_5^2}\left[\frac{q\left(t^2b_6+g_n^2b_5^2b_6-tg_n\left(1+b_6^2\right)\right)}{g_nb_6}\right.\right.$$
$$-\frac{qt^2b_5(1-b_1g_n)(1-b_2g_n)(1-b_3g_n)(1-b_4g_n)}{g_n(f_ng_n-1)(tg_n-b_5)}$$
$$\left.\left.+\frac{q(t-b_1b_5)(t-b_2b_5)(t-b_3b_5)(t-b_4b_5)}{(tg_n-b_5)(f_nb_5-t)}\right]\right\}.$$

Furthermore

$$\mathfrak{P}_{n,+}=a_n\left[\frac{\hat{\gamma}_n}{b_6\gamma_n}-\frac{b_6\gamma_n}{\hat{\gamma}_n}\right],\qquad \mathfrak{P}_{n,-}=a_n\left[\frac{\hat{\gamma}_{n-1}}{b_6\gamma_{n-1}}-\frac{b_6\gamma_{n-1}}{\hat{\gamma}_{n-1}}\right].$$

Proof. Using the leading orders in the expansions (2.27), (2.29), i.e., the $[x]$ terms, with definitions (2.31) we deduce

$$r_{1,+}=\frac{\hat{\gamma}_n}{b_6\gamma_n},\qquad r_{1,-}=\frac{b_6\gamma_{n-1}}{\hat{\gamma}_{n-1}}.$$

Using the leading orders in the expansions (2.28), i.e., the $[x^0]$ terms, with definition (2.31) and (2.26) we deduce

$$p_+ = a_n\left[\frac{\hat{\gamma}_n}{b_6\gamma_n} - \frac{b_6\gamma_n}{\hat{\gamma}_n}\right], \qquad p_- = a_n\left[\frac{\hat{\gamma}_{n-1}}{b_6\gamma_{n-1}} - \frac{b_6\gamma_{n-1}}{\hat{\gamma}_{n-1}}\right].$$

Using the coefficient of the $[x^7]$ term in the $(1,1)$ element of the A-B compatibility relations (2.32), along with the solution of (3.22) for $f_n(q^{-1}t)$ and (3.25) for $g_n(qt)$ we deduce

$$\begin{aligned}\left(1-b_5^2\right)\frac{r_{0,+}}{r_{1,+}} &= qb_5\left(\frac{t}{f_n} - b_5\right)g_n + \left(\frac{b_5qt}{f_n} - 1\right)\hat{g}_n \\ &\quad + \left[1 + qb_5^2 - b_5\left(b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1}\right)qt\right]\frac{1}{f_n} + \frac{\left(1+b_6^2\right)}{b_6}b_5^2qt,\end{aligned}$$

whereas if we examine the $[x^7]$ term in the $(2,2)$ element of the A-B compatibility relations in the same way then we find

$$\begin{aligned}\left(1-b_5^2\right)\frac{r_{0,-}}{r_{1,-}} &= qb_5\left(b_5 - \frac{t}{f_n}\right)g_n + \left(1 - \frac{b_5qt}{f_n}\right)\hat{g}_n \\ &\quad + \left[b_5\left(b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1}\right)qt - qb_5^2 - 1\right]\frac{1}{f_n} - \frac{\left(1+b_6^2\right)}{b_6}qt.\end{aligned}$$

Into both of these expressions we can employ (3.25) for $\hat{g}_n$ and make a partial fraction expansion with respect to f_n. ■

4 Reconciliation with the Lax pairs of Sakai and Yamada

4.1 Sakai Lax Pair

In [15] Sakai constructed a Lax pair for the $E_6^{(1)}$ q-Painlevé equations using a degeneration of a two-variable case of the Garnier system based upon the Lax pairs for the $D_5^{(1)}$ q-Painlevé system [14]. Subsequently Murata [11] gave more details for this Lax pair. We intend to establish a correspondence between our Lax pair and that of Sakai. We will carry this out in a sequence of simple steps rather than as a single step as this will reveal how similar they are.

Our first step is to give a variation on the parameterisation of the spectral and deformation matrices to that given in Section 3. In this alternative formulation, we seek a spectral matrix $\tilde{A}(x;t)$ (actually identical to the Cayley transform A_n^*) with the specifications

$$\tilde{A}(x;t) = A_0 + A_1x + A_2x^2 + A_3x^3, \tag{4.1}$$

and

(i) the determinant is

$$b_6\left(b_1x-1\right)\left(b_2x-1\right)\left(b_3x-1\right)\left(b_4x-1\right)\left(x-b_6t\right)\left(b_6x-t\right),$$

(ii) A_3 is diagonal with entries $\kappa_1 = -b_5b_6$ and $\kappa_2 = -b_6/b_5$,

(iii) $A_0 = b_6t\mathbb{1}$,

(iv) the root of the (1,2) entry of $\tilde{A}(x;t)$ with respect to x is λ,

(v) $\tilde{A}(\lambda;t)$ is lower triangular with diagonal entries $-b_5b_6\lambda z_+$ and $-\frac{b_6\lambda z_-}{b_5}$ where $b_6^2z_-z_+\lambda^2 = \det\tilde{A}(\lambda;t)$.

Any such matrix is in the general form

$$\tilde{A}(x;t) = tb_6 I - \begin{pmatrix} b_5 b_6 x[z_1 + (x-\alpha)(x-\lambda)] & \dfrac{b_6 w x\,(x-\lambda)}{b_5} \\ \dfrac{b_5 b_6 x(x\gamma+\delta)}{w} & \dfrac{b_6 x[z_2 + (x-\beta)(x-\lambda)]}{b_5} \end{pmatrix},$$

where the properties specify the variables

$$\begin{aligned}(1-b_5^2)\alpha &= \frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}+\left(\frac{1}{b_6}+b_6\right)t \\ &\quad - (b_1+b_2+b_3+b_4)b_5\frac{t}{\lambda} - b_5\left(\frac{1}{b_6}+b_6\right)\frac{1}{\lambda}+\frac{b_5^2 z_1}{\lambda}+\frac{z_2}{\lambda}-2\lambda,\end{aligned} \tag{4.2}$$

$$\begin{aligned}(1-b_5^2)\beta &= -\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}\right)b_5^2-\left(\frac{1}{b_6}+b_6\right)b_5^2 t \\ &\quad + (b_1+b_2+b_3+b_4)b_5\frac{t}{\lambda} + b_5\left(\frac{1}{b_6}+b_6\right)\frac{1}{\lambda}-\frac{b_5^2 z_1}{\lambda}-\frac{z_2}{\lambda}+2b_5^2\lambda,\end{aligned} \tag{4.3}$$

$$\begin{aligned}\gamma &= -(b_3b_4+b_2b_3+b_2b_4+b_1b_2+b_1b_3+b_1b_4) - \left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}\right)\left(\frac{1}{b_6}+b_6\right)t \\ &\quad - t^2+\alpha\beta+z_1+z_2+2(\alpha+\beta)\lambda+\lambda^2,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\delta &= b_1+b_2+b_3+b_4+\left[(b_3b_4+b_2b_3+b_2b_4+b_1b_2+b_1b_3+b_1b_4)\left(\frac{1}{b_6}+b_6\right)-\left(\frac{1}{b_5}+b_5\right)\right]t \\ &\quad +\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}\right)t^2 - z_1(\beta+\lambda)-z_2(\alpha+\lambda)+(-2\alpha\beta+\gamma)\lambda-(\alpha+\beta)\lambda^2.\end{aligned} \tag{4.5}$$

The z_1 and z_2 are related to $z_\pm$ by

$$z_1 = z_+ + \frac{t}{b_5\lambda} \qquad \text{and} \qquad z_2 = z_- + \frac{tb_5}{\lambda}. \tag{4.6}$$

In addition

$$w = \frac{1-qb_5^2}{q}a_n.$$

We seek a deformation matrix $\tilde{B}(x;t)$ of the form

$$\tilde{B}(x;t) = \frac{x}{(x-b_6qt)\left(x-b_6^{-1}qt\right)}(x\mathbb{1}+B_0), \qquad \text{where} \qquad B_0 = \begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2}\end{bmatrix}. \tag{4.7}$$

This leads to the compatibility relation

$$\tilde{B}(qx;t)\tilde{A}(x;t) = \tilde{A}(x;qt)\tilde{B}(x;t). \tag{4.8}$$

This relation is just a rewriting of (2.32) whereby all the factors of χ are placed into the denominator of $\tilde{B}$ by the above definition.

Lemma 3. *The overdetermined system* (4.8)*, with* (4.1) *and* (4.7) *is satisfied if the coupled $E_6^{(1)}$ q-Painlevé equations* (3.22) *and* (3.25) *are satisfied.*

Proof. Examining the coefficient of x^6 in the numerator of the (1,2) entry of (4.8) we find

$$r_{1,2} = \frac{q}{1-qb_5^2}(\hat{w}-w). \tag{4.9}$$

Now we seek two alternative expressions for $r_{1,2}$ – one involving quantities at the advanced time qt and another involving those at the unshifted time t. The first of these is found from solving for the (1,2) entry of the residue of (4.8) at $x = b_6qt$ simultaneously with the (1,2) entry of the residue of (4.8) at $x = b_6^{-1}qt$. This yields

$$r_{1,2} = \frac{-qt\hat{w}\big(qtb_6 - \hat{\lambda}\big)\big(qt - b_6\hat{\lambda}\big)}{b_5\left\{qt\big(1 - b_5b_6\hat{z}_1\big) + \big(qtb_6 - \hat{\lambda}\big)\big[b_6 + qtb_5\big(qt - b_6\hat{\lambda}\big)\big]\right\}}. \tag{4.10}$$

The second expression for $r_{1,2}$ is found from solving for the (1,2) entry of the residue of (4.8) at $x = b_6t$ simultaneously with the (1,2) entry of the residue of (4.8) at $x = b_6^{-1}t$. This gives

$$r_{1,2} = \frac{-qtw(b_6t - \lambda)(t - b_6\lambda)}{tb_5 - tb_6z_2 + b_5b_6(b_6t - \lambda) + t(b_6t - \lambda)(t - b_6\lambda)}. \tag{4.11}$$

Combining (4.9) and (4.10) or (4.9) and (4.11), and employing the change of variables (3.18) and (3.19) with (3.20) and (3.21), we can solve for $\hat{w}$ in two ways. Assuming w is non-zero it cancels out, leaving an expression for $\hat{\mathcal{Z}}$ in terms of $\mathcal{Z}$ and λ. This is equivalent to the first $E_6^{(1)}$ q-Painlevé equation (3.22).

To find the second equation we solve (4.8) for $\tilde{A}(x;t)$

$$\tilde{A}(x;t) = \tilde{B}(qx;t)^{-1}\tilde{A}(x;qt)\tilde{B}(x;t),$$

and use this to find the zero of $\tilde{A}(x;t)_{12}$, i.e., $g(t)$. In addition to $r_{1,2}$ we now require $r_{2,2}$ (even though the denominator of $\tilde{A}(x;t)_{12}$ depends on $r_{1,1}$, $r_{1,2}$, $r_{2,1}$, $r_{2,2}$ identities resulting from the compatibility conditions imply that this will trivialise – see the subsequent observation). The entry $r_{2,2}$ has already been found, along with $r_{1,2}$, from the arguments given earlier and this is

$$\begin{aligned} r_{2,2} = \hat{\lambda} - \frac{qt\big(1 + b_6^2\big)}{b_6} - \big(b_6qt - \hat{\lambda}\big)\big(qt - b_6\hat{\lambda}\big) \\ \times \frac{b_6 + q^2t^2b_5\big(1 + b_6^2\big) - b_5b_6qt\big(\hat{\alpha} + \hat{\lambda}\big)}{b_6^2\big(\hat{\lambda} - b_6qt\big) - b_6qt + b_5b_6^2qt\hat{z}_1 - b_5b_6qt\big(b_6qt - \hat{\lambda}\big)\big(qt - b_6\hat{\lambda}\big)}. \end{aligned} \tag{4.12}$$

The numerator of $\tilde{A}(x;t)_{12}$ appears to be a polynomial of degree 6 in x, however it has trivial zeros matching those of the denominator

$$\frac{q^2b_5}{b_6^2}(x - b_6t)(x - b_6qt)(b_6x - t)(b_6x - qt)\hat{w},$$

so that their ratio is in fact polynomial of degree 2. Into $\tilde{A}(x;t)_{12}$ we first substitute for $r_{1,2}$ using (4.9), then for $r_{2,2}$ using (4.12), and thirdly for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$ using (4.2), (4.3), (4.4), (4.5) at up-shifted times, respectively. Into the resulting expression we employ (4.6) for $\hat{z}_1$, $\hat{z}_2$ along with (3.18), (3.19) at the up-shifted time to bring the whole expression in terms of $\hat{\lambda}$ and $\hat{\mathcal{Z}}$. The relevant zero of the ensuing expression (the other zero is $x = 0$) then gives $\lambda = g$ in terms of $\hat{\lambda} = \hat{g}$ and f, or equivalently by (3.25). ■

Now we recount the formulation given by Sakai [15] and Murata [11]. Their Lax pairs are

$$Y(qx;t) = A(x,t)Y(x;t), \qquad Y(x;qt) = B(x,t)Y(x;t),$$

satisfying the compatibility condition $A(x,qt)B(x,t) = B(qx,t)A(x,t)$. The spectral matrix is parameterised in the following way

$$A(x,t) = \begin{pmatrix} \kappa_1W(x,t) & \kappa_2wL(x,t) \\ \kappa_1w^{-1}X(x,t) & \kappa_2Z(x,t) \end{pmatrix} = A_0 + A_1x + A_2x^2 + A_3x^3,$$

subject to the key properties

(i) the determinant of $A(x,t)$ is

$$\kappa_1\kappa_2(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5t)(x-a_6t),$$

(ii) A_3 is diagonal with entries κ_1 and $\kappa_2 = q\kappa_1$,

(iii) A_0 has eigenvalues $\theta_1 t$ and $\theta_2 t$,

(iv) the single root of the (1,2) entry of $A(x,t)$ in x is λ,

(v) $A(\lambda,t)$ is lower triangular with diagonal entries $\kappa_1\mu_1$ and $\kappa_2\mu_2$.

Given these requirements, the entries of $A(x,t)$ are specified by

$$\begin{aligned}
&L(x,t) = x-\lambda,\\
&Z(x,t) = \mu_2 + (x-\lambda)\big[\delta_2 + x^2 + x(\gamma+\lambda)\big],\\
&W(x,t) = \mu_1 + (x-\lambda)\big[\delta_1 + x^2 + x(-\gamma-e_1+\lambda)\big],\\
&X(x,t) = \big[WZ - (x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5t)(x-a_6t)\big]L^{-1},
\end{aligned}$$

where

$$\begin{aligned}
&(\kappa_1-\kappa_2)\delta_1 = \lambda^{-1}[\kappa_1\mu_1+\kappa_2\mu_2-\theta_1t-\theta_2t] - \kappa_2\big[\gamma(\gamma+e_1)+2\lambda^2-\lambda e_1+e_2\big],\\
&(\kappa_1-\kappa_2)\delta_2 = -\lambda^{-1}[\kappa_1\mu_1+\kappa_2\mu_2-\theta_1t-\theta_2t] + \kappa_1\big[\gamma(\gamma+e_1)+2\lambda^2-\lambda e_1+e_2\big],\\
&\mu_1\mu_2 = (\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)(\lambda-a_5t)(\lambda-a_6t),\\
&\theta_1\theta_2 = a_1a_2a_3a_4a_5a_6q\kappa_1^2.
\end{aligned}$$

Here e_j is the j^{th} elementary symmetric function of the indeterminates $\{a_1,a_2,a_3,a_4,a_5t,a_6t\}$. Despite the expression for $X(x,t)$, it is a quadratic polynomial in x. In Murata's notation we have $\tilde{\mu} = \mu_1$, $\mu = \mu_2$, $\tilde{\delta} = \delta_1$ and $\delta = \delta_2$. The deformation matrix $B(x,t)$ is a rational function in x of the form

$$B(x,t) = \frac{x(x\mathbb{1}+B_0)}{(x-a_5qt)(x-a_6qt)}.$$

Next we consider the first transformation of the Sakai linear problem with the following definition:

$$\mathcal{Y}(x,t) = \mathcal{S}(x,t)^{-1}Y(x,t),$$

and

$$\mathcal{S} = \begin{pmatrix} 1 & 0\\ s_1+s_2x & x\end{pmatrix}.$$

The transformed spectral linear problem is

$$\mathcal{Y}(qx,t) = \mathcal{A}(x,t)\mathcal{Y}(x,t),$$

with a transformed spectral matrix

$$\mathcal{A}(x,t) = \mathcal{S}(qx,t)^{-1}A(x,t)\mathcal{S}(x,t).$$

We fix the parameters of the transformation by the requirement that the coefficient of x^{-1} in the (2,1) entry of $\mathcal{S}$ is zero (only the (2,1) entry is non-zero) and also that the coefficient of x^0 in the (2,1) entry of $\mathcal{S}$ is zero. Thus we find

$$s_1 = \frac{1}{2q\kappa_1w\lambda}\big[(\theta_2-\theta_1)t + \kappa_1(\mu_1 - q\mu_2 + (q\delta_2-\delta_1)\lambda)\big],$$

$$
\begin{aligned}
q(q\theta_1-\theta_2)ws_2 = {} & \left(\frac{2\theta_1\theta_2}{\kappa_1}t - q\theta_1\mu_2 - \theta_2\mu_1\right)\frac{1}{\lambda^2} - q\kappa_1 e_5 t^{-1}\frac{1}{\lambda} \\
& - e_1\theta_2 + (q\theta_1-\theta_2)\gamma + (q\theta_1+\theta_2)\lambda.
\end{aligned}
$$

The new spectral matrix can be parameterised by the polynomial

$$
\mathcal{A}(x,t) = \mathcal{A}_0 + \mathcal{A}_1 x + \mathcal{A}_2 x^2 + \mathcal{A}_3 x^3,
$$

and possesses the following properties

(i) the determinant of $\mathcal{A}(x,t)$ is

$$
\kappa_1^2(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5t)(x-a_6t),
$$

(ii) $\mathcal{A}_3 = \kappa_1\mathbb{1}$,

(iii) $\mathcal{A}_0$ is diagonal with entries $\theta_1 t$ and $q^{-1}\theta_2 t$,

(iv) the roots of the (1,2) entry of $\mathcal{A}$ in x are 0 and λ,

(v) $\mathcal{A}(\lambda,t)$ is lower triangular with diagonal entries $\kappa_1\mu_1$ and $\kappa_2\mu_2$.

Any such matrix admits the general form

$$
\mathcal{A}(x,t) = \begin{pmatrix} \theta_1 t + \kappa_1 x[(x-\lambda)(x-a)+\nu_1] & q\kappa_1 w x(x-\lambda) \\ \kappa_1 w^{-1} x(xc+d) & q^{-1}\theta_2 t + \kappa_1 x[(x-\lambda)(x-b)+\nu_2] \end{pmatrix},
$$

where the properties given above fix the introduced parameters as

$$
\begin{aligned}
(q\theta_1-\theta_2)a &= \left[\theta_2\nu_1 + q\theta_1\nu_2 + q\kappa_1 e_5 t^{-1}\right]\lambda^{-1} + q\theta_1 e_1 - 2q\theta_1\lambda, \\
(q\theta_1-\theta_2)b &= -\left[\theta_2\nu_1 + q\theta_1\nu_2 + q\kappa_1 e_5 t^{-1}\right]\lambda^{-1} - \theta_2 e_1 + 2\theta_2\lambda, \\
qc &= ab + 2(a+b)\lambda + \lambda^2 - e_2 + \nu_1 + \nu_2, \\
qd &= -(a+b)\lambda^2 - 2ab\lambda - a\nu_2 - b\nu_1 + (qc-\nu_1-\nu_2)\lambda + e_3 + \frac{q\theta_1+\theta_2}{q\kappa_1}t.
\end{aligned}
$$

The variables, ν_1 and ν_2 are defined by

$$
\nu_1 = \frac{\kappa_1\mu_1 - \theta_1 t}{\kappa_1\lambda} \qquad \text{and} \qquad \nu_2 = \frac{q\kappa_1\mu_2 - \theta_2 t}{q\kappa_1\lambda}.
$$

The transformed deformation matrix $\mathcal{B}$ is computed using

$$
\mathcal{B}(x,t) = \mathcal{S}(x,qt)^{-1}B(x,t)\mathcal{S}(x,t),
$$

and has the form

$$
\mathcal{B} = \frac{x(x\mathcal{B}_0 + \mathbb{1})}{(x-a_5qt)(x-a_6qt)}.
$$

We define a new variable ν using

$$
\mu_2 \equiv \frac{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)}{\lambda-\check{\nu}},
$$

and by implication

$$
\mu_1 \equiv (\lambda-a_5t)(\lambda-a_6t)(\lambda-\check{\nu}).
$$

Using identical techniques to those employed in the proof of Lemma 3, we can show that the compatibility relation leads to the evolution equations

$$(\lambda-\check{\nu})(\lambda-\nu)=\frac{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)}{(\lambda-a_5t)(\lambda-a_6t)},$$
$$\left(1-\frac{\nu}{\hat{\lambda}}\right)\left(1-\frac{\nu}{\lambda}\right)=\frac{a_5a_6}{q}\frac{(\nu-a_1)(\nu-a_2)(\nu-a_3)(\nu-a_4)}{(a_5a_6t\nu+\theta_1/q\kappa_1)(a_5a_6t\nu+\theta_2/q\kappa_1)}.$$

To make the full correspondence between our system and this one we must consider a further transformation, given by the linear solution

$$\mathfrak{Y}(x,t)=\frac{\left[\vartheta_q\left(q^{-1}x\right)\right]^3}{e_{q,t}(x)}\mathcal{Y}\left(x^{-1},t^{-1}\right).$$

The prefactors are elliptic functions defined in terms of the q-factorial by

$$\vartheta_q(z)=\left(q,-qz,-z^{-1};q\right)_\infty,\qquad e_{q,t}(z)=\frac{\vartheta_q(z)\vartheta_q\left(t^{-1}\right)}{\vartheta_q\left(zt^{-1}\right)},$$

with properties

$$\vartheta_q(qz)=qz\vartheta_q(z),\qquad e_{q,t}(qz)=te_{q,t}(z),\qquad e_{q,qt}(z)=ze_{q,t}(z).$$

This is the solution satisfying the linear equations

$$\mathfrak{Y}\left(q^{-1}x,t\right)=\mathfrak{A}(x,t)\mathfrak{Y}(x,t),\qquad \mathfrak{Y}\left(x,q^{-1}t\right)=\mathfrak{B}(x,t)\mathfrak{Y}(x,t).$$

The transformed spectral matrix $\mathfrak{A}$ is given by

$$\mathfrak{A}(x,t)=tx^3\mathcal{A}\left(x^{-1},t^{-1}\right)=\mathfrak{A}_3+\mathfrak{A}_2x+\mathfrak{A}_1x^2+\mathfrak{A}_0x^3, \tag{4.13}$$

which swaps the roles of the leading matrices around $x=0$ and $x=\infty$. This spectral matrix has the properties

(i) the determinant of $\mathfrak{A}(x,t)$ is
$$\kappa_1^2(1-a_1x)(1-a_2x)(1-a_3x)(1-a_4x)(t-a_5x)(t-a_6x),$$
(ii) $\mathfrak{A}_3=\kappa_1t\mathbb{1}$,
(iii) $\mathfrak{A}_0$ is diagonal with entries θ_1 and $q^{-1}\theta_2$,
(iv) the roots of the (1,2) entry of $\mathfrak{A}(x,t)$ in x are 0 and λ^{-1},
(v) $\mathfrak{A}(\lambda,t)$ is lower triangular with diagonal entries $\kappa_1\mu_1t\lambda^{-3}$ and $\kappa_1\mu_2t\lambda^{-3}$.

The transformed deformation matrix has the form

$$\mathfrak{B}(x,t)=\frac{x}{(t-a_5qx)(t-a_6qx)}(x\mathbb{1}+\mathfrak{B}_0). \tag{4.14}$$

Since the compatibility relation between (4.13) and (4.14) is rationally equivalent to that for $\mathcal{Y}$, the evolution equations are the same.

It is clear that $\mathfrak{Y}$ and Y_n satisfy equivalent linear problems and that the following correspondences hold:

$$q\mapsto q^{-1},\qquad t\mapsto t^{-1},\qquad \lambda(t)\mapsto\frac{1}{g(t^{-1})},\qquad \nu(t)\mapsto f\left(t^{-1}\right),$$
$$\kappa_1\mapsto b_6,\qquad a_i\mapsto b_i\quad i=1,2,3,4,\qquad a_5\mapsto\frac{1}{b_6},\qquad \theta_1\mapsto -b_5b_6,\qquad q^{-1}\theta_2\mapsto -\frac{b_6}{b_5}.$$

4.2 Reconciliation with the Lax pair of Yamada [18]

In his derivation of a Lax pair for the $E_6^{(1)}$ q-Painlevé system Yamada employed the degeneration limits of $E_8^{(1)}$ q-Painlevé $\to$ $E_7^{(1)}$ q-Painlevé $\to$ $E_6^{(1)}$ q-Painlevé. In doing so he retained eight parameters $b_1,\dots,b_8$ constrained by $qb_1b_2b_3b_4 = b_5b_6b_7b_8$, and his $E_6^{(1)}$ q-Painlevé equation was given by the mapping of the variables

$$t \mapsto q^{-1}t, \qquad f,g \mapsto \bar{f},\bar{g},$$

subject to the coupled first-order system (see his (36))

$$\frac{(fg-1)(\bar{f}g-1)}{f\bar{f}} = q\frac{(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)}{b_5b_6(b_7g-t)(b_8g-t)}, \tag{4.15}$$

$$\frac{(f\underline{g}-1)(fg-1)}{g\underline{g}} = \frac{(b_1-f)(b_2-f)(b_3-f)(b_4-f)}{(f-b_5t)(f-b_6t)}. \tag{4.16}$$

The Lax pairs constructed by the degeneration limits were given as a coupled second-order q-difference equation in a scalar variable $Y(z,t)$ (see his (37))

$$\begin{aligned}
&\frac{(b_1q-z)(b_2q-z)(b_3q-z)(b_4q-z)t^2}{q(qf-z)z^4}\left[Y(q^{-1}z) - \frac{gz}{t^2(gz-q)}Y(z)\right] \\
&\quad + \left[\frac{q(b_1g-1)(b_2g-1)(b_3g-1)(b_4g-1)}{g(fg-1)z^2(gz-q)} - \frac{b_5b_6(b_7g-t)(b_8g-t)}{fgz^3}\right]Y(z) \\
&\quad + \frac{(b_5t-z)(b_6t-z)}{t^2z^2(f-z)}\left[Y(qz) - \frac{t^2(gz-1)}{gz}Y(z)\right] = 0,
\end{aligned} \tag{4.17}$$

and a second-order, mixed q-difference equation,

$$\frac{gz}{t^2}Y(z) + (q-gz)Y\left(q^{-1}z\right) - q^{-2}gz(qf-z)\bar{Y}\left(q^{-1}z\right) = 0. \tag{4.18}$$

In order to bring (4.15) and (4.16) into correspondence with our form of the $E_6^{(1)}$ q-Painlevé system (see (1.1) and (1.2)) we will employ the following transformation of Yamada's variables

$$t \mapsto t^{-1}, \qquad z \mapsto z^{-1}, \qquad f,g \mapsto g^{-1},f^{-1}, \qquad \tilde{Y}(z) = Y\left(z^{-1}\right),$$

and the specialisations of the parameters

$$b_5 \mapsto b_6^{-1}, \qquad b_6 \mapsto b_6, \qquad b_7 \mapsto qb_5, \qquad b_8 \mapsto b_5^{-1},$$

so that $b_5b_6 = 1$ and $b_7b_8 = q$. Under these transformations we deduce that (4.15) becomes (1.2) and (4.16) becomes (1.1). Furthermore the pure second-order divided-difference equation (4.17) becomes

$$\begin{aligned}
&\frac{\prod_{j=1}^{4}(1-b_jz)}{t^2z(z-g)}\tilde{Y}(z) + \left\{-\frac{\prod_{j=1}^{4}(1-b_jz)}{z(z-g)(1-fz)} + \frac{z\prod_{j=1}^{4}(b_j-f)}{f(1-fg)(1-fz)} - \frac{z(f-b_5qt)(b_5f-t)}{b_5qt^2f}\right. \\
&\qquad \left. - \frac{(z-b_6qt)(b_6z-qt)(q-fz)}{b_6qt^2z(z-qg)}\right\}\tilde{Y}\left(q^{-1}z\right) + \frac{(z-b_6qt)(b_6z-qt)}{b_6z(z-qg)}\tilde{Y}\left(q^{-2}z\right) = 0,
\end{aligned} \tag{4.19}$$

and the mixed divided-difference equation (4.18) becomes

$$\frac{qt^2}{fz}\tilde{Y}\big(q^{-1}z;t\big) - q\frac{(1-fz)}{fz}\tilde{Y}(z;t) - \frac{(z-g)}{fgz^2}\tilde{Y}(z;qt) = 0. \tag{4.20}$$

Having put Yamada's Lax pairs into a suitable form we now seek to make a correspondence with our own theory and results. A single mixed divided-difference equation can be constructed from the matrix Lax pairs ((2.10) and (2.25)). For generic semi-classical systems on a q-lattice grid we can deduce either

$$\begin{aligned}&-\frac{1}{W+\Delta yV}\frac{1}{\mathfrak{P}_+}p_n(x;qt) + \frac{1}{(W+\Delta yV)(R+\Delta uS)}\left[-\frac{\mathfrak{W}_+}{\mathfrak{T}_+}+\frac{\mathfrak{R}_+}{\mathfrak{P}_+}\right]p_n(x;t)\\ &\qquad + \frac{1}{R+\Delta uS}\frac{1}{\mathfrak{T}_+}p_n(qx;t) = 0,\end{aligned}$$

or an alternative,

$$\begin{aligned}&-\frac{\mathfrak{T}_+(x)}{(W-\Delta yV)(x)}p_n(qx;qt) + \frac{1}{(W-\Delta yV)(x)(R+\Delta uS)(qx)}\\ &\qquad\times\big[\mathfrak{T}_+(x)\mathfrak{R}_+(qx)+\mathfrak{P}_+(qx)\mathfrak{W}_-(x)\big]p_n(qx;t) - \frac{\mathfrak{P}_+(qx)}{(R+\Delta uS)(qx)}p_n(x;t) = 0,\end{aligned} \tag{4.21}$$

which we will work with. Using the spectral and deformation data (3.2), (3.7) and the explicit evaluations of the deformation matrix (3.30) and spectral matrix (3.29), we compute the coefficients of the above equation

$$\begin{aligned}-(R+\Delta uS)(qx)\mathfrak{T}_+(x) &= a_n\frac{1-qb_5^2}{b_5}x(x-b_6t)(x-g),\\ -(W-\Delta yV)(x)\mathfrak{P}_+(qx) &= \frac{a_n\gamma_n}{\hat{\gamma}_n}\frac{b_6\big(1-qb_5^2\big)}{b_5}\frac{t(1-b_2x)(1-b_3x)(x-b_6t)}{b_5qt-f},\\ \mathfrak{T}_+(x)\mathfrak{R}_+(qx)+\mathfrak{P}_+(qx)\mathfrak{W}_-(x) &= \frac{a_n\gamma_n}{\hat{\gamma}_n}\frac{b_6\big(1-qb_5^2\big)}{b_5}\frac{(x-b_6t)(b_6x-t)(1-fx)}{b_5qt-f}.\end{aligned}$$

Now we set $p_n = FU$ where F is a gauge factor and U is the new independent variable, into (4.21) and make a direct comparison with (4.20). Comparing the coefficients of $U(x;t)$ and $U(qx;t)$ in this later equation we deduce that

$$\frac{F(qx,t)}{F(x,t)} = \frac{1}{t^2}\frac{(1-b_2x)(1-b_3x)}{1-b_6xt^{-1}}.$$

A solution is given by

$$F(x,t) = e_{q,t^{-2}}(x)\frac{\big(b_6xt^{-1};q\big)_\infty}{(b_2x,b_3x;q)_\infty}C(x,t),$$

where C is a q-constant function, $C(qx,t) = C(x,t)$. Now comparing the coefficients of $U(qx;qt)$ and $U(qx;t)$ in the previous equation we find that

$$\frac{F(qx,qt)}{F(qx,t)} = \frac{\gamma_n}{\hat{\gamma}_n}\frac{b_6(b_6x-t)}{qg(b_5qt-f)x^2}.$$

Substituting our solution into this equation we find a complete cancellation of all the x dependent factors resulting in a pure q-difference equation in t

$$\frac{\hat{\gamma}_n\hat{C}}{\gamma_nC} = \frac{b_6qt}{g(f-b_5qt)}.$$

Thus we just need a solution $C(t)$ independent of x, however we only require the existence of a non-zero, bounded solution rather than knowledge of a specific solution. In conclusion we find that our new mixed, divided-difference equation is now

$$t^2U(x;t)-(1-fx)U(qx;t)-\frac{x-g}{qgx}U(qx;qt)=0,$$

which is clearly proportional to (4.20) with the identification $U(x;t)=\tilde{Y}(q^{-1}x;t)$.

A second-order q-difference equation in the spectral variable x for one of the components, say p_n, was given in (2.22), and for q-linear grids can be simplified as

$$\begin{aligned}&\frac{W+\Delta yV}{\mathfrak{T}_+}(x)p_n(qx)-\left[\frac{\mathfrak{W}_+}{\mathfrak{T}_+}(x)+\frac{\mathfrak{W}_-}{\mathfrak{T}_+}\left(q^{-1}x\right)\right]p_n(x)\\&\qquad+\frac{W-\Delta yV}{\mathfrak{T}_+}\left(q^{-1}x\right)p_n\left(q^{-1}x\right)=0.\end{aligned}\tag{4.22}$$

From the explicit solution of the gauge factor we note

$$\frac{F(qx,t)}{F(x,t)}=\frac{(1-b_2x)(1-b_3x)}{t(t-b_6x)},\qquad \frac{F\left(q^{-1}x,t\right)}{F(x,t)}=\frac{t\left(t-b_6q^{-1}x\right)}{\left(1-b_2q^{-1}x\right)\left(1-b_3q^{-1}x\right)}.$$

Substituting the change of variables into (4.22) we compute that

$$\begin{aligned}&\frac{W+\Delta yV}{\mathfrak{T}_+}(x)\frac{F(qx,t)}{F(x,t)}=\frac{1}{(q-1)u_1a_n}\frac{b_6}{t}\frac{\prod_{j=1}^{4}(1-b_jx)}{x(x-g)},\\&\frac{W-\Delta yV}{\mathfrak{T}_+}\left(q^{-1}x\right)\frac{F\left(q^{-1}x,t\right)}{F(x,t)}=\frac{1}{(q-1)u_1a_n}\frac{t(x-b_6qt)(b_6x-qt)}{x(x-qg)}.\end{aligned}$$

In addition, using the explicit representations of the diagonal elements of A_n^*, i.e., $\mathfrak{W}_\pm$ (see (3.28), (3.29)) we compute that

$$\begin{aligned}-\frac{1}{b_6t}\left[\frac{\mathfrak{W}_+}{x(x-g)}+\left.\frac{\mathfrak{W}_-}{x(x-g)}\right|_{q^{-1}x}\right]&=-\frac{\prod_{j=1}^{4}(1-b_jx)}{x(x-g)(1-fx)}+\frac{x\prod_{j=1}^{4}(f-b_j)}{f(1-fg)(1-fx)}\\&\quad-\frac{x(f-b_5qt)(b_5f-t)}{b_5qt^2f}-\frac{(x-b_6qt)(b_6x-qt)(q-fx)}{b_6qt^2x(x-qg)}.\end{aligned}$$

In summary we find

$$\begin{aligned}&\frac{\prod_{j=1}^{4}(1-b_jx)}{t^2x(x-g)}U(qx)+\left[-\frac{\prod_{j=1}^{4}(1-b_jx)}{x(x-g)(1-fx)}+\frac{x\prod_{j=1}^{4}(f-b_j)}{f(1-fg)(1-fx)}-\frac{x(f-b_5qt)(b_5f-t)}{b_5qt^2f}\right.\\&\qquad\left.-\frac{(x-b_6qt)(b_6x-qt)(q-fx)}{b_6qt^2x(x-qg)}\right]U(x)+\frac{(x-b_6qt)(b_6x-qt)}{b_6x(x-qg)}U\left(q^{-1}x\right)=0.\end{aligned}\tag{4.23}$$

Thus we can see that (4.23) agrees exactly with (4.19) and the identification noted above.

Acknowledgements

This research has been supported by the Australian Research Council's Centre of Excellence for Mathematics and Statistics of Complex Systems. We are grateful for the clarifications by Kenji Kajiwara of results given in [6] and [5] and the assistance of Yasuhiko Yamada in explaining the results of his work [18]. We also appreciate the assistance of Jason Whyte in the preparation of this manuscript.

References

[1] Birkhoff G.D., General theory of linear difference equations, *Trans. Amer. Math. Soc.* **12** (1911), 243–284.

[2] Birkhoff G.D., The generalized Riemann problem for linear differential equations and the Allied problems for linear difference and q-difference equations, *Trans. Amer. Math. Soc.* **49** (1913), 521–568.

[3] Ismail M.E.H., Classical and quantum orthogonal polynomials in one variable, *Encyclopedia of Mathematics and its Applications*, Vol. 98, Cambridge University Press, Cambridge, 2005.

[4] Jimbo M., Sakai H., A q-analog of the sixth Painlevé equation, *Lett. Math. Phys.* **38** (1996), 145–154, chao-dyn/9507010.

[5] Kajiwara K., Masuda T., Noumi M., Ohta Y., Yamada Y., Construction of hypergeometric solutions to the q-Painlevé equations, *Int. Math. Res. Not.* **2005** (2005), no. 24, 1439–1463, nlin.SI/0501051.

[6] Kajiwara K., Masuda T., Noumi M., Ohta Y., Yamada Y., Hypergeometric solutions to the q-Painlevé equations, *Int. Math. Res. Not.* **2004** (2004), no. 47, 2497–2521, arXiv:nlin.SI/0403036.

[7] Koekoek R., Lesky P.A., Swarttouw R.F., Hypergeometric orthogonal polynomials and their q-analogues, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2010.

[8] Koekoek R., Swarttouw R.F., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1998, `http://aw.twi.tudelft.nl/~koekoek/askey/`.

[9] Magnus A.P., Associated Askey–Wilson polynomials as Laguerre–Hahn orthogonal polynomials, in Orthogonal Polynomials and their Applications (Segovia, 1986), *Lecture Notes in Math.*, Vol. 1329, Springer, Berlin, 1988, 261–278.

[10] Magnus A.P., Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points, *J. Comput. Appl. Math.* **65** (1995), 253–265, math.CA/9502228.

[11] Murata M., Lax forms of the q-Painlevé equations, *J. Phys. A: Math. Theor.* **42** (2009), 115201, 17 pages, arXiv:0810.0058.

[12] Papageorgiou V.G., Nijhoff F.W., Grammaticos B., Ramani A., Isomonodromic deformation problems for discrete analogues of Painlevé equations, *Phys. Lett. A* **164** (1992), 57–64.

[13] Ramani A., Grammaticos B., Tamizhmani T., Tamizhmani K.M., Special function solutions of the discrete Painlevé equations, *Comput. Math. Appl.* **42** (2001), 603–614.

[14] Sakai H., A q-analog of the Garnier system, *Funkcial. Ekvac.* **48** (2005), 273–297.

[15] Sakai H., Lax form of the q-Painlevé equation associated with the $A_2^{(1)}$ surface, *J. Phys. A: Math. Gen.* **39** (2006), 12203–12210.

[16] Sakai H., Rational surfaces associated with affine root systems and geometry of the Painlevé equations, *Comm. Math. Phys.* **220** (2001), 165–229.

[17] Witte N.S., Semi-classical orthogonal polynomial systems on non-uniform lattices, deformations of the Askey table and analogs of isomonodromy, arXiv:1204.2328.

[18] Yamada Y., Lax formalism for q-Painlevé equations with affine Weyl group symmetry of type $E_n^{(1)}$, *Int. Math. Res. Not.* **2011** (2011), no. 17, 3823–3838, arXiv:1004.1687.