Routh Reduction by Stages

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Abstract. This paper deals with the Lagrangian analogue of symplectic or point reduction by stages. We develop Routh reduction as a reduction technique that preserves the Lagrangian nature of the dynamics. To do so we heavily rely on the relation between Routh reduction and cotangent symplectic reduction. The main results in this paper are: (i) we develop a class of so called magnetic Lagrangian systems and this class has the property that it is closed under Routh reduction; (ii) we construct a transformation relating the magnetic Lagrangian system obtained after two subsequent Routh reductions and the magnetic Lagrangian system obtained after Routh reduction w.r.t. to the full symmetry group.

Key words: symplectic reduction; Routh reduction; Lagrangian reduction; reduction by stages

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Contents

1   Introduction

It is not a surprise that the bigger part of the recent literature on the geometric description of mechanical systems deals in some way or another with symmetry. The benefits of exploiting symmetry are indeed self-evident: The equations of motion of a mechanical system which exhibits a symmetry can be reduced to a new set of equations with fewer unknowns, possibly easier to solve. In particular, the Hamiltonian description of such systems has attracted most of the attention in the literature, and the important role played by Poisson manifolds and symplectic structures has been strongly emphasized (see e.g. [12] and references therein). Less well-known is the process of symmetry reduction for Lagrangian systems. Much like on the Hamiltonian side, there are in fact two different paths that lead to two different Lagrangian reduction theories. Roughly speaking, the invariance of the Lagrangian leads via Noether's theorem to a set of conserved quantities (the momenta). Whether or not one takes these conserved quantities into account in the reduction process leads to either the Routh or the Lagrange–Poincaré reduction method (see e.g. [4, 5, 6, 10, 14, 15]).

In this paper we deal with Routh's reduction procedure. In a way, one restricts the attention to only those solutions of the system with a prescribed value of momentum. The price one has to pay is that the new symmetry group (after restriction) is in general only a subgroup of the symmetry group of the original Lagrangian system. In [10] it was pointed out that Routh reduction can be interpreted as a special case of symplectic reduction, which moreover preserves the Lagrangian nature of the system.

In the current paper we will investigate those aspects of the theory that are related to reduction in multiple stages. In case the symmetry group of the system has a normal subgroup, one can indeed first perform a Routh reduction by means of this subgroup. It is then natural to ask whether the reduced system is invariant under some residual, as yet to be determined group, and whether a subsequent second Routh reduction leads to results which are equivalent to direct reduction by the full group. The answer to the same question, but for the above mentioned Lagrange–Poincaré reduction theory, is by now well-known [4]. It can be understood in the following sense: After each reduction step one remains in the category of so-called 'Lagrangian systems on Lie algebroids' and symmetry reduction can be phrased in terms of Lie algebroid morphisms (see e.g. [5] and the references therein). Equivalently, one may use a connection to decompose the Lie algebroid structure at each stage. The corresponding category is then the so-called category of 'Lagrange–Poincaré bundles' from [4], while the corresponding 'Lagrange–Poincaré morphisms' do the reduction.

It is natural to ask whether or not such a category of systems exists also for Routh reduction (it is mentioned as an open problem in [12]). The candidate we propose (in Section 3) is the class of what we call 'magnetic Lagrangian systems'. We show in Sections 4 and 5 that this class has the property that it is closed under Routh reduction, i.e. that after each step of a reduction in stages the reduced system remains in the class of magnetic Lagrangian systems. As in [10], we heavily rely on a generalized version of cotangent symplectic reduction. In Section 6 we apply the framework of magnetic Lagrangian systems to reduction in several stages, where the first stage consists of Routh reduction by a normal subgroup of the overall symmetry group. We end the paper with some illustrative examples in the final section.

2   Preliminaries on Routh reduction

Definition 1. A Lagrangian system is a pair (Q, L) with Q a manifold and L a function on TQ. The manifold Q is the configuration space and L is called the Lagrangian. The dimension of Q is called the number of degrees of freedom of the Lagrangian system. A Lagrangian system is

of mechanical type if for arbitrary $v_q \in T_q Q$, the Lagrangian can be written as

$$L(v_q) = \frac{1}{2} \langle\!\langle v_q, v_q \rangle\!\rangle_Q - V(q),$$

with $\langle\!\langle \cdot, \cdot \rangle\!\rangle_Q$ a Riemannian metric on Q and V a function on Q, called the potential energy of the Lagrangian. The function $\frac{1}{2} \langle\!\langle v_q, v_q \rangle\!\rangle_Q$ is called the kinetic energy.

We are interested in the Euler–Lagrange equations associated to a Lagrangian system. These equations are necessary conditions for a curve $q(t) : I \subset \mathbb{R} \to Q$ to extremize the action integral $\int_I L(q(t), \dot{q}(t)) dt$. In a local coordinate chart (q^i) on Q the Lagrangian L is a function of (q^i, v^i) and the Euler–Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad \dot{q}^i = v^i, \qquad i = 1, \ldots, n = \dim Q.$$

Routh reduction is a reduction technique for Lagrangian systems that are invariant under the action of a symmetry group. In classical textbooks, it is a step-by-step procedure that describes the construction of a new Lagrangian function with fewer degrees of freedom.

Before we can formulate Routh reduction, we fix notations for the action of a Lie group on an arbitrary manifold M and related concepts.

Actions of Lie groups and principal bundles. When a Lie group G is given, $\mathfrak{g}$ denotes the Lie algebra of G and exp the exponential map from $\mathfrak{g}$ to G. The adjoint action of G on $\mathfrak{g}$ is denoted by Ad and the coadjoint action of G on $\mathfrak{g}^*$ by Ad*.

Definition 2.

1. A right action of a group G on a manifold M is denoted by $\Psi^M : G \times M \to M; (g, m) \mapsto \Psi_g^M(m) = mg$. Throughout this paper we only consider free and proper actions of Lie groups on manifolds.

2. The infinitesimal action is given by $\psi_m^M : \mathfrak{g} \to T_m M; \xi \mapsto d/d\epsilon|_0(m \exp \epsilon \xi)$.

3. For any element $\xi \in \mathfrak{g}$ one can consider the fundamental vector field ξ_M, defined pointwise as $\xi_M(m) = \psi_m^M(\xi)$.

4. The map defined pointwise as the dual to ψ_m^M is denoted by $(\psi^M)^* : T^*M \to \mathfrak{g}^*$.

5. The lifted action Ψ^{TM} on TM of Ψ^M is given by $\Psi^{TM} : G \times TM \to TM; (g, v_m) \mapsto T\Psi_g^M(v_m)$.

6. The lifted action Ψ^{T^*M} on T^*M is given by $\Psi^{T^*M} : G \times T^*M \to T^*M; (g, \alpha_m) \mapsto T^*\Psi_{g^{-1}}^M(\alpha_m)$.

With these notations, ξ_{TM} denotes a fundamental vector field on TM determined by the lifted action. It follows that ξ_{TM} is the complete lift of ξ_M.

Every right action Ψ^M gives rise to a left action Φ^M: $\Phi_g^M(m) = \Psi_{g^{-1}}^M(m)$. We only consider right actions. This is not a true restriction since one may reformulate the main results for left actions if needed.

The orbit space M/G of a free and proper action is a manifold and $\pi : M \to M/G$ carries the structure of a principal G-bundle. Throughout the paper $[m]_G$ denotes a point in the orbit space M/G, i.e. it is the orbit through $m \in M$. Every tangent vector in the kernel of $T\pi$ is of the form $\xi_M(m)$ for some $\xi \in \mathfrak{g}$. These vectors are called vertical and form a distribution which we call the vertical distribution $V\pi = \ker T\pi$.

Definition 3. A principal connection on a principal G-bundle is a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ satisfying two conditions:

1) it is equivariant, i.e. $\Psi_g^* \mathcal{A} = \mathrm{Ad}_{g^{-1}} \cdot \mathcal{A}$ for any $g \in G$, and

2) for $\xi \in \mathfrak{g}$ arbitrary, $\mathcal{A}(\xi_M) = \xi$.

The kernel of $\mathcal{A}$ determines a right invariant distribution on M which is a complement of the vertical distribution. It is therefore called the *horizontal* distribution of $\mathcal{A}$ and is typically denoted by $H^{\mathcal{A}} \subset TM$. On the other hand, any right invariant distribution H satisfying $H \oplus V\pi = TQ$ determines a principal connection, see [8].

We denote by $\langle f, \mathcal{A} \rangle$, where f is a $\mathfrak{g}^*$-valued function on M, the 1-form on M pointwise defined by

$$v_m \mapsto \langle f(m), \mathcal{A}(m)(v_m) \rangle \in \mathbb{R}.$$

The cotangent vector $\langle f, \mathcal{A} \rangle(m) \in T_m^* M$ is often denoted by $\langle f(m), \mathcal{A}(m) \rangle$. In particular, if $f = \mu \in \mathfrak{g}^*$ is constant, then $\langle \mu, \mathcal{A} \rangle$ is a 1-form on M.

Routh reduction.

Definition 4. Let (Q, L) denote a Lagrangian system and assume that the configuration space is equipped with an action Ψ^Q. The Lagrangian system (Q, L) is G-invariant if L is invariant under the lifted action Ψ^{TQ}, i.e. $L(\Psi_g^{TQ}(v_q)) = L(v_q)$ for arbitrary $v_q \in TQ$ and $g \in G$.

If a mechanical Lagrangian system is G-invariant then the kinetic energy metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle_Q$ and the potential energy V are both invariant under the pull-back of Ψ_g^Q, for arbitrary g.

Definition 5.

1. For a Lagrangian system (Q, L), the Legendre transform $\mathbb{F}L : TQ \to T^*Q$ is the fibre derivative of L, i.e. for arbitrary $v_q, w_q \in TQ$

$$\langle \mathbb{F}L(v_q), w_q \rangle = \left. \frac{d}{du} \right|_{u=0} L(v_q + u w_q).$$

The Lagrangian is hyperregular if $\mathbb{F}L$ is a diffeomorphism.

2. For a G-invariant Lagrangian system (Q, L), the momentum map $J_L : TQ \to \mathfrak{g}^*$ is the map $(\psi^Q)^* \circ \mathbb{F}L$, i.e. for arbitrary $v_q \in TQ$ and $\xi \in \mathfrak{g}$

$$\langle J_L(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle.$$

3. Given a G-invariant mechanical Lagrangian system (Q, L), and a point q in Q. The inertia tensor $\mathbb{I}_q$ is a metric on $\mathfrak{g}$ defined by $\mathbb{I}_q(\xi, \eta) = \langle\!\langle \xi_Q(q), \eta_Q(q) \rangle\!\rangle_Q$.

The momentum map associated to a G-invariant Lagrangian system satisfies the following two important properties:

1. It is conserved along the solutions of the Euler–Lagrange equations, i.e. if $q(t)$ is a solution to the Euler–Lagrange equations, then $\frac{d}{dt}(J_L(\dot{q}(t))) = 0$.

2. It is equivariant w.r.t. the action Ψ^{TQ} and the coadjoint action Ad^* on $\mathfrak{g}^*$, i.e.

$$J_L(\Psi_g^{TQ}(v_q)) = \mathrm{Ad}_g^*(J_L(v_q))$$

for arbitrary $g \in G$ and $v_q \in TQ$.

Definition 6. Given a G-invariant Lagrangian system (Q, L) and an arbitrary vector $v_q \in TQ$.

1. The map $J_L|_{v_q} : \mathfrak{g} \to \mathfrak{g}^*$ is the map $\xi \mapsto J_L(v_q + \xi_Q(q))$.

2. The Lagrangian system (Q, L) is G-regular if $J_L|_{v_q}$ is a diffeomorphism for every $v_q \in TQ$.

Every G-invariant mechanical Lagrangian system is G-regular. To show this, remark that for a mechanical Lagrangian system

$$\langle J_L|_{v_q}(\xi), \eta \rangle = \langle\!\langle v_q, \eta_Q(q) \rangle\!\rangle_Q + \langle\!\langle \xi_Q(q), \eta_Q(q) \rangle\!\rangle_Q.$$

Hence $J_L|_{v_q}(\xi) = J_L|_{v_q}(0) + \mathbb{I}_q(\xi)$ is an affine map, whose linear part is determined by the inertia metric on $\mathfrak{g}$.

Proposition 1. *Consider a G-invariant and G-regular Lagrangian system (Q, L) and fix a regular value $\mu \in \mathfrak{g}^*$ of the momentum map J_L. Let G_μ denote the isotropy subgroup of μ w.r.t. the coadjoint action of G on $\mathfrak{g}^*$, i.e. $g \in G_\mu$ iff $\mathrm{Ad}_g^* \mu = \mu$.*

1. *The submanifold $i_\mu : J_L^{-1}(\mu) \to TQ$ is G_μ-invariant and the restricted action of G_μ on $J_L^{-1}(\mu)$ is free and proper.*

2. *The quotient manifold $J_L^{-1}(\mu)/G_\mu$ is diffeomorphic to the fibred product $T(Q/G) \times_{Q/G} Q/G_\mu$ over Q/G.*

The first statement is a direct consequence of the equivariance of J_L. We postpone a proof of the second statement (Proposition 7, page 15).

We are now ready to describe a preliminary version of Routh reduction where the symmetry group G is Abelian and the bundle $Q \to Q/G$ is trivial, so that we can choose a connection $\mathcal{A}$ with vanishing curvature. Since G is Abelian, we have that the isotropy subgroup G_ν for every $\nu \in \mathfrak{g}^*$ is the entire group G, and in particular $T(Q/G) \times_{Q/G} Q/G_\nu$ is nothing but $T(Q/G)$.

Theorem 1 (Routh reduction – Abelian version)**.** *Let G be Abelian and let $\mathcal{A}$ be a connection on Q with vanishing curvature. The Routh reduction procedure of a G-invariant and G-regular Lagrangian system (Q, L) consists of the following steps.*

1. *Fix a regular value μ of the momentum map J_L and consider the submanifold $J_L^{-1}(\mu)$.*

2. *Compute the restriction of the G-invariant function $L - \langle \mu, \mathcal{A} \rangle$ to the level set $J_L^{-1}(\mu)$. Let $\tilde{L}$ denote its projection to the quotient manifold $T(Q/G) \cong J_L^{-1}(\mu)/G$.*

Every solution of the Euler–Lagrange equations of the Lagrangian system $(Q/G, \tilde{L})$ is the projection of a solution of the Euler–Lagrange equations of the original system (Q, L) with momentum μ. Conversely, every solution of the Euler–Lagrange equations of the Lagrangian system (Q, L) with momentum μ projects to a solution of the Euler–Lagrange equations of the system $(Q/G, \tilde{L})$.

One may also find a description of global Abelian Routh reduction in [3].

Example: the spring pendulum. The system consists of a point particle with mass m moving in a horizontal plane, and attached to the origin by means of a spring with spring constant k. We choose polar coordinates (r, θ) for this system, so that the mechanical Lagrangian is given by $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}kr^2$. This system is clearly invariant under translations in the θ-direction, given by $\Psi_a(r, \theta) = (r, \theta + a)$. The momentum map for this action is $J_L = mr^2\dot{\theta}$ and the Euler–Lagrange equations are

$$m\ddot{r} - mr\dot{\theta}^2 + kr = 0,$$
$$\frac{d}{dt}(mr^2\dot{\theta}) = 0.$$

Let us fix a regular value $0 \neq \mu = mr^2\dot{\theta}$ for the momentum map and let $\mathcal{A} = d\theta$ be the standard connection with vanishing curvature. The Routhian is obtained from

$$\tilde{L}(r, \dot{r}) = (L - \mu\dot{\theta})|_{mr^2\dot{\theta}=\mu} = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}kr^2 - \frac{1}{2}\frac{\mu^2}{mr^2}.$$

The Euler–Lagrange equation for $\tilde{L}$ is $m\ddot{r} = -kr + \frac{\mu^2}{mr^3}$ and solutions of this equations are in correspondence to solutions of the Euler–Lagrange equations for L with momentum μ.

Routh reduction can be extended to more general contexts, including non-Abelian group actions [14], Lagrangians of non-mechanical type [2, 6], Lagrangians invariant up to a total time derivative [10] and Lagrangians that are not G-regular [11]. Additional complications arise in these cases: for instance, if the bundle $Q \to Q/G$ is not trivial, the reduced system is typically subjected to an additional force term associated to the curvature of the chosen connection. On the other hand, if the symmetry group is not Abelian, the quotient space $J_L^{-1}(\mu)/G_\mu$ no longer has the structure of a tangent bundle; instead it is diffeomorphic to $T(Q/G) \times_{Q/G} Q/G_\mu$. The interpretation of the reduced system as a Lagrangian system then requires additional definitions, which are postponed to the following section.

Our main interest in this paper is Routh reduction by stages, where the assumption is that the Lagrangian system obtained after applying a first Routh reduction carries additional symmetry, so that we can reapply Routh reduction. The system obtained after one Routh reduction is formulated on a fibred product of the type $T(Q/G) \times_{Q/G} Q/G_\mu$. This is the fibred product of a bundle $Q/G_\mu \to Q/G$ with the tangent bundle to the base space. This observation is our main motivation to extend Routh reduction to Lagrangian systems defined on such fibred products: in order to develop Routh reduction by stages, we have to be able to reduce Lagrangian systems obtained after a first reduction.

Routh reduction is closely related to symplectic reduction on the associated cotangent bundles (see [10, 14]). In the next section we will introduce the concept of a "magnetic Lagrangian system" and we will emphasize its symplectic formulation. The concept of magnetic Lagrangian systems is the analogue of Hamiltonian systems one encounters in magnetic cotangent bundle reduction [12].

3   Magnetic Lagrangian systems

A magnetic Lagrangian system is a Lagrangian system with configuration space the total space of a bundle $\epsilon : P \to Q$ and where the Lagrangian is independent of the velocities tangent to the fibres of ϵ. Additionally the system is subjected to a force term that is of magnetic type. It might help to keep in mind that in the case of a Routh reduced Lagrangian system, P corresponds to Q/G_μ and the fibration ϵ is given by the projection $Q/G_\mu \to Q/G$.

Definition 7. A magnetic Lagrangian system is a triple $(\epsilon : P \to Q, L, \mathcal{B})$ where $\epsilon : P \to Q$ is a fibre bundle, L is a smooth function on the fibred product $TQ \times_Q P$ and $\mathcal{B}$ is a closed 2-form on P. We say that P is the configuration manifold of the system and that L is the Lagrangian.

A coordinate chart (q^i, p^a), $i = 1, \ldots, n = \dim Q$, $a = 1, \ldots, k = \dim P - \dim Q$, adapted to the fibration $\epsilon : P \to Q$ determines a coordinate chart (q^i, v^i, p^a) on $TQ \times_Q P$, and the Lagrangian L is then a function depending on (q^i, v^i, p^a). By definition, L is independent of the velocities in the fibre coordinates p^a and therefore it determines a singular Lagrangian when interpreted as a function on TP. Locally, the Euler–Lagrange equations for this singular Lagrangian are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \mathcal{B}_{ij}\dot{q}^j + \mathcal{B}_{ia}\dot{p}^a, \qquad i = 1, \ldots, n,$$

$$-\frac{\partial L}{\partial p^a} = -\mathcal{B}_{ia}\dot{q}^i + \mathcal{B}_{ab}\dot{p}^b, \qquad a = 1, \ldots, k.$$

Here we used the following coordinate expression of the 2-form $\mathcal{B}$ is $\frac{1}{2}\mathcal{B}_{ij}dq^i \wedge dq^j + \mathcal{B}_{ia}dq^i \wedge dp^a + \frac{1}{2}\mathcal{B}_{ab}dp^a \wedge dp^b$. These Euler–Lagrange equations have a geometric interpretation (see Proposition 2). First we introduce additional notations.

Definition 8. Assume a magnetic Lagrangian system $(\epsilon : P \to Q, L, \mathcal{B})$ is given.

1. $T_P Q$ denotes the fibred product $TQ \times_Q P$ and (v_q, p), where $v_q \in TQ$ and $p \in P$ such that $\epsilon(p) = q$, is a point in $T_P Q$. Similarly, $T_P^* Q$ denotes the fibred product $T^* Q \times_Q P$ and (α_q, p), with $\epsilon(p) = q$, is an arbitrary element in $T_P^* Q$.

2. $V\epsilon$ denotes the distribution on P of tangent vectors vertical to ϵ.

3. $\hat{\epsilon} : TP \to T_P Q$ is the projection fibred over P that maps $v_p \in TP$ onto $(T\epsilon(v_p), p) \in T_P Q$.

4. $\rho_1 : T_P Q \to TQ$ is the projection that maps $(v_q, p) \in T_P Q$ onto $v_q \in TQ$.

5. $\rho_2 : T_P Q \to P$ is the projection that maps $(v_q, p) \in T_P Q$ onto $p \in P$.

6. $\epsilon_1 : T_P^* Q \to T^* Q$ is the projection that maps $(\alpha_q, p) \in T_P^* Q$ onto $\alpha_q \in T^* Q$.

7. $\epsilon_2 : T_P^* Q \to P$ is the projection that maps $(\alpha_q, p) \in T_P^* Q$ onto $p \in P$.

8. The Legendre transform $\mathbb{F}L : T_P Q \to T_P^* Q$ maps $(v_q, p) \in T_P Q$ to $(\alpha_q, p) \in T_P^* Q$ where $\alpha_q \in T_q^* Q$ is determined from

$$\langle \alpha_q, w_q \rangle = \left. \frac{d}{du} \right|_{u=0} L(v_q + u w_q, p),$$

for arbitrary $w_q \in T_q Q$.

9. The energy E_L is a function on $T_P Q$ defined by $E_L(v_q, p) = \langle \mathbb{F}L(v_q, p), (v_q, p) \rangle - L(v_q, p)$. (Here the contraction of an element $(\alpha_q, p) \in T_P^* Q$ with $(v_q, p) \in T_P Q$ is defined naturally as $\langle (\alpha_q, p), (v_q, p) \rangle = \langle \alpha_q, v_q \rangle$.)

10. By means of the Legendre transform we can pull-back the presymplectic 2-form $\epsilon_1^* \omega_Q + \epsilon_2^* \mathcal{B}$ on $T_P^* Q$ to a presymplectic 2-form $\mathbb{F}L^*(\epsilon_1^* \omega_Q + \epsilon_2^* \mathcal{B})$ on $T_P Q$. The latter is denoted by $\Omega^{L,\mathcal{B}}$ (Here $\omega_Q = d\theta_Q$, with θ_Q the Poincaré–Cartan 1-form on $T^* Q$ and a presymplectic 2-form is understood to be a closed 2-form, not necessarily of constant rank.)

Proposition 2. *Given a curve $p(t)$ in P, and let $\gamma(t)$ denote the curve in $T_P Q$ equal to $(\dot{q}(t), p(t)) \in T_P Q$ with $q(t) = \epsilon(p(t))$. The curve $p(t)$ in P is a solution to the Euler–Lagrange equations for the magnetic Lagrangian system $(\epsilon : P \to Q, L, \mathcal{B})$ iff $\gamma(t)$ is a solution to the presymplectic equation*

$$i_{\dot{\gamma}(t)} \Omega^{L,\mathcal{B}}(\gamma(t)) = -dE_L(\gamma(t)).$$

Locally, the presymplectic equation coincides with the previously mentioned Euler–Lagrange equations, since

$$\Omega^{L,\mathcal{B}} = d\left(\frac{\partial L}{\partial v^i}\right) \wedge dq^i + \frac{1}{2} \mathcal{B}_{ij} dq^i \wedge dq^j + \mathcal{B}_{ia} dq^i \wedge dp^a + \frac{1}{2} \mathcal{B}_{ab} dp^a \wedge dp^b,$$

$$dE_L = v^i d\left(\frac{\partial L}{\partial v^i}\right) + \frac{\partial L}{\partial v^i} dv^i - dL = v^i d\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial p^a} dp^a.$$

Definition 9. A magnetic Lagrangian system $(\epsilon : P \to Q, L, \mathcal{B})$

1) is hyperregular if $\mathbb{F}L$ is a diffeomorphism and if the restriction of $\mathcal{B}$ to $V\epsilon$ is nondegenerate,

2) is of mechanical type if $L(v_q, p) = \frac{1}{2} \langle\!\langle (v_q, p), (v_q, p) \rangle\!\rangle_{\rho_1} - V(p)$ with $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\rho_1}$ is a metric on the vector bundle $\rho_1 : T_P Q \to P$ and V is a function on P.

Note that if $\mathcal{B}$ is nondegenerate, then the typical fibre of P necessarily has to be even-dimensional. In a local coordinate chart the nondegeneracy condition on $\mathcal{B}$ is expressed by $\det \mathcal{B}_{ab} \neq 0$.

Proposition 3. *If a magnetic Lagrangian system* $(\epsilon : P \to Q, L, \mathcal{B})$ *is hyperregular, the 2-form* $\Omega^{L,\mathcal{B}} = \mathbb{F}L^*(\epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B})$ *determines a symplectic structure on* T_PQ.

Proof. Assume that the magnetic Lagrangian system is hyperregular. The nondegeneracy of $\Omega^{L,\mathcal{B}}$ is easily checked if we work in a coordinate chart adapted to the fibration:

$$\Omega^{L,\mathcal{B}} = d\left(\frac{\partial L}{\partial v^i}\right) \wedge dq^i + \frac{1}{2}\mathcal{B}_{ij}dq^i \wedge dq^j + \mathcal{B}_{ia}dq^i \wedge dp^a + \frac{1}{2}\mathcal{B}_{ab}dp^a \wedge dp^b.$$

Since $\mathbb{F}L$ is a diffeomorphism, the 1-forms $d\left(\frac{\partial L}{\partial v^i}\right)$, dq^i and dp^a provide pointwise a basis on $T^*(T_PQ)$. One can now use standard arguments to prove the nondegeneracy. ∎

We conclude that a hyperregular magnetic Lagrangian system has a symplectic structure although the Lagrangian itself is singular when interpreted as a function on TP. The energy E_L is the Hamiltonian. This is an important observation.

Remark 1. Throughout the paper we only consider magnetic Lagrangian systems that are hyperregular. This is not a true restriction. The results remain valid for general magnetic Lagrangian systems: instead of relating Routh reduction to symplectic reduction, it is possible to relate it to presymplectic reduction [7].

Remark 2. We conclude this section with a remark on equivalent magnetic Lagrangian systems. Roughly speaking, two Lagrangian systems are equivalent if the resulting dynamics coincide, more specifically if they produce the same Euler–Lagrange equations. It is well known that for a Lagrangian system the Euler–Lagrange equations do not change when the Lagrangian is augmented with a total time derivative of a function on the configuration space. For magnetic Lagrangian systems this gauge freedom can be extended.

Consider a 1-form α along ϵ, i.e. a section of $\epsilon_2 : T_P^*Q \to P$, or in other words a linear function on T_PQ given by $(v_q, p) \mapsto \langle \alpha(p), (v_q, p) \rangle$. When $\alpha : P \to T_P^*Q$ is composed with $T^*\epsilon \circ \epsilon_1 : T_P^*Q \to T^*P$, it determines a 1-form on P which, with a slight abuse of notation, we denote by $\epsilon^*\alpha$. Now consider the function $L'(v_q, p) = L(v_q, p) - \langle \alpha(p), (v_q, p) \rangle$ and the gyroscopic 2-form $\mathcal{B}' = \mathcal{B} + d\epsilon^*\alpha$. Together they define a new magnetic Lagrangian system $(\epsilon : P \to Q, L', \mathcal{B}')$ which is equivalent to $(\epsilon : P \to Q, L, \mathcal{B})$.

Lemma 1. *The magnetic Lagrangian systems* $(\epsilon, L', \mathcal{B}')$ *and* $(\epsilon, L, \mathcal{B})$ *are equivalent.*

This is a reformulation of a well-known result in classical mechanics, saying that an exact gyroscopic force can be taken into account by means of a velocity dependent potential [19].

Proof. We show that any solutions to the Euler–Lagrange equations of $(\epsilon, L', \mathcal{B}')$ is a solution to the Euler–Lagrange equations of $(\epsilon, L, \mathcal{B})$. We work in a local coordinate neighborhood (q^i, p^a) as before. We fix a coordinate expression for $\alpha = \alpha_i(q^j, p^a)dq^i$. The Euler–Lagrange equations for $(\epsilon, L', \mathcal{B}')$ equal, with $\dot{q}^i = v^i$ and $\dot{p}^a = v^a$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial v^i}\right) - \frac{\partial L'}{\partial q^i} = \mathcal{B}'_{ij}v^j + \mathcal{B}'_{ia}v^a, \qquad i = 1, \ldots, n,$$

$$-\frac{\partial L'}{\partial p^a} = -\mathcal{B}'_{ia}v^i + \mathcal{B}'_{ab}v^b, \qquad a = 1, \ldots, k.$$

It now remains to substitute the definition of L' and $\mathcal{B}'$, i.e. $L' = L - \alpha_i v^i$ and

$$\mathcal{B}'_{ij} = \mathcal{B}_{ij} + \left(\frac{\partial \alpha_j}{\partial q^i} - \frac{\partial \alpha_i}{\partial q^j}\right), \qquad \mathcal{B}'_{ia} = \mathcal{B}_{ia} - \frac{\partial \alpha_i}{\partial p^a}, \qquad \mathcal{B}'_{ab} = \mathcal{B}_{ab}. \qquad \blacksquare$$

4 Magnetic Lagrangian systems with symmetry

4.1 Symplectic reduction: a brief introduction

An overall reference for this section is e.g. [12]. Let (M, ω) be a symplectic manifold on which G acts on the right, $\Psi^M : M \times G \to M$. Given a function $f : M \to \mathfrak{g}^*$, then f_ξ for $\xi \in \mathfrak{g}$ denotes the real valued function on M defined by $f_\xi(m) = \langle f(m), \xi \rangle$.

Definition 10.

1. The action Ψ^M is said to be *canonical* if $(\Psi_g^M)^* \omega = \omega$ for all $g \in G$.

2. A map $J : M \to \mathfrak{g}^*$ is a momentum map if $i_{\xi_M} \omega = -dJ_\xi$, for $\xi \in \mathfrak{g}$ arbitrary.

3. If M is connected, the *non-equivariance 1-cocycle* σ of the momentum map J equals

$$\sigma : \ G \to \mathfrak{g}^* : \ g \mapsto J\big(mg^{-1}\big) - \mathrm{Ad}^*_{g^{-1}}(J(m)),$$

   where m is arbitrary in M.

The definition of σ is independent of m (see [1]). Recall that a 1-cocycle with values in $\mathfrak{g}^*$ statisfies, for $g, h \in G$ arbitrary,

$$\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h).$$

Definition 11. Let σ be a 1-cocycle with values in $\mathfrak{g}^*$. The affine action of G on $\mathfrak{g}^*$ with 1-cocycle σ is given by $(g, \mu) \mapsto \mathrm{Ad}^*_g \mu + \sigma(g^{-1})$ for arbitrary $\mu \in \mathfrak{g}^*$ and $g \in G$.

The momentum map J is equivariant with respect to the affine action with 1-cocle from Definition 10: $J(mg) = \mathrm{Ad}^*_g J(m) + \sigma(g^{-1})$. For a fixed element $\mu \in \mathfrak{g}^*$, the Lie group $G_\mu < G$ denotes the isotropy subgroup of μ w.r.t. the affine action, i.e. $g \in G_\mu$ if $\mu = \mathrm{Ad}^*_g \mu + \sigma(g^{-1})$ or equivalently if

$$\mu - J(mg) = \mathrm{Ad}^*_g(\mu - J(m)).$$

Theorem 2 (symplectic reduction SR). *Let (M, ω) be a symplectic manifold, with G acting canonically on M. Let J be a momentum map for this action with non-equivariance cocycle σ. Assume that μ is a regular value of J, and denote by G_μ the isotropy group of μ under the affine action of G on $\mathfrak{g}^*$. Then the pair (M_μ, ω_μ) is a symplectic manifold, with $M_\mu = J^{-1}(\mu)/G_\mu$ and with ω_μ a 2-form on M_μ uniquely determined from $i_\mu^* \omega = \pi_\mu^* \omega_\mu$, with $i_\mu : J^{-1}(\mu) \to M$ and $\pi_\mu : J^{-1}(\mu) \to M_\mu = J^{-1}(\mu)/G_\mu$.*

Any Hamiltonian h on M which is invariant under the action of G induces a function h_μ on M_μ satisfying $\pi_\mu^* h_\mu = i_\mu^* h$. The Hamiltonian vector field X_h is tangent to $J^{-1}(\mu)$ and the corresponding vector field on $J^{-1}(\mu)$ is π_μ-related to the Hamiltonian vector field X_{h_μ} on M_μ.

4.2 Invariant magnetic Lagrangian systems and momentum maps

In order to define invariant magnetic Lagrangian systems, we start from a free and proper action of a Lie group G on both P and Q such that they commute with ϵ: for arbitrary $g \in G$,

$$\epsilon \circ \Psi_g^P = \Psi_g^Q \circ \epsilon. \tag{1}$$

In other words, G acts on $\epsilon : P \to Q$ by bundle automorphisms. The projections of the principal bundles are denoted by $\pi^Q : Q \to Q/G$ and $\pi^P : P \to P/G$. These actions induce lifted right actions on $T_P Q$ and $T_P^* Q$:

1) $\Psi_g^{T_P Q}(v_q, p) := (T\Psi_g^Q(v_q), \Psi_g^P(p))$ for $(v_q, p) \in T_P Q$ and $g \in G$,

2) $\Psi_g^{T_P^* Q}(\alpha_q, p) := (T^*\Psi_{g^{-1}}^Q(\alpha_q), \Psi_g^P(p))$ for $(\alpha_q, p) \in T_P^* Q$ and $g \in G$.

Definition 12. A magnetic Lagrangian system $(\epsilon : P \to Q, L, \mathcal{B})$ is G-invariant if

1) $\mathcal{B}$ is invariant under the action of Ψ^P: $(\Psi_g^P)^*\mathcal{B} = \mathcal{B}$ for all $g \in G$,

2) the Lagrangian L is invariant under the lifted action $\Psi^{T_P Q}$ on $T_P Q$, i.e. $L \circ \Psi_g^{T_P Q} = L$ for all $g \in G$.

Let $(\epsilon : P \to Q, L, \mathcal{B})$ be a G-invariant hyperregular magnetic Lagrangian system and consider the symplectic 2-form $\epsilon_1^* \omega_Q + \epsilon_2^* \mathcal{B}$ on $T_P^* Q$.

Proposition 4. *The action $\Psi^{T_P^* Q}$ is canonical and the Legendre transform $\mathbb{F}L : T_P Q \to T_P^* Q$ is equivariant, i.e. $\mathbb{F}L \circ \Psi_g^{T_P Q} = \Psi_g^{T_P^* Q} \circ \mathbb{F}L$. As a consequence the action $\Psi^{T_P Q}$ is canonical w.r.t. $\Omega^{L, \mathcal{B}}$.*

Proof. Note that $\epsilon_1 \circ \Psi_g^{T_P^* Q} = T^*\Psi_{g^{-1}}^Q \circ \epsilon_1$ and $\epsilon_2 \circ \Psi_g^{T_P^* Q} = \Psi_g^P \circ \epsilon_2$. Since the action $\Psi_g^{T^* Q} = T^*\Psi_{g^{-1}}^Q$ on $T^* Q$ is canonical w.r.t. ω_Q and since $\mathcal{B}$ is a G-invariant 2-form on P, we conclude that $\Psi^{T_P^* Q}$ is canonical. To show the equivariance of $\mathbb{F}L$ we use its definition: let $g \in G$, $v_q, w_q \in TQ$ be arbitrary, then

$$\langle \mathbb{F}L \left(\Psi_g^{T_P Q}(v_q, p) \right), \Psi_g^{T_P Q}(w_q, p) \rangle = \frac{d}{du}\bigg|_{u=0} L(\Psi_g^{T_P Q}(v_q + u w_q))$$

$$= \frac{d}{du}\bigg|_{u=0} L(v_q + u w_q) = \langle \mathbb{F}L(v_q, p), (w_q, p) \rangle. \qquad \blacksquare$$

In order to define a momentum map, we introduce a concept that is similar to the so-called $\mathcal{B}\mathfrak{g}$-potential in [12].

Definition 13. A $\mathfrak{g}^*$-valued function δ on P is a $\mathcal{B}\mathfrak{g}$-potential for the magnetic 2-form $\mathcal{B}$ if $i_{\xi_P}\mathcal{B} = d\delta_\xi$ for any $\xi \in \mathfrak{g}$.

Recall that δ_ξ is defined pointwise as $\delta_\xi(p) = \langle \delta(p), \xi \rangle$. If $\mathcal{B}$ is G-invariant and admits a $\mathcal{B}\mathfrak{g}$-potential δ, then it follows that $d((\Psi_g^P)^*\delta_\xi - \delta_{\mathrm{Ad}_g \xi}) = 0$. If P is connected, this induces a $\mathfrak{g}^*$-valued 1-cocycle on G:

$$\sigma_\delta : \ G \to \mathfrak{g}^*; \qquad g \mapsto (\Psi_{g^{-1}}^P)^*\delta(p) - \mathrm{Ad}_{g^{-1}}^* \cdot \delta(p),$$

with p arbitrary in P. The infinitesimal version of this 1-cocycle is denoted by $\Sigma_\delta(\xi, \eta) = -\langle T_e \sigma_\delta(\xi), \eta \rangle$. It is a 2-cocycle on the Lie algebra and satisfies $\Sigma_\delta(\xi, \eta) = -\mathcal{B}(\xi_P, \eta_P) - \delta_{[\xi, \eta]} = \xi_P(\delta_\eta) - \delta_{[\xi, \eta]}$. For the following proposition, recall from Definition 2 that $(\psi^Q)^* : T^* Q \to \mathfrak{g}^*$ is the dual of the infinitesimal action of G on Q.

Proposition 5. *The map $J_\delta = (\psi^Q)^* \circ \epsilon_1 - \delta \circ \epsilon_2$ is a momentum map for the symplectic manifold $(T_P^* Q, \epsilon_1^* \omega_Q + \epsilon_2^* \mathcal{B})$ with non-equivariance 1-cocycle $-\sigma_\delta$. Due to the equivariance of $\mathbb{F}L$, the map $J_{L, \delta} = J_\delta \circ \mathbb{F}L$ is a momentum map for the symplectic manifold $(T_P Q, \Omega^{L, \mathcal{B}})$ with non-equivariance 1-cocycle $-\sigma_\delta$.*

Proof. The map $(\psi^Q)^* : T^* Q \to \mathfrak{g}^*$ is an equivariant momentum map for the symplectic manifold $(T^* Q, \omega_Q)$. It is straightforward that the combined map $J_\delta = (\psi^Q)^* \circ \epsilon_1 - \delta \circ \epsilon_2$ is a momentum map for the lifted action on $T_P^* Q$ w.r.t. the symplectic form $\epsilon_1^* \omega_Q + \epsilon_2^* \mathcal{B}$ with non-equivariance cocycle $-\sigma_\delta$. The rest of the statement follows by construction. $\blacksquare$

We conclude that the symplectic structures $(T_P Q, \Omega^{L, \mathcal{B}})$ and $(T_P^* Q, \epsilon_1^* \omega_Q + \epsilon_2^* \mathcal{B})$ associated to a G-invariant and hyperregular magnetic Lagrangian system with a $\mathcal{B}\mathfrak{g}$-potential δ admit a momentum map with cocycle $-\sigma_\delta$ and are amenable to symplectic reduction.

5 Routh reduction for magnetic Lagrangian systems

5.1 Magnetic cotangent bundle reduction

In this section we study the reduction of the symplectic manifold $(T_P^*Q, \epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B})$ determined from the magnetic 2-form $\mathcal{B}$ of a G-invariant hyperregular magnetic Lagrangian system with $\mathcal{B}\mathfrak{g}$-potential δ. From Proposition 5, it follows that the function J_δ is a momentum map with non-equivariance cocycle $-\sigma_\delta$. As usual, G_μ denotes the isotropy subgroup of $\mu \in \mathfrak{g}^*$ for the affine action.

Proposition 6. *Fix a connection $\mathcal{A}^Q$ on the bundle $\pi^Q : Q \to Q/G$ and a regular value μ of the momentum map J_δ.*

1. *$g \in G_\mu$ iff $\mu + \delta(pg) = \mathrm{Ad}_g^*(\mu + \delta(p))$.*

2. *$\mathcal{A}^P = \epsilon^*\mathcal{A}^Q$ is a principal connection on $\pi^P : P \to P/G$.*

3. *The 1-form $\langle \mu + \delta, \mathcal{A}^P \rangle$ on P is G_μ-invariant.*

4. *The 2-form $\mathcal{B} + d(\langle \mu + \delta, \mathcal{A}^P \rangle)$ is G_μ-invariant and reducible to a 2-form on P/G_μ.*

5. *The quotient manifold $J_\delta^{-1}(\mu)/G_\mu$ is diffeomorphic to $T_{P/G_\mu}^*(Q/G)$.*

Proof. Recall the pointwise definition of $\langle \mu+\delta, \mathcal{A}^P \rangle$: given $v_p \in TP$, then $\langle \mu+\delta, \mathcal{A}^P \rangle(p)(v_p) := \langle \mu + \delta(p), \mathcal{A}^P(p)(v_p) \rangle$. Below, $\langle \mu + \delta(p), \mathcal{A}^P(p) \rangle \in T_p^*P$ denotes the cotangent vector $\langle \mu + \delta, \mathcal{A}^P \rangle(p)$. We continue with the proof.

1. This is straightforward from the definition of σ_δ.

2. The $\mathfrak{g}^*$-valued 1-form $\mathcal{A}^P$ determines a principal connection if it satisfies $\mathcal{A}^P(\xi_P) = \xi$ for all $\xi \in \mathfrak{g}$, and if $(\Psi_g^P)^*\mathcal{A}^P = \mathrm{Ad}_{g^{-1}} \cdot \mathcal{A}^P$ for all $g \in G$:

$$\mathcal{A}^P(\xi_P)(p) = \mathcal{A}^Q(\epsilon(p))(T\epsilon(\xi_P(p))) = \mathcal{A}^Q(\epsilon(p))(\xi_Q(\epsilon(p))) = \xi \qquad \text{and}$$
$$(\Psi_g^P)^*\mathcal{A}^P = (\Psi_g^P)^*\epsilon^*\mathcal{A}^Q = \epsilon^*(\Psi_g^Q)^*\mathcal{A}^Q = \mathrm{Ad}_{g^{-1}} \cdot \epsilon^*\mathcal{A}^Q = \mathrm{Ad}_{g^{-1}} \cdot \mathcal{A}^P.$$

3. The pull-back 1-form $(\Psi_g^P)^*\langle \mu + \delta, \mathcal{A}^P \rangle$ equals

$$\langle \mu + \delta(pg), ((\Psi_g^P)^*\mathcal{A}^P)(p) \rangle = \langle \mathrm{Ad}_g^*(\mu + \delta(p)), \mathrm{Ad}_{g^{-1}} \cdot \mathcal{A}^P(p) \rangle.$$

4. That $\mathcal{B} + d(\langle \mu + \delta, \mathcal{A}^P \rangle)$ is G_μ-invariant is a straightforward consequence of 3. It is projectable to P/G_μ since the contraction with any fundamental vector field ξ_P with $\xi \in \mathfrak{g}_\mu$ vanishes:

$$i_{\xi_P}\left(d(\langle \mu + \delta, \mathcal{A}^P \rangle) + \mathcal{B}\right) = \mathcal{L}_{\xi_P}(\langle \mu + \delta, \mathcal{A}^P \rangle) - d(\langle \mu + \delta, \xi \rangle) + d\delta_\xi$$
$$= \langle \xi_P(\mu + \delta), \mathcal{A}^P \rangle + \langle \mu + \delta, \mathrm{ad}_{-\xi} \cdot \mathcal{A}^P \rangle - d\delta_\xi + d\delta_\xi$$
$$= \langle \xi_P(\delta) - \mathrm{ad}_\xi^*\mu - \mathrm{ad}_\xi^*\delta, \mathcal{A}^P \rangle = 0,$$

where we used the fact that $\xi \in \mathfrak{g}_\mu$, or equivalently $\mathrm{ad}_\xi^*\mu = i_\xi\Sigma_\delta$.

5. Similar to cotangent bundle reduction, we use a 'shift map' to construct the required diffeomorphism. Recall that $V\pi^Q$ denotes the bundle of tangent vectors vertical to $\pi^Q : Q \to Q/G$. The subbundle $V^0\pi^Q$ of T^*Q is defined as the annihilator of $V\pi^Q$. Below we introduce a *shift map* $\phi_\mu^\mathcal{A}$ between $J_\delta^{-1}(\mu)$ and $V_P^0\pi^Q = V^0\pi^Q \times_Q P$ and we show that it is equivariant w.r.t. to actions of G_μ obtained by restriction of Ψ^{T_PQ} to $J_\delta^{-1}(\mu)$ and $V_P^0\pi^Q$. This is sufficient for $\phi_\mu^\mathcal{A}$ to project to a diffeomorphism $[\phi_\mu^\mathcal{A}]$ between the quotient spaces. This provides us with the desired diffeomorphism because the quotient of $V_P^0\pi^Q$ is well known: $V_P^0\pi^Q/G_\mu = T_{P/G_\mu}^*(Q/G)$.

Definition of the shift map. In the following we use a slight abuse of notations: if we write $\langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle$ for some $p \in P$ and $q = \epsilon(p)$, then this is the cotangent vector in T_q^*Q determined by $v_q \mapsto \langle \mu + \delta(p), \mathcal{A}^Q(q)(v_q) \rangle$. Let $\phi_\mu^{\mathcal{A}}$ be the map

$$\phi_\mu^{\mathcal{A}} : \; J_\delta^{-1}(\mu) \to V_P^0 \pi_Q; \qquad (\alpha_q, p) \mapsto \left(\alpha_q - \langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle, p \right).$$

It is well defined, i.e. $\phi_\mu^{\mathcal{A}}(\alpha_q, p) \in V_P^0 \pi^Q$ for any (α_q, p) in the level set $J_\delta^{-1}(\mu)$ since

$$\langle \phi_\mu^{\mathcal{A}}(\alpha_q, p), (\xi_Q(q), p) \rangle = \langle \alpha_q, \xi_Q(q) \rangle - \langle \mu + \delta(p), \mathcal{A}^Q(q)(\xi_Q(q)) \rangle$$
$$= \langle J_\delta(\alpha_q, p), \xi \rangle - \langle \mu, \xi \rangle = 0.$$

Equivariance of the shift map. For arbitrary $g \in G_\mu$,

$$\phi_\mu^{\mathcal{A}}\left(\Psi_g^{T_P^*Q}(\alpha_q, p) \right) = \left(T^* \Psi_{g^{-1}}^Q(\alpha_q) - \langle \mu + \delta(pg), \mathcal{A}^Q(qg) \rangle, pg \right)$$
$$= \left(T^* \Psi_{g^{-1}}^Q \left(\alpha_q - \langle \mu + \delta(pg), \mathrm{Ad}_{g^{-1}} \cdot \mathcal{A}^Q(q) \rangle \right), pg \right)$$
$$= \left(T^* \Psi_{g^{-1}}^Q \left(\alpha_q - \langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle \right), pg \right) = \Psi_g^{T_P^*Q}\left(\phi_\mu^{\mathcal{A}}(\alpha_q, p) \right).$$

The projection of $\phi_\mu^{\mathcal{A}}$ is thus well defined, and it is denoted by $[\phi_\mu^{\mathcal{A}}]$, i.e. $[\phi_\mu^{\mathcal{A}}] : J_\delta^{-1}(\mu)/G_\mu \to V_P^0 \pi^Q / G_\mu = T_{P/G_\mu}^*(Q/G)$. ∎

Definition 14. $\tilde{\mathcal{B}}$ is the 2-form on P/G_μ obtained after reducing $\mathcal{B} + d(\langle \mu + \delta, \mathcal{A}^P \rangle)$.

In the following proposition we put a symplectic structure on the fibred product $T_{P/G_\mu}^*(Q/G) = T^*(Q/G) \times_{Q/G} P/G_\mu$. The notations are similar to those on T_P^*Q.

Definition 15. We define the projections

1) $\tilde{\epsilon}_1 : T_{P/G_\mu}^*(Q/G) \to T^*(Q/G)$,

2) $\tilde{\epsilon}_2 : T_{P/G_\mu}^*(Q/G) \to P/G_\mu$.

The canonical symplectic 2-form on $T^*(Q/G)$ is denoted by $\omega_{Q/G}$.

Completely analogous to the construction on T_P^*Q, we introduce $\tilde{\epsilon}_1^* \omega_{Q/G} + \epsilon_2^* \tilde{\mathcal{B}}$ as a 2-form on $T_{P/G_\mu}^*(Q/G)$.

Figure 1. Cotangent bundle reduction (notations as in Theorem 2 with $M = T_P^*Q$).

Theorem 3 (Generalized magnetic cotangent bundle reduction). *Let μ denote a regular value of the momentum map J_δ for the symplectic structure $(T_P^*Q, \epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B})$ with isotropy subgroup G_μ, and let $\mathcal{A}^Q$ be a principal connection on $Q \to Q/G$. Then the symplectically reduced manifold (M_μ, ω_μ) is symplectomorphic to $(T_{P/G_\mu}^*(Q/G), \tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}}))$, with symplectomorphism $[\phi_\mu^{\mathcal{A}}]$.*

Although this result is a straightforward extension of standard magnetic cotangent bundle reduction in [12], we include a proof because its method turns out to be useful later on.

Proof. Let i_0 denote the injection $V_P^0\pi^Q \to T_P^*Q$ and π_μ^0 the projection $V_P^0\pi^Q \to T_{P/G_\mu}^*(Q/G)$. We know that $[\phi_\mu^{\mathcal{A}}]$ is a diffeomorphism, and therefore it only remains to show that the 2-form $\tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}}$ is pull-backed to ω_μ under this map (and therefore, implicitly $\tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}}$ will be nondegenerate).

We use the fact that ω_μ is uniquely determined by $i_\mu^*\omega_Q = \pi_\mu^*\omega_\mu$, with $i_\mu : J_\delta^{-1}(\mu) \to T_P^*Q$ the natural inclusion and $\pi_\mu : J_\delta^{-1}(\mu) \to M_\mu$ the projection to the quotient space. Due to the uniqueness property, it is therefore sufficient to show that

$$\pi_\mu^*\big([\phi_\mu^{\mathcal{A}}]^*(\tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}})\big) = i_\mu^*(\epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B}). \tag{2}$$

The left-hand side of (2). Since $[\phi_\mu^{\mathcal{A}}]\circ\pi_\mu = \pi_\mu^0\circ\phi_\mu^{\mathcal{A}}$ the left-hand side of the above equation is $(\pi_\mu^0 \circ \phi_\mu^{\mathcal{A}})^*\big(\tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}}\big)$.

The right-hand side of (2). We first show the equality of two 1-forms:

$$i_\mu^*(\epsilon_1^*\theta_Q) \qquad \text{and} \qquad \big(\phi_\mu^{\mathcal{A}}\big)^*\Big(i_0^*\big(\epsilon_1^*\theta_Q + \epsilon_2^*\langle\mu + \delta, \mathcal{A}^P\rangle\big)\Big).$$

Let $(\alpha_q, p) \in J_\delta^{-1}(\mu)$, $V_{(\alpha_q,p)} \in T(T_P^*Q)$ a tangent vector to $J_\delta^{-1}(\mu)$ and $v_q \in T_qQ$ denotes the projection of $V_{(\alpha_q,p)}$. Then, the first 1-form equals

$$i_\mu^*(\epsilon_1^*\theta_Q)(\alpha_q, p)(V_{(\alpha_q,p)}) = \theta_Q(\alpha_q)(T\epsilon_1(V_{(\alpha_q,p)})) = \langle\alpha_q, v_q\rangle.$$

The second 1-form evaluated on this tangent vector gives

$$\begin{aligned}
(\phi_\mu^{\mathcal{A}})^*&\Big(i_0^*\big(\epsilon_1^*\theta_Q + \epsilon_2^*\langle\mu + \delta, \mathcal{A}^P\rangle\big)\Big)(\alpha_q, p)(V_{(\alpha_q,p)}) \\
&= \theta_Q\big(\alpha_q - \langle\mu + \delta(p), \mathcal{A}^Q(q)\rangle\big)\big(T\epsilon_1(T(\phi_\mu^{\mathcal{A}})(V_{(\alpha_q,p)}))\big) + \langle\mu + \delta(p), \mathcal{A}^Q(q)(v_q)\rangle \\
&= \langle\alpha_q, v_q\rangle.
\end{aligned}$$

The two 1-forms are identical and after taking the exterior derivative and adding $\epsilon_2^*\mathcal{B}$, we obtain that the right-hand side of (2) equals

$$i_\mu^*(\epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B}) = \big(\phi_\mu^{\mathcal{A}}\big)^*\Big(i_0^*\big(\epsilon_1^*\omega_Q + \epsilon_2^*(d\langle\mu + \delta, \mathcal{A}^P\rangle + \mathcal{B})\big)\Big).$$

Since (i) $(\tilde{\epsilon}_2 \circ \pi_\mu^0)^*\tilde{\mathcal{B}} = (\epsilon_2 \circ i_0)^*(d\langle\mu + \delta, \mathcal{A}^P\rangle + \mathcal{B})$, and (ii) $(\tilde{\epsilon}_1 \circ \pi_\mu^0)^*\theta_{Q/G} = i_0^*\epsilon_1^*\theta_Q$, one easily verifies that the left-hand side equals the right-hand side in (2). ∎

We conclude with a result on the behavior of a symplectomorphism under symplectic reduction. Assume two symplectic manifolds (M, Ω) and (M', Ω') and a symplectomorphism $f : M \to M'$ (i.e. a map for which $f^*\Omega' = \Omega$) are given. We assume in addition that both M and M' are equipped with a canonical free and proper action of G. Let $J : M \to \mathfrak{g}^*$ and $J' : M' \to \mathfrak{g}^*$ denote corresponding momentum maps for these actions on M and M' respectively. We say that f is equivariant if $f(mg) = f(m)g$ for arbitrary $m \in P$, $g \in G$. Note that the non-equivariance cocycles for J and J' are equal up to a coboundary. Without loss of generality we may assume that $f^*J' = J$ and that the non-equivariance cocycles coincide. This in turn guarantees that the affine actions on $\mathfrak{g}^*$ coincide and that the isotropy groups of an element $\mu \in \mathfrak{g}^*$ coincide for both affine actions. Finally, fix a regular value $\mu \in \mathfrak{g}^*$ for both J and J'.

Theorem 4. *If f is an equivariant symplectic diffeomorphism $M \to M'$, such that $J' = J \circ f$, then under symplectic reduction, the symplectic manifolds (M_μ, Ω_μ) and (M'_μ, Ω'_μ) are symplectically diffeomorphic under the map*

$$[f_\mu]: \ M_\mu \to M'_\mu; \qquad [m]_{G_\mu} \mapsto [f(m)]_{G_\mu}.$$

Proof. This is a straightforward result. Since f is a diffeomorphism for which $J' = J \circ f$, the restriction f_μ of f to $J^{-1}(\mu)$ determines a diffeomorphism from $J^{-1}(\mu)$ to $J'^{-1}(\mu)$. The equivariance implies that f_μ reduces to a diffeomorphism $[f_\mu]$ from $M_\mu = J^{-1}(\mu)/G_\mu$ to $M'_\mu = J'^{-1}(\mu)/G_\mu$. It is our purpose to show that $[f_\mu]^*\Omega'_\mu = \Omega_\mu$, or since both π_μ and π'_μ are projections, that $\pi_\mu^*\Omega_\mu = f_\mu^*(\pi_\mu'^*\Omega'_\mu)$. The determining property for Ω_μ is $\pi_\mu^*\Omega_\mu = i_\mu^*\Omega$ (and similarly for Ω'_μ). From diagram chasing we have that $i_\mu^*\Omega = f_\mu^*(i_\mu'^*\Omega')$. Then

$$\pi_\mu^*\Omega_\mu = i_\mu^*\Omega = f_\mu^*(i_\mu'^*\Omega') = f_\mu^*(\pi_\mu'^*\Omega'_\mu) = \pi_\mu^*([f_\mu]^*\Omega'_\mu),$$

since $\pi'_\mu \circ f_\mu = [f_\mu] \circ \pi_\mu$ by definition. This concludes the proof. ∎

5.2   Route reduction for magnetic Lagrangian systems

In Proposition 5 we have introduced the momentum map $J_{L,\delta} = J_\delta \circ \mathbb{F}L$ for a G-invariant hyperregular magnetic Lagrangian system $(\epsilon: P \to Q, L, \mathcal{B})$, i.e.

$$\langle J_{L,\delta}(v_q, p), \xi \rangle = \langle \mathbb{F}L(v_q, p), (\xi_Q(q), p) \rangle - \delta_\xi(p).$$

We know from Proposition 4 that the Legendre transform $\mathbb{F}L$ is equivariant. After restriction to $J_{L,\delta}^{-1}(\mu)$ it reduces to a symplectic diffeomorphism between the symplectically reduced spaces $J_{L,\delta}^{-1}(\mu)/G_\mu$ and $J_\delta^{-1}(\mu)/G_\mu$ (see also Theorem 4). The following diagram summarizes these previous observations.

$$
\begin{array}{ccc}
(T_P Q, \Omega_Q^{L,\mathcal{B}}) & \xrightarrow{\ \mathbb{F}L\ } & (T_P^*Q, \epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B}) \\[4pt]
{\scriptstyle\text{SR}}\Big\downarrow & & {\scriptstyle\text{SR}+[\phi_\mu^{\mathcal{A}}]}\Big\downarrow \\[4pt]
(J_{L,\delta}^{-1}(\mu)/G_\mu, \Omega_\mu) & \xrightarrow{[\phi_\mu^{\mathcal{A}}]\circ[(\mathbb{F}L)_\mu]} & (T_{P/G_\mu}^*(Q/G), \tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}})
\end{array}
$$

Figure 2. Diagram relating tangent and cotangent reduction (SR = symplectic reduction).

We are now only three steps away from a description of Routh reduction for magnetic Lagrangian systems. The final goal is to describe the symplectic reduced manifold $(J_{L,\delta}^{-1}(\mu)/G_\mu, \Omega_\mu)$ as a symplectic manifold associated with a new magnetic Lagrangian system, with Lagrangian say L_μ.

 1. The first step is the construction of a diffeomorphism $\Delta_\mu: T_{P/G_\mu}(Q/G) \to J_{L,\delta}^{-1}(\mu)/G_\mu$.

This is crucial because the Hamiltonian dynamics determined by the Euler–Lagrange equations reduce to Hamiltonian dynamics on a manifold of the form $T_{P/G_\mu}(Q/G)$. If we can characterize this reduced dynamics as the Euler–Lagrange equations of a magnetic Lagrangian system with configuration manifold $P/G_\mu \to Q/G$, we have developed a Routh reduction technique for magnetic Lagrangian systems. This characterization consists of the two remaining steps mentioned before:

2. We show that the composition $[\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu] \circ \Delta_\mu$ equals the fibre derivative of some Lagrangian function $\tilde{L}$ on $T_{P/G_\mu}(Q/G)$, and this implies that the symplectic 2-form $\Delta_\mu^* \Omega_\mu$ on $T_{P/G_\mu}(Q/G)$ is of the form

$$\mathbb{F}\tilde{L}^*(\tilde{\epsilon}_1^* \omega_{Q/G} + \tilde{\epsilon}_2^* \tilde{\mathcal{B}}),$$

which is the symplectic structure associated to the *reduced* magnetic Lagrangian system $(P/G_\mu \to Q/G, \tilde{L}, \tilde{\mathcal{B}})$;

3. We show that the reduction of the energy E_L in the symplectic reduction scheme is precisely the energy $E_{\tilde{L}}$ on $T_{P/G_\mu}(Q/G)$. This guarantees that the Euler–Lagrange equations associated to the reduced Lagrangian $\tilde{L}$ are related to the Euler–Lagrange equations for L.

The new reduced Lagrangian $\tilde{L}$ is what we call the Routhian (it is often alternatively denoted by R or R^μ). This is summarized in the diagram in Fig. 3.

$$
\begin{array}{ccc}
(T_PQ, \Omega_Q^{L,\mathcal{B}}) & \xrightarrow{\ \mathbb{F}L\ } & (T_P^*Q, \epsilon_1^*\omega_Q + \epsilon_2^*\mathcal{B}) \\[4pt]
{\scriptstyle \mathrm{SR}+\Delta_\mu^{-1}} \Big\downarrow & & {\scriptstyle \mathrm{SR}+[\phi_\mu^{\mathcal{A}}]} \Big\downarrow \\[4pt]
(T_{P/G_\mu}(Q/G), \Omega^{\tilde{L},\tilde{\mathcal{B}}}) & \xrightarrow{\ \mathbb{F}\tilde{L}\ } & (T_{P/G_\mu}^*(Q/G), \tilde{\epsilon}_1^*\omega_{Q/G} + \tilde{\epsilon}_2^*\tilde{\mathcal{B}})
\end{array}
$$

Figure 3. A symplectic view on Routh reduction.

Step 1: the definition of $\Delta_\mu : T_{P/G_\mu}(Q/G) \to J_{L,\delta}^{-1}(\mu)/G_\mu$. First consider the map $\Pi_\mu : J_{L,\delta}^{-1}(\mu)/G_\mu \to T_{P/G_\mu}(Q/G)$ which is obtained from the G_μ-invariance of the map

$$J_{L,\delta}^{-1}(\mu) \to T_{P/G_\mu}(Q/G); (v_q, p) \mapsto (T\pi^Q(v_q), [p]_{G_\mu}).$$

We will define the map Δ_μ as the inverse of Π_μ. In general however, Π_μ is not invertible. The following definition is the analogue of G-regularity for a classical Lagrangian.

Definition 16. The Lagrangian L of a G-invariant magnetic Lagrangian system is called G-regular if the map $J_{L,\delta}|_{(v_q,p)} : \mathfrak{g} \to \mathfrak{g}^*; \xi \mapsto J_{L,\delta}(v_q + \xi_Q(q), p)$ is a diffeomorphism for all $(v_q, p) \in T_PQ$.

Every mechanical G-invariant magnetic Lagrangian system is G-regular because $J_{L,\delta}|_{(v_q,p)}$ is an affine map modeled on the metric on $\mathfrak{g}$ obtained from the kinetic energy metric.

Proposition 7. Π_μ *is a diffeomorphism if the Lagrangian is G-regular.*

Proof. We construct the inverse for Π_μ. Choose an element $(v_{[q]_G}, [p]_{G_\mu})$ in $T_{P/G_\mu}(Q/G)$ and fix a point $(v_q, p) \in T_PQ$ that projects onto $(v_{[q]_G}, [p]_{G_\mu})$, i.e. $T\pi^Q(v_q) = v_{[q]}$ and $p \in [p]_{G_\mu}$. Due to the G-regularity of L, there exists a unique element ξ in $\mathfrak{g}$ such that $J_{L,\delta}(v_q + \xi_Q(q), p) = \mu$. The orbit of the point $(v_q + \xi_Q(q), p)$ under the G_μ-action determines an element in $J_{L,\delta}^{-1}(\mu)/G_\mu$. We will show that this construction is independent of the chosen point (v_q, p), and therefore defines a map $\Delta_\mu : T_{P/G_\mu}(Q/G) \to J_{L,\delta}^{-1}(\mu)/G_\mu$ which is the inverse to Π_μ.

To show that the construction is independent of the chosen point (v_q, p) in T_PQ, we choose any other point (v_q', p') in T_PQ projecting onto $(v_{[q]_G}, [p]_{G_\mu})$. The point is of the form $(v_q', p') = (\Psi_g^{TQ}(v_q + \eta_Q(q)), pg)$ for some $g \in G_\mu$ and $\eta \in \mathfrak{g}$. We now repeat the previous construction applied to (v_q', p'): we consider a unique element $\xi' \in \mathfrak{g}$ for which $J_{L,\delta}(\Psi_g^{TQ}(v_q + \eta_Q(q)) + \xi_Q'(qg), pg) = \mu$. Due to the equivariance of $J_{L,\delta}$, we conclude that

$$\mu = J_{L,\delta}(\Psi_g^{TQ}(v_q + \eta_Q(q)) + \xi_Q'(qg), pg) = \mathrm{Ad}_g^* \cdot J_{L,\delta}(v_q + (\eta + \mathrm{Ad}_g\xi')_Q(q), p) - \sigma_\delta(g^{-1}).$$

Since $g \in G_\mu$, we may conclude that $J_{L,\delta}(v_q + (\eta + \mathrm{Ad}_g \xi')_Q(q), p) = \mu$ and therefore $\eta + \mathrm{Ad}_g \xi' = \xi$. This implies that $(v_q' + \xi_Q'(q), p') = \Psi_g^{T_P Q}(v_q + \xi_Q(q), p)$ for $g \in G_\mu$ such that $p' = pg$. This concludes the proof. ∎

The previous proposition guarantees that for a given G-regular Lagrangian and a bundle adapted coordinate chart (x^r, g^m) in $Q \to Q/G$ and (x^r, g^m, p^a) in $P \to Q$, the functions $(x^r, \dot{x}^r, g^m, p^a)$ determine a coordinate chart in $J_{L,\delta}^{-1}(\mu)$. This is useful in the following.

Step 2: the reduced symplectomorphism. We will now introduce a new Lagrangian $\tilde{L}$ on $T_{P/G_\mu}(Q/G)$ with the property that its Legendre transform $\mathbb{F}\tilde{L} : T_{P/G_\mu}(Q/G) \to T^*_{P/G_\mu}(Q/G)$ is precisely the reduced Legendre transform $[(\mathbb{F}L)_\mu]$, more specifically $\mathbb{F}\tilde{L} = [\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu] \circ \Delta_\mu$.

As is already described in [10], the structure of the Lagrangian $\tilde{L}$ is completely determined by the cotangent bundle reduction scheme, and in particular by the shift map $\phi_\mu^{\mathcal{A}}$ from $J_\delta^{-1}(\mu) \to V_P^0 \pi^Q$. To make this statement more precise, we remark that $[\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu]$ is the quotient map of $\phi_\mu^{\mathcal{A}} \circ \mathbb{F}L$. Thus, if $[(v_q, p)]_{G_\mu}$ is arbitrary in $J_{L,\delta}^{-1}(\mu)/G_\mu$, then $[\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu]([(v_q, p)]_{G_\mu})$ is the quotient of

$$\mathbb{F}L(v_q, p) - \left(\langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle, p \right).$$

The latter is the fibre derivative of a new function on $T_P Q$, namely $L(v_q, p) - \langle \mu + \delta(p), \mathcal{A}^Q(q)(v_q) \rangle$. This function is clearly G_μ-invariant.

Definition 17. The function $\tilde{L}$ is defined as the pull-back under Δ_μ of the quotient map of the restriction to $J_{L,\delta}^{-1}(\mu)$ of the function

$$(v_q, p) \mapsto L(v_q, p) - \left\langle \mu + \delta(p), \mathcal{A}^Q(q)(v_q) \right\rangle.$$

Lemma 2. $\mathbb{F}\tilde{L} = [\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu] \circ \Delta_\mu$.

Proof. Fix elements $(v_{[q]_G}, [p]_{G_\mu}) \in T_{P/G_\mu}(Q/G)$ and fix a representative $(v_q, p) \in J_{L,\delta}^{-1}(\mu)$ in the orbit $\Delta_\mu(v_{[q]_G}, [p]_{G_\mu}) \in J_{L,\delta}^{-1}(\mu)/G_\mu$. By definition of the maps involved, we have

$$
\begin{aligned}
\left([\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu] \circ \Delta_\mu\right)(v_{[q]_G}, [p]_{G_\mu}) &= (\pi_\mu^0 \circ \phi_\mu^{\mathcal{A}})(\mathbb{F}L(v_q, p)) \\
&= \pi_\mu^0\big(\mathbb{F}L(v_q, p) - \left(\langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle, p\right)\big).
\end{aligned}
$$

Next we study the fibre derivative of the map $\tilde{L}$. Fix a point $(w_{[q]_G}, [p]_{G_\mu})$ and we compute

$$\left\langle \mathbb{F}\tilde{L}(v_{[q]_G}, [p]_{G_\mu}), (w_{[q]_G}, [p]_{G_\mu}) \right\rangle = \left. \frac{d}{du} \right|_{u=0} \tilde{L}(v_{[q]_G} + u w_{[q]_G}, [p]_{G_\mu}).$$

We construct a curve $u \mapsto \zeta(u)$ in $J_{L,\delta}^{-1}(\mu)$ that projects onto the curve $u \mapsto \Delta_\mu(v_{[q]_G} + u w_{[q]_G}, [p]_{G_\mu})$ in $J_{L,\delta}^{-1}(\mu)/G_\mu$ such that $\zeta(0) = (v_q, p)$ and $\dot{\zeta}(0)$ is vertical to the projection $\rho_2 \circ i_\mu : J_{L,\delta}^{-1}(\mu) \to P$ (recall that $\rho_2 : T_P Q \to P$ is the projection onto the second factor in the fibred product).

The existence of such a curve is best shown in a specific coordinate chart (see above) for $J_L^{-1}(\mu)$: $(x^r, \dot{x}^r, g^m, p^a)$ where the index $r = 1, \ldots, \dim Q/G$, $m = 1, \ldots, \dim G$ and $a = 1, \ldots, \dim P - \dim Q$. Note that $(x^r, \dot{x}^r)$ is a coordinate chart in $T(Q/G)$. In these coordinates, we let $(v_q, p) = (x_0^r, \dot{x}_0^r, g_0^m, p_0^a)$ and $w_{[q]_G} = (x_0^r, w_0^r)$. We define the curve $\zeta(u)$ to be the curve $u \mapsto (x_0^r, \dot{x}_0^r + u w_0^i, g_0^m, p_0^a)$. Then the tangent vector to $T\rho_1(\dot{\zeta}(0))$ is the vertical lift of some $w_q \in T_q Q$ with $T\pi^Q(w_q) = w_{[q]_G}$.

Finally, from the definition of $\tilde{L}$ and the fibre derivative $\mathbb{F}\tilde{L}$ we obtain

$$\left\langle \mathbb{F}\tilde{L}(v_{[q]_G}, [p]_{G_\mu}), (w_{[q]_G}, [p]_{G_\mu}) \right\rangle = \left. \frac{d}{du} \right|_{u=0} \left(L - \langle \mu + \delta, \mathcal{A}^Q \rangle \right)(\zeta(u))$$

$$= \langle \mathbb{F}L(v_q, p) - (\langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle, p), (w_q, p) \rangle$$

$$= \left\langle \pi_\mu^0 \big(\mathbb{F}L(v_q, p) - (\langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle, p) \big), (w_{[q]_G}, [p]_{G_\mu}) \right\rangle,$$

since $\mathbb{F}L(v_q, p) - (\langle \mu + \delta(p), \mathcal{A}^Q(q) \rangle, p) \in V_P^0 \pi^Q$. This concludes the proof. ∎

Step 3: the reduced energy Hamiltonian. The third and last step concerns the specific reduced dynamics. We have to relate the energy E_L on $T_P Q$ to the energy of the Routhian $\tilde{L}$ on $T_{P/G_\mu}(Q/G)$. In the following Lemma we again use the notations from Theorem 2 applied to the reduction of the symplectic structure on $(T_P Q, \Omega^{L,\mathcal{B}})$.

Lemma 3. *The energy $E_{\tilde{L}}$ is the reduced Hamiltonian, i.e. it satisfies:*

$$(\Pi_\mu \circ \pi_\mu)^* E_{\tilde{L}} = i_\mu^* E_L,$$

with $\Pi_\mu \circ \pi_\mu : J_{L,\delta}^{-1}(\mu) \to J_{L,\delta}^{-1}(\mu)/G_\mu \to T_{P/G_\mu}(Q/G)$ and $i_\mu : J_{L,\delta}^{-1}(\mu) \to T_P Q$.

Proof. Let $(v_q, p) \in J_{L,\delta}^{-1}(\mu)$, such that $(\Pi_\mu \circ \pi_\mu)(v_q, p) = (v_{[q]_G}, [p]_{G_\mu})$. Then

$$i_\mu^* E_L(v_q, p) = \langle \mathbb{F}L(v_q, p), (v_q, p) \rangle - L(v_q, p)$$

$$= \langle (\phi_\mu^{\mathcal{A}} \circ (\mathbb{F}L)_\mu)(v_q, p), (v_q, p) \rangle - \big(L(v_q, p) - \langle \mu + \delta(p), \mathcal{A}^Q(q)(v_q) \rangle \big)$$

$$= \left\langle \big([\phi_\mu^{\mathcal{A}}] \circ [(\mathbb{F}L)_\mu] \circ \Delta_\mu \big)(v_{[q]_G}, [p]_{G_\mu}), (v_{[q]_G}, [p]_{G_\mu}) \right\rangle - \tilde{L}(v_{[q]_G}, [p]_{G_\mu}).$$

Using the result from Lemma 2 this concludes the proof. ∎

Routh reduction. The previous three steps are summarized in the following theorem.

Theorem 5 (Routh reduction for magnetic Lagrangian systems). *Let $(\epsilon : P \to Q, L, \mathcal{B})$ be a hyperregular, G-invariant and G-regular magnetic Lagrangian system and let δ be a $\mathcal{B}\mathfrak{g}$-potential of the magnetic term $\mathcal{B}$ with 1-cocycle σ_δ.*

1. *Let $\mu \in \mathfrak{g}^*$ be a regular value of the momentum map $J_{L,\delta}$ and let G_μ be the isotropy subgroup of μ w.r.t. the affine action on $\mathfrak{g}^*$ with 1-cocycle $-\sigma_\delta$, i.e. $g \in G_\mu$ if and only if $\mu = \mathrm{Ad}_g^* \mu - \sigma_\delta(g^{-1})$.*

2. *Fix a connection $\mathcal{A}^Q$ on $\pi^Q : Q \to Q/G$ and let $\mathcal{A}^P$ be the corresponding connection on $P \to P/G$. Compute the restriction of the G_μ-invariant function $L(v_q, p) - \langle \mu + \delta(p), \mathcal{A}^Q(q)(v_q) \rangle$ to $J_{L,\delta}^{-1}(\mu)$ and let $\tilde{L}$ be its quotient to $T_{P/G_\mu}(Q/G) \cong J_{L,\delta}^{-1}(\mu)/G_\mu$.*

3. *Compute $\tilde{\mathcal{B}}$ as the projection to P/G_μ of the 2-form $\mathcal{B} + d\langle \mu + \delta, \mathcal{A}^P \rangle$.*

4. *Consider the magnetic Lagrangian system: $(\tilde{\epsilon}_\mu : P/G_\mu \to Q/G, \tilde{L}, \tilde{\mathcal{B}})$.*

This reduced magnetic Lagrangian system is hyperregular and every solution $p(t) \in P$ to the Euler–Lagrange equations for $(\epsilon, L, \mathcal{B})$ with momentum μ projects under $P \to P/G_\mu$ to a solution of the Euler–Lagrange equations for $(\tilde{\epsilon}_\mu : P/G_\mu \to Q/G, \tilde{L}, \tilde{\mathcal{B}})$. Conversely, every solution to the Euler–Lagrange equations for $(\tilde{\epsilon}_\mu : P/G_\mu \to Q/G, \tilde{L}, \tilde{\mathcal{B}})$ is the projection of a solution to the Euler–Lagrange equations for $(\epsilon, L, \mathcal{B})$ with momentum μ.

It is possible to say more about the structure of $\tilde{\mathcal{B}}$ and its relation to the connection $\mathcal{A}^Q$. This and reconstruction aspects fall out of the scope of this paper. We refer to [11] where these topics are described in more detail. They carry over to this more general framework in a straightforward way.

5.3   Reduction of magnetic Lagrangian systems on Lie groups

Consider a magnetic Lagrangian system on $P = Q = G$, i.e. the configuration space is a Lie group G. We start from a function ℓ on $\mathfrak{g}$ and with it we associate a Lagrangian L on TG by right multiplication $L(g, v_g) = \ell(v_g g^{-1})$. By definition L is invariant under the right action of G on itself. We assume that a magnetic 2-form $\mathcal{B}$ is given which is invariant under right multiplication and admits a $\mathcal{B}\mathfrak{g}$-potential $\delta : G \to \mathfrak{g}^*$. We first rephrase some definitions in this specific setting.

1. The 1-cocycle $\sigma_\delta : G \to \mathfrak{g}^*$ satisfies $\sigma_\delta(g) = \delta(hg^{-1}) - \mathrm{Ad}^*_{g^{-1}}\delta(h)$, for arbitrary $h \in G$. If we let $h = e$, then $\sigma_\delta(g) = \delta(g^{-1}) - \mathrm{Ad}^*_{g^{-1}}\delta(e)$ or equivalently, $\mathrm{Ad}^*_g\sigma_\delta(g) = \mathrm{Ad}^*_g\delta(g^{-1}) - \delta(e)$. Similarly if we let $h = g$, then $\sigma_\delta(g) = \delta(e) - \mathrm{Ad}^*_{g^{-1}}\delta(g)$. Since δ is determined up to a constant, we may assume without loss of generality that $\delta(e) = 0$.

2. The associated 2-cocycle $\Sigma_\delta(\xi, \eta) = \xi_G(\delta_\eta) - \delta_{[\xi,\eta]}$.

3. We use the right identification of TG with $G \times \mathfrak{g}$, i.e. (g, v_g) is mapped to $(g, v_g g^{-1}) \in G \times \mathfrak{g}$. The right action of G on TG equals right multiplication in the first factor of $G \times \mathfrak{g}$ under this identification.

4. We use the Maurer–Cartan principal connection on $G \to G/G$: $\mathcal{A}(g)(v_g) = g^{-1}v_g$. In the right identification, the connection corresponds to the map $(g, \xi) \in G \times \mathfrak{g} \mapsto \mathrm{Ad}_{g^{-1}}\xi \in \mathfrak{g}$.

5. The momentum map $J_{L,\delta} : TG \to \mathfrak{g}^*$ equals $J_{L,\delta}(g, \xi g) = \mathrm{Ad}^*_g\mathbb{F}\ell(\xi) - \delta(g)$, for $(g, \xi) \in G \times \mathfrak{g}$ arbitrary. If L is G-regular then $\mathbb{F}\ell$ is invertible, i.e. there exists a function $\chi : \mathfrak{g}^* \to \mathfrak{g}$ such that $\mathbb{F}\ell(\chi(\nu)) = \nu$.

6. The affine action on $\mathfrak{g}^*$ is $(g, \mu) \mapsto \mathrm{Ad}^*_g\mu - \sigma_\delta(g^{-1})$.

7. The isotropy group G_μ consists of group elements g such that $\mu + \delta(g) = \mathrm{Ad}^*_g\mu$.

8. The quotient G/G_μ (right coset space) can be identified with $\tilde{\mathcal{O}}_\mu$, i.e. $[g]_{G_\mu} \in G/G_\mu$ is mapped onto $\nu = \mathrm{Ad}^*_{g^{-1}}\mu - \sigma_\delta(g) = \mathrm{Ad}^*_{g^{-1}}(\mu + \delta(g))$. A tangent vector to G/G_μ at $[g]_{G_\mu}$ which is the projection of ξg is mapped to a vector $\dot{\nu} = -\mathrm{ad}^*_\xi\nu + i_\xi\Sigma_\delta$ in $\mathfrak{g}^*$.

Lemma 4. *The 2-form $\mathcal{B} + d(\langle\mu + \delta, \mathcal{A}\rangle)$ reduces to the Kirillov–Kostant–Souriau symplectic 2-form on $\tilde{\mathcal{O}}_\mu \cong G/G_\mu$.*

Proof. Let g be arbitrary, and let $v_g = \xi g$, $w_g = \eta g$ be two tangent vectors in T_gG with $\xi, \eta \in \mathfrak{g}$ arbitrary. Note that $v_g = \xi'_G(g)$ with $\xi' = \mathrm{Ad}_{g^{-1}}\xi$, and similarly $w_g = \eta'_G(g)$ with $\eta' = \mathrm{Ad}_{g^{-1}}\eta$. Then

$$\mathcal{B}(g)(v_g, w_g) = \mathcal{B}(g)(\xi'_G(g), \eta'_G(g)) = \eta'_G(\delta_{\xi'})(g).$$

On the other hand

$$d(\langle\mu + \delta, \mathcal{A}\rangle)(g)(v_g, w_g) = \xi'_G(\langle\mu + \delta, \eta'\rangle)(g) - \eta'_G(\langle\mu + \delta, \xi'\rangle)(g) - \langle\mu + \delta(g), [\xi', \eta']\rangle.$$

Before continuing, we compute the equivariance of the 2-cocycle $\Sigma_\delta(\xi', \eta')$:

$$\Sigma_\delta(\xi', \eta') = -\frac{d}{ds}\langle\sigma_\delta(g^{-1}\exp s\xi g), \eta'\rangle = -\frac{d}{ds}\langle\sigma_\delta(g^{-1}\exp s\xi) + \mathrm{Ad}^*_{\exp -s\xi g}\sigma_\delta(g), \eta'\rangle$$

$$= -\frac{d}{ds}\langle\sigma_\delta(g^{-1}) + \mathrm{Ad}^*_g\sigma_\delta(\exp s\xi) + \mathrm{Ad}^*_g\mathrm{Ad}^*_{\exp -s\xi}\sigma_\delta(g), \mathrm{Ad}_{g^{-1}}\eta\rangle$$

$$= \Sigma_\delta(\xi, \eta) + \langle\sigma_\delta(g), [\xi, \eta]\rangle.$$

Summarizing, we have

$$\left(\mathcal{B} + d(\langle\mu + \delta, \mathcal{A}\rangle)\right)(g)(v_g, w_g) = \xi'_G(\langle\mu + \delta, \eta'\rangle)(g) - \langle\mu + \delta(g), [\xi', \eta']\rangle$$

$$= \Sigma_\delta(\xi', \eta') + \delta_{[\xi',\eta']}(g) - \langle \mu + \delta(g), [\xi', \eta'] \rangle$$
$$= \Sigma_\delta(\xi, \eta) - \langle \mathrm{Ad}^*_{g^{-1}}\mu - \sigma_\delta(g), [\xi, \eta] \rangle.$$

This 2-form is reducible to a 2-form on $\tilde{\mathcal{O}}_\mu$. If we use the isomorphism $G/G_\mu \to \tilde{\mathcal{O}}_\mu$ introduced above, the 2-form reduces to $\tilde{\mathcal{B}}(\nu)(\dot\nu, \dot\nu') = \langle \dot\nu, \eta \rangle$ with $\eta \in \mathfrak{g}$ such that $\dot\nu' = -\mathrm{ad}^*_\eta\nu + i_\eta\Sigma_\delta$. ∎

We conclude by computing the Routhian $\tilde{L}$ as a function on $\tilde{\mathcal{O}}_\mu$. By definition, it equals the reduction to $\tilde{\mathcal{O}}_\mu$ of $L - \langle \mu + \delta, \mathcal{A}\rangle|_{J_{L,\delta}=\mu}$. In the right identification, the level set $J_{L,\delta}(g, v_g) = \mu$ is precisely $\mathrm{Ad}^*_{g^{-1}}(\mu + \delta(g)) = \mathbb{F}\ell(\xi)$, with $\xi g = v_g$. If we set $\nu = \mathrm{Ad}^*_{g^{-1}}(\mu + \delta(g)) \in \tilde{\mathcal{O}}_\mu$, the fixed momentum condition is $\xi = \chi(\nu)$. The Routhian $\tilde{L}(\nu)$ becomes in the right identification

$$\tilde{L}(\nu) = \ell(\chi(\nu)) - \langle \nu, \chi(\nu) \rangle.$$

By application of the chain rule, it easily follows that $\langle d\tilde{L}(\nu), \dot\nu' \rangle = -\langle \dot\nu', \chi(\nu) \rangle$. The reduced Euler–Lagrange equations are

$$i_{\dot\nu}\tilde{\mathcal{B}}(\nu) = d\tilde{L}(\nu) \qquad \text{or} \qquad \dot\nu = -\mathrm{ad}^*_{\chi(\nu)}\nu + i_{\chi(\nu)}\Sigma_\delta. \tag{3}$$

For later purpose, we remark that for a left action and ℓ originating from a left invariant Lagrangian, the reduced equations are $\dot\nu = \mathrm{ad}^*_{\chi(\nu)}\nu - i_{\chi(\nu)}\Sigma_\delta$ (here the 1-cocycle satisfies $\sigma_\delta(g) = \delta(g)$).

6   Routh reduction by stages

In reduction by stages, we study the reduction of a G-invariant system (symplectic or Lagrangian) under the action of the full group G and under the induced action w.r.t. a normal subgroup $K \lhd G$. We shall adopt as far as possible the notations used in [12]. A detailed construction of the following definitions is found in this reference.

Definition 18.

1. The Lie-algebra of K is $\mathfrak{K}$ and i denotes the injection $i : \mathfrak{K} \to \mathfrak{g}$ with dual $i^* : \mathfrak{g}^* \to \mathfrak{K}^*$.

2. The group G acts on $\mathfrak{K}$ by restriction of the adjoint action. The induced action of G on $\mathfrak{K}^*$ is denoted by the same symbol $\mathrm{Ad}^* : G \times \mathfrak{K}^* \to \mathfrak{K}^*$.

3. μ denotes an element in $\mathfrak{g}^*$ and $\nu \in \mathfrak{K}^*$. Then G_μ is the isotropy subgroup of μ under the Ad^*-action of G on $\mathfrak{g}^*$; G_ν is the isotropy subgroup of ν under the Ad^*-action of G on $\mathfrak{K}^*$ obtained as the dual of the restricted Ad-action of G on $\mathfrak{K}$; and K_ν is the isotropy of ν w.r.t. to standard coadjoint action of K on $\mathfrak{K}^*$. These groups satisfy $G_\nu \cap K = K_\nu$ and K_ν is normal in G_ν.

4. $\mathfrak{g}_\nu$ and $\mathfrak{K}_\nu$ denote the Lie algebras of G_ν and K_ν respectively. $\bar{G}_\nu$ denotes the quotient group G_ν/K_ν and its Lie algebra equals $\bar{\mathfrak{g}}_\nu = \mathfrak{g}_\nu/\mathfrak{K}_\nu$.

5. The projections onto the quotient groups are denoted by $r : G \to \bar{G} = G/K$ and $r_\nu : G_\nu \to \bar{G}_\nu$, and on the level of the Lie algebra: $r' : \mathfrak{g} \to \bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{K}$ and $r'_\nu : \mathfrak{g}_\nu \to \bar{\mathfrak{g}}_\nu$. The inclusion map $G_\nu \to G$ induces a map $k_\nu : \mathfrak{g}_\nu \to \mathfrak{g}$, with its dual $k_\nu^* : \mathfrak{g}^* \to \mathfrak{g}_\nu^*$.

6. ρ denotes an element in $\bar{\mathfrak{g}}_\nu^*$.

In [12] symplectic reduction by stages is performed under the condition of a so-called 'stages hypothesis'. An element $\mu \in \mathfrak{g}^*$ is said to satisfy the stages hypothesis if for any $\mu' \in \mathfrak{g}^*$ satisfying $\mu|_\mathfrak{K} = \mu'|_\mathfrak{K} = \nu$ and $\mu|_{\mathfrak{g}_\nu} = \mu'|_{\mathfrak{g}_\nu} = \bar\mu$, there exists an element $k \in K_\nu$ and $g \in (G_\nu)_{\bar\mu}$ such that $\mathrm{Ad}^*_{kg}\mu' = \mu$. The stages hypothesis is a condition on a chosen momentum value and

depends only on the symmetry group G. It was already clear in [12] that the hypothesis is automatically satisfied if G is a central extension or if G is a semi-direct product group. In the recent contribution [16] it has been pointed out that the hypothesis is in fact always satisfied, and that it can be taken out of the reduction by stages statements altogether. Taking advantage of this result, in this paper, we will not make further reference to the stages hypothesis.

6.1 Symplectic reduction by stages

Theorem 6 (Symplectic reduction by stages [12]). *Let (M, ω) be a symplectic manifold with a canonical G-action Ψ^M with an equivariant momentum map J_G.*

1. *Fix a regular value $\mu \in \mathfrak{g}^*$ of the momentum map and perform symplectic reduction to obtain the symplectic manifold (M_μ, ω_μ).*

2. *The restriction of the action Ψ^M to K is canonical and the map $J_K = i^* \circ J_G : M \to \mathfrak{K}^*$ determines an equivariant momentum map for this induced action. Fix a regular value ν of J_K and perform symplectic reduction to obtain the symplectic manifold (M_ν, ω_ν).*

3. *The level set $J_K^{-1}(\nu)$ is G_ν-invariant.*

4. *The group $\bar{G}_\nu = G_\nu/K_\nu$ acts on M_ν by projecting the restricted action of G_ν on $J_K^{-1}(\nu)$. This induced action Ψ^{M_ν} is free, proper and canonical on (M_ν, ω_ν). Assume that K_ν is connected.*

5. *Fix an element $\bar{\nu}$ in $\mathfrak{g}_\nu^*$ such that the restriction of $\bar{\nu}|_{\mathfrak{K}_\nu}$ equals $\nu|_{\mathfrak{K}_\nu}$. There is a well-defined momentum map $J_{\bar{G}_\nu} : M_\nu \to \bar{\mathfrak{g}}_\nu^*$ for the induced action Ψ^{M_ν}. This momentum map is determined from J_G and $\bar{\nu}$ and has a non-equivariance cocycle: $J_{\bar{G}_\nu}$ satisfies*

   $$(r_\nu')^* \circ J_{\bar{G}_\nu} \circ \pi_\nu = k_\nu^* \circ J_G \circ i_\nu - \bar{\nu}.$$

6. *Let $\rho \in \bar{\mathfrak{g}}_\nu^*$ be a regular value for the momentum map $J_{\bar{G}_\nu}$ and let $(\bar{G}_\nu)_\rho$ be the isotropy subgroup of ρ w.r.t. the affine action of $\bar{G}_\nu$ on $\bar{\mathfrak{g}}_\nu^*$. Perform symplectic reduction to obtain the symplectic manifold $((M_\nu)_\rho, (\omega_\nu)_\rho)$ with $(M_\nu)_\rho = J_{\bar{G}_\nu}^{-1}(\rho)/(\bar{G}_\nu)_\rho$.*

If ρ is chosen such that $(r_\nu')^*\rho = \mu|_{\mathfrak{g}_\nu} - \bar{\nu}$, then there exists a symplectic diffeomorphism

$$F : (M_\mu, \omega_\mu) \to ((M_\nu)_\rho, (\omega_\nu)_\rho).$$

For our purpose it is also important to understand the reduction of a G-invariant Hamiltonian h on M. We assume that all conditions in Theorem 6 are satisfied. First note that, by definition of the momentum maps, we have an inclusion j_μ of $J_G^{-1}(\mu)$ in $J_K^{-1}(\nu)$. Recall that we use π_ν for the projection $J_K^{-1}(\nu) \to M_\nu$. It was shown in [12] that the image of $\pi_\nu \circ j_\mu : J_G^{-1}(\mu) \to M_\nu$ is contained in $J_{\bar{G}_\nu}^{-1}(\rho)$. Moreover, this map is equivariant w.r.t. the action of G_μ on $J_G^{-1}(\mu)$ and $(\bar{G}_\nu)_\rho$ on $J_{\bar{G}_\nu}^{-1}(\rho)$ (this makes sense, since G_μ projects to a subset of $(\bar{G}_\nu)_\rho$). The quotient of $\pi_\nu \circ j_\mu$ is the symplectic diffeomorphism F mentioned in the previous theorem (see also Fig. 4).

Let h be a G-invariant Hamiltonian on M and let h_μ be the function on M_μ obtained from $\pi_\mu^* h_\mu = i_\mu^* H$. On the other hand we let h_ν be the function satisfying $\pi_\nu^* h_\nu = i_\nu^* h$. This function is $\bar{G}_\nu$-invariant: $h_\nu([m]_{K_\nu}[g]_{K_\nu}) = h(i_\nu(mg)) = h(i_\nu(m)g)$, with $m \in J_K^{-1}(\nu)$ and $g \in G_\nu$ arbitrary. Note that $(\pi_\nu \circ j_\mu)^* h_\nu = i_\mu^* h$.

The Hamiltonian h_ν is a $\bar{G}_\nu$-invariant function on (M_ν, ω_ν). Applying the second symplectic reduction to this manifold, we obtain a new reduced Hamiltonian $(h_\nu)_\rho$ on $(M_\nu)_\rho$.

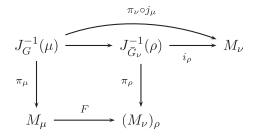**Proposition 8.** $F^*((h_\nu)_\rho) = h_\mu$.

Figure 4. Commuting diagram relating the different reduced symplectic manifolds.

Proof. We rely on the commuting diagram in Fig. 4:

$$\pi_\mu^*(F^*((h_\nu)_\rho)) = (\pi_\nu \circ j_\mu)^* h_\nu = i_\mu^*(h).$$

This uniquely characterizes $F^*((h_\nu)_\rho)$ as the function h_μ. ∎

6.2   Routh reduction by stages

Routh reduction by stages is symplectic reduction by stages applied to the symplectic struc-
ture of the initial Lagrangian system. In this section we show that the symplectic structures
and energy hamiltonians in the different stages can in fact be associated to specific magnetic
Lagrangians systems, and eventually gives us Routh reduction by stages. The symplectic re-
duction by stages then provides us a diffeomorphism relating the solutions of the different
Euler–Lagrange equations for the Lagrangian systems in the final stages.

   We start with a hyperregular Lagrangian L, invariant under the action of a Lie group G. We
assume that this Lagrangian satisfies a regularity condition which is more stringent than mere
G-regularity.

Definition 19. The Lagrangian L is said to be G-hyperregular if for any $v_q \in TQ$ and any
subspace $\mathfrak{K}' < \mathfrak{g}$ with injection $i' : \mathfrak{K}' \to \mathfrak{g}$, the mapping $i'^* \circ J_L|_{v_q} \circ i' : \mathfrak{K}' \to \mathfrak{K}'^*$ defined by
$\xi \mapsto i'^*\big(J_L(v_q + (i'(\xi))_Q(q))\big)$ is invertible.

   Lagrangians of mechanical type are G-hyperregular. Let K be a normal subgroup of G, $\mathfrak{K}$ the
Lie algebra of K and $i : \mathfrak{K} \to \mathfrak{g}$ the canonical injection. Due to the hyperregularity the invariant
Lagrangian L is both G- and K-hyperregular, and both G- and K-invariant. By definition of J_L,
the map $i^* \circ J_L = i^* \circ (\psi^{TQ})^* \circ \mathbb{F}L$ is the momentum map for the K-action.

Theorem 7 (Routh reduction by stages). *Assume (Q, L) is a hyperregular, G-hyperregular and
G-invariant Lagrangian system. Let K denote a normal subgroup of G.*

  1. *Let $\mu \in \mathfrak{g}^*$ be a regular value of the momentum map J_L and $\mathcal{A}^0$ a G-connection on Q. Let
     $(Q/G_\mu \to Q/G, L_0, \mathcal{B}_0)$ be the magnetic Lagrangian system obtained by performing Routh
     reduction with respect to G.*

  2. *Fix a regular value $\nu \in \mathfrak{K}^*$ of the momentum map $i^* \circ J_L$ for the K-action and a K-
     connection $\mathcal{A}^1$ on Q. Assume that $\mathcal{A}^1$ is G-equivariant w.r.t. the action of G on Q and $\mathfrak{K}^*$.
     Consider the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ obtained by performing
     Routh reduction with respect to K.*

  3. *$\bar{G}_\nu$ acts on Q/K_ν and Q/K by projecting the induced action of G_ν on Q. These induced
     actions are free and proper.*

  4. *Assume that K_ν is connected. Fix an element $\bar{\nu} \in \mathfrak{g}_\nu^*$ such that $\bar{\nu}|_{\mathfrak{K}_\nu} = \nu|_{\mathfrak{K}_\nu}$. Then
     the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ is $\bar{G}_\nu$-invariant, $\bar{G}_\nu$-regular and*

admits a $\mathcal{B}_1\bar{\mathfrak{g}}_\nu$-potential δ_1 entirely determined by the choice of $\bar{\nu}$. The potential satisfies, for arbitrary $q \in Q$

$$(r'_\nu)^*\big(\delta_1([q]_{K_\nu})\big) = -(\psi^Q \circ k_\nu)^*\big(\langle \nu, \mathcal{A}^1(q)\rangle\big) + \bar{\nu}.$$

Let J_1 denote the momentum map associated with δ_1.

5. Fix a regular value $\rho \in \bar{\mathfrak{g}}_\nu^*$ for the momentum map J_1 and let $(\bar{G}_\nu)_\rho$ be the isotropy subgroup of ρ w.r.t. the affine action of $\bar{G}_\nu$ on $\bar{\mathfrak{g}}_\nu$. Fix a $\bar{G}_\nu$-connection on Q/K. Consider the magnetic Lagrangian system $((Q/K_\nu)/(\bar{G}_\nu)_\rho \to (Q/K)/\bar{G}_\nu), L_2, \mathcal{B}_2)$ obtained by performing Routh reduction with respected to $\bar{G}_\nu$.

If ρ is chosen such that $(r'_\nu)^*\rho = \mu|_{\mathfrak{g}_\nu} - \bar{\nu}$, then every solution $\gamma(t) \in Q/G_\mu$ to the Euler–Lagrange equations for $(Q/G_\mu \to Q/G, L_0, \mathcal{B}_0)$ is mapped to a solution in $(Q/K_\nu)/(\bar{G}_\nu)_\rho$ to the Euler–Lagrange equations for $((Q/K_\nu)/(\bar{G}_\nu)_\rho \to (Q/K)/\bar{G}_\nu, L_2, \mathcal{B}_2)$. Conversely, a solution in $(Q/K_\nu)/(\bar{G}_\nu)_\rho$ to the Euler–Lagrange equations for $((Q/K_\nu)/(\bar{G}_\nu)_\rho \to (Q/K)/\bar{G}_\nu, L_2, \mathcal{B}_2)$ is the projection of a solution in Q/G_μ to the Euler–Lagrange equations for $(Q/G_\mu \to Q/G, L_0, \mathcal{B}_0)$.

Proof. 1 and 2 are obtained by applying Routh reduction. 3 follows from [12, p. 152]: we know that the quotient groups $\bar{G}_\nu = G_\nu/K_\nu$ acts in a free and proper way on the quotient space Q/K_ν. The group $\bar{G}_\nu$ is a subgroup of $\bar{G}$ and acts freely and properly and Q/K. We now show 4.

    $\bar{G}_\nu$-**Invariance of the Routh reduced system** $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$.

Lemma 5. If the connection $\mathcal{A}^1$ is chosen such that it is equivariant w.r.t. the action of the full group G, i.e. if

$$(\Psi_g^Q)^*\mathcal{A}^1 = \mathrm{Ad}_{g^{-1}}\mathcal{A}^1,$$

then the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ is $\bar{G}_\nu$-invariant and δ_1 is a $\mathcal{B}_1\bar{\mathfrak{g}}_\nu$-potential.

Proof. We first show that L_1 is $\bar{G}_\nu$-invariant. For that purpose, we choose an arbitrary $\bar{g} \in \bar{G}_\nu$ and let $g \in G_\nu$ be a representative. Similar we choose a point $(v_{[q]_K}, [q]_{K_\nu}) \in T_{Q/K_\nu}(Q/K)$ such that it is the projection of $v_q \in (i \circ J_L)^{-1}(\nu) \subset TQ$. By definition of the quotient action on $T_{Q/K_\nu}(Q/K)$, the action of $\bar{g}$ on an element $(v_{[q]_K}, [q]_{K_\nu})$ equals the projection of $v_q g$. We now check the invariance of L_1 at an arbitrary point in $T_{Q/K_\nu}(Q/K)$:

$$L_1\left(\Psi_{\bar{g}}^{T_{Q/K_\nu}(Q/K)}\big(v_{[q]_K}, [q]_{K_\nu}\big)\right) = L(\Psi_g^{TQ}(v_q)) - \langle \nu, \mathcal{A}^1(qg)(T\Psi_g^Q(v_q))\rangle$$

$$= L(v_q) - \langle \mathrm{Ad}_{g^{-1}}^*\nu, \mathcal{A}^1(q)(v_q)\rangle = L_1\big(v_{[q]_K}, [q]_{K_\nu}\big).$$

Next, we check the $\bar{G}_\nu$-invariance of $\mathcal{B}_1$. Recall that $\mathcal{B}_1$ is the projection to Q/K_ν of the 2-form $d\langle \nu, \mathcal{A}^1\rangle$ on Q. We first consider the equivariance of this 2-form under G_ν. Let $g \in G_\nu$ be arbitrary, then

$$(\Psi_g^Q)^*(d\langle \nu, \mathcal{A}^1\rangle) = d\langle \mathrm{Ad}_{g^{-1}}^*\nu, \mathcal{A}^1\rangle = d\langle \nu, \mathcal{A}^1\rangle.$$

We thus obtain G_ν-invariance for $d\langle \nu, \mathcal{A}^1\rangle$, and we may conclude that $(\Psi_{\bar{g}}^{Q/K_\nu})^*\mathcal{B}_1 = \mathcal{B}_1$ holds on Q/K_ν.

    The third and final step is the definition of the $\mathcal{B}_1\bar{\mathfrak{g}}_\nu$-potential. We consider an element $\bar{\xi} = [\xi]_{\mathfrak{K}_\nu} \in \bar{\mathfrak{g}}_\nu = \mathfrak{g}_\nu/\mathfrak{K}_\nu$ and let $\xi \in \mathfrak{g}_\nu$ be a representative. Then, by definition of $\mathcal{B}_1$, the 1-form $i_{\bar{\xi}_{Q/K_\nu}}\mathcal{B}_1$ is the projection to Q/K_ν of the 1-form $i_{\xi_Q}d(\langle \nu, \mathcal{A}^1\rangle)$ on Q (i.e. ξ_Q projects to $\bar{\xi}_{Q/K_\nu}$). Again we concentrate on the 1-form on Q:

$$i_{\xi_Q}d\langle \nu, \mathcal{A}^1\rangle = \mathcal{L}_{\xi_Q}(\langle \nu, \mathcal{A}^1\rangle) - d\big(i_{\xi_Q}\langle \nu, \mathcal{A}^1\rangle\big).$$

Since $\langle \nu, \mathcal{A}^1 \rangle$ is G_ν-invariant, we conclude that $i_{\xi_Q} d\langle \nu, \mathcal{A}^1 \rangle = -d\big(i_{\xi_Q}\langle \nu, \mathcal{A}^1 \rangle\big)$. The exact 1-from on the right gives a strong hint of the structure of the $\bar{\mathfrak{g}}_\nu$-potential. Assume now that we fixed an element $\bar{\nu} \in \mathfrak{g}_\nu^*$ such that $\bar{\nu}|_{\mathfrak{K}_\nu} = \nu|_{\mathfrak{K}_\nu}$.

The function δ on Q, defined by

$$-\delta_\xi(q) = \langle \nu, \mathcal{A}^1(q)(\xi_Q(q)) \rangle - \langle \bar{\nu}, \xi \rangle$$

is our candidate for the $\mathcal{B}_1 \bar{\mathfrak{g}}_\nu$-potential. This statement makes sense provided that δ_ξ projects to a function on Q/K_ν and that it only depends on the equivalence class $\bar{\xi} = \xi + \mathfrak{K}_\nu$ of $\xi \in \mathfrak{g}_\nu$. The latter is a straightforward consequence of the fact that $\mathcal{A}^1$ is a principal K-connection. The K_ν-invariance is more involved, and we rely on a result in [12]. For any k in K_ν, we have

$$-\delta_\xi(qk) = \langle \nu, \mathcal{A}^1(qk)(\xi_Q(qk)) \rangle - \langle \bar{\nu}, \xi \rangle = \langle \nu, \mathcal{A}^1(q)((\mathrm{Ad}_k \xi)_Q(q)) \rangle - \langle \bar{\nu}, \xi \rangle.$$

Therefore δ_ξ is constant on the orbits of K_ν in Q if $\langle \nu, \mathcal{A}^1(q)((\xi - \mathrm{Ad}_k \xi)_Q(q)) \rangle$ vanishes for all k. To show this we introduce a function f on K_ν given by $f(k) = \langle \nu, \mathcal{A}^1(q)((\xi - \mathrm{Ad}_k \xi)_Q(q)) \rangle$ and we use similar arguments as in [12, p. 156]. If we can show that $f(e) = 0$, $df|_e = 0$ and $f(k_1 k_2) = f(k_1) + f(k_2)$ for arbitrary $k_{1,2} \in K_\nu$, we may conclude that $f = 0$ (since K_ν is assumed connected).

The first condition $f(e) = 0$ is trivial. To check the second condition: let $\kappa \in \mathfrak{K}_\nu$ be arbitrary, then

$$df|_e(\kappa) = \langle \nu, \mathcal{A}^1(q)(-\mathrm{ad}_\kappa \xi)_Q(q)) \rangle = -\langle \nu, \mathrm{ad}_\kappa \xi \rangle = -\langle \mathrm{ad}_\kappa^* , \xi \rangle = 0.$$

Above, we have used the fact that K_ν is normal in G_ν and that, as a consequence, the Lie bracket $[\kappa, \xi]$ is in $\mathfrak{K}_\nu$. Therefore the contraction of the corresponding fundamental vector field with $\mathcal{A}^1$ is precisely $[\kappa, \xi]$. Next, we check the third condition and compute $f(k_1 k_2)$. Given the identity

$$\xi - \mathrm{Ad}_{k_1} \mathrm{Ad}_{k_2} \xi = \xi - \mathrm{Ad}_{k_1} \xi + \mathrm{Ad}_{k_1}(\xi - \mathrm{Ad}_{k_2} \xi)$$

and the fact that $k_1 \in K_\nu$,

$$\begin{aligned}
f(k_1 k_2) &= \langle \nu, \mathcal{A}^1(q)((\xi - \mathrm{Ad}_{k_1 k_2} \xi)_Q(q)) \rangle \\
&= \langle \nu, \mathcal{A}^1(q)((\xi - \mathrm{Ad}_{k_1} \xi)_Q(q)) \rangle + \langle \nu, \mathcal{A}^1(q)(\mathrm{Ad}_{k_1}(\xi - \mathrm{Ad}_{k_2} \xi)_Q(q)) \rangle \\
&= f(k_1) + \langle \mathrm{Ad}_{k_1}^* \nu, \mathcal{A}^1(q)((\xi - \mathrm{Ad}_{k_2} \xi)_Q(q)) \rangle = f(k_1) + f(k_2).
\end{aligned}$$

This completes the proof: the $\mathfrak{g}_\nu^*$-valued function δ is shown to be projectable to a $\bar{\mathfrak{g}}_\nu^*$-valued function on Q/K_ν. This is the sought-after potential δ_1: for arbitrary $q \in Q$, we have

$$(r_\nu')^*\big(\delta_1([q]_{K_\nu})\big) = -(\psi^Q \circ k_\nu)^*\big(\langle \nu, \mathcal{A}^1(q) \rangle\big) + \bar{\nu}. \qquad \blacksquare$$

Symplectic structure of $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ and symplectic reduction by stages.

Lemma 6. *Apply symplectic reduction by stages to the symplectic structure associated to the G-invariant Lagrangian system (Q, L). Identify the symplectically reduced manifold M_ν with the symplectic structure on $T_{Q/K_\nu}(Q/K)$ induced by the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$. Then:*

1. *The action $\Psi^{T_{Q/K_\nu}(Q/K)}$ of $\bar{G}_\nu$ on $T_{Q/K_\nu}(Q/K)$ is precisely the induced action on the first reduced space M_ν in symplectic reduction by stages.*

2. *For a chosen $\bar{\nu} \in \mathfrak{g}_\nu^*$, the momentum map $J_1 : T_{Q/K_\nu}(Q/K) \to \bar{\mathfrak{g}}_\nu^*$ associated with the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ corresponds to the induced momentum map $J_{\bar{G}_\nu}$ from symplectic reduction by stages.*

Proof. 1. The momentum map for the K action is precisely $J_K := i_{\bar{\mathfrak{K}}}^* \circ J_L$, with $J_L : TQ \to \mathfrak{g}^*$. By definition, the induced action of $\bar{G}_\nu$ on $J_K^{-1}(\nu)/K_\nu$ is obtained by projecting the action of G_ν on $J_K^{-1}(\nu)$. If we take into account that we realize the quotient manifold $J_K^{-1}(\nu)/K_\nu$ as $T_{Q/K_\nu}(Q/K)$, the induced action on $T_{Q/K_\nu}(Q/K)$ is obtained by projection of the action of G_ν on TQ under the projection $TQ \to T_{Q/K_\nu}(Q/K)$. This is precisely the action we have introduced above.

2. The *induced momentum map* $J_{\bar{G}_\nu}$ is defined in the following way (we consider it directly as a function on $T_{Q/K_\nu}(Q/K)$ instead of on M_ν):

$$\left\langle J_{\bar{G}_\nu}\big(v_{[q]_K}, [q]_{K_\nu}\big), \bar{\xi} \right\rangle = \langle J_L(v_q), k_\nu(\xi) \rangle - \langle \bar{\nu}, \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q \rangle - \langle \bar{\nu}, \xi \rangle, \tag{4}$$

where $\xi \in \mathfrak{g}_\nu$ is arbitrary and projects to $\bar{\xi} \in \bar{\mathfrak{g}}_\nu$, v_q projects to $(v_{[q]_K}, [q]_{K_\nu})$. By definition of the momentum map of the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$, we have

$$\left\langle J_1\big(v_{[q]_K}, [q]_{K_\nu}\big), \bar{\xi} \right\rangle = \left\langle \mathbb{F}L_1\big(v_{[q]_K}, [q]_{K_\nu}\big), \big((\bar{\xi})_{Q/K}, [p]_{K_\nu}\big) \right\rangle - (\delta_1)_{\bar{\xi}}([p]_{K_\nu}). \tag{5}$$

We now show that the right-hand side of (5) equals the right-hand side of (4). We therefore use the definition of $\mathbb{F}L_1$ and δ_1 as being the projection of maps upstairs:

$$\left\langle \mathbb{F}L_1\big(v_{[q]_K}, [q]_{K_\nu}\big), \big((\bar{\xi})_{Q/K}, [q]_{K_\nu}\big) \right\rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle - \langle \nu, \mathcal{A}^1(q)(\xi_Q(q)) \rangle,$$
$$(\delta_1)_{\bar{\xi}}([q]_{K_\nu}) = \delta_\xi(p) = -\langle \nu, \mathcal{A}^1(q)(\xi_Q(q)) \rangle + \langle \bar{\nu}, \xi \rangle. \qquad \blacksquare$$

Finally, before we can reapply Routh reduction for the second stage, we need to check that L_1 is $\bar{G}_\nu$-regular.

Lemma 7. *The magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ is $\bar{G}_\nu$-regular.*

Proof. We have to show that, for any $(v_{[q]_K}, [q]_{K_\nu})$ the map

$$\bar{\mathfrak{g}}_\nu \to \bar{\mathfrak{g}}_\nu^*; \qquad \bar{\xi} \mapsto J_1(v_{[q]_K} + \bar{\xi}_{Q/K}([q]_K), [q]_{K_\nu})$$

is invertible. Let v_q determine a tangent vector in $J_K^{-1}(\nu)$ representing $(v_{[q]_K}, [q]_{K_\nu})$. Let η denote an arbitrary element in $\bar{\mathfrak{g}}_\nu^*$. Due to the assumed G-hyperregularity, there is a unique $\xi \in \mathfrak{g}_\nu$ such that $k_\nu^*(J_L(v_q + \xi_Q(q))) = (r_\nu')^* \eta + \bar{\nu}$. The projection $\bar{\xi} = r_\nu'(\xi)$ of ξ defines the inverse element for η, since it is such that

$$k_\nu^*(J_L(v_q + \xi_Q(q))) - \bar{\nu} = (r_\nu')^* J_1(v_{[q]_K} + \bar{\xi}_{Q/K}) = (r_\nu')^* \eta. \qquad \blacksquare$$

Symplectic and Routh reduction by stages. Summarizing the above lemmas, we conclude that the magnetic Lagrangian system $(Q/K_\nu \to Q/K, L_1, \mathcal{B}_1)$ is amenable to Routh reduction and that the symplectic structure and momentum map associated to this Lagrangian system correspond to the symplectic structure and momentum map encountered in symplectic reduction by stages. If ρ is chosen such that the compatibility relation $(r_\nu')^* \rho = \mu|_{\mathfrak{g}_\nu} - \bar{\nu}$ holds, then from symplectic reduction by stages we have that the symplectic structures associated to $(Q/G_\mu \to Q/G, L_0, \mathcal{B}_0)$ and $((Q/K_\nu)/(\bar{G}_\nu)_\rho \to (Q/K)/\bar{G}_\nu, L_2, \mathcal{B}_2)$ are symplectically diffeomorphic by means of the symplectic diffeomorphism F introduced earlier. From Proposition 8 it follows that $F^* E_{L_2} = E_{L_0}$ and therefore the corresponding Hamiltonian vector fields are F-related. We define a map $\tau : Q/G_\mu \to (Q/K_\nu)/(\bar{G}_\nu)_\rho$ as $\tau([q]_{G_\mu}) = \big[[q]_{K_\nu}\big]_{(\bar{G}_\nu)_\rho}$. The map is well-defined since G_μ is a subgroup of G_ν and since $r_\nu(G_\mu) \subset (\bar{G}_\nu)_\rho$.

Lemma 8. *The symplectic diffeomorphism F is fibred over τ.*

$$T_{Q/G_\mu}(Q/G) \xrightarrow{\ F\ } T_{(Q/K_\nu)/(\bar{G}_\nu)_\rho}((Q/K)/\bar{G}_\nu)$$

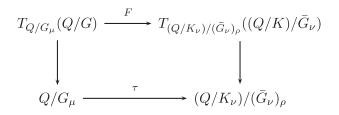$$Q/G_\mu \xrightarrow{\ \tau\ } (Q/K_\nu)/(\bar{G}_\nu)_\rho$$

Figure 5. Fibration of the symplectic diffeomorphism F.

Proof. We recall the definition of the map F: fix an element $(v_{[q]_G}, [q]_{G_\mu})$ and let $v_q \in J_L^{-1}(\mu)$ be a representative. The point $F(v_{[q]_G}, [q]_{G_\mu})$ is obtained by taking the consecutive quotients of v_q. In particular, the component of the final quotient in the configuration space $(Q/K_\nu)/(\bar{G}_\nu)$ of the magnetic Lagrangian system, is precisely the image τ. ∎

Since F is a diffeomorphism, τ is onto. The Hamiltonian vector field on $T_{Q/G_\mu}(Q/G)$ and $T_{(Q/K_\nu)/(\bar{G}_\nu)_\rho}((Q/K)/\bar{G}_\nu)$ are F-related. Their integral curves project onto solutions of the Euler–Lagrange equations. This concludes the proof of Theorem 7. ∎

7   Examples

7.1   Elroy's Beanie

This system appears in e.g. [13]. It consists of two planar rigid bodies that are connected in their center of mass. The system moves in the plane and it is subject to some conservative force with potential V. The configuration space is $SE(2) \times S^1$, with coordinates (x, y, θ, ψ). Here (x, y) is the position of the center of mass, θ is the rotation of the first rigid body, and ψ the relative rotation of the second body w.r.t. the first. The kinetic energy of the system is $SE(2)$-invariant and we will suppose that the potential is invariant as well. This means in fact that only the relative position of the two bodies matters for the dynamics of the system. The Lagrangian is of the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\psi})^2 - V(\psi).$$

The Euler–Lagrange equations of the system are, in normal form,

$$\ddot{x} = 0, \quad \ddot{y} = 0, \qquad \ddot{\theta} = \frac{1}{I_1}V', \qquad \ddot{\psi} = -\frac{I_1 + I_2}{I_1 I_2}V'.$$

The symmetry group and the principal connection. An element of $SE(2)$, the special Euclidean group, can be represented by a matrix of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

The identity of the group is $(x = 0,\ y = 0,\ \theta = 0)$ and the multiplication is given by

$$(x_1, y_1, \theta_1) * (x_2, y_2, \theta_2) = (x_2\cos\theta_1 - y_2\sin\theta_1 + x_1, x_2\sin\theta_1 + y_2\cos\theta_1 + y_1, \theta_1 + \theta_2).$$

The matrices

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

form a basis for the Lie algebra, for which $[e_1, e_2] = 0$, $[e_1, e_3] = e_2$ and $[e_2, e_3] = -e_1$. A corresponding basis of fundamental vector fields on Q is

$$\tilde{e}_1 = \frac{\partial}{\partial x}, \qquad \tilde{e}_2 = \frac{\partial}{\partial y}, \qquad \tilde{e}_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta},$$

and a basis of invariant vector fields is

$$\hat{e}_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \qquad \hat{e}_2 = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}, \qquad \hat{e}_3 = \frac{\partial}{\partial \theta}.$$

One can easily verify that the Lagrangian is invariant under the $SE(2)$-action. There is a trivial principal connection on $P = Q = SE(2) \times S^1 \to Q/SE(2) = S^1$, which locally takes the form

$$(dx + yd\theta)e_1 + (dy - xd\theta)e_2 + d\theta e_3.$$

The momentum map J_L is given by:

$$J_L = m\dot{x}e^1 + m\dot{y}e^2 + \big(m(x\dot{y} - y\dot{x}) + I_1\dot{\theta} + I_2(\dot{\theta} + \dot{\psi})\big)e^3.$$

In what follows we perform two Routh reductions on the Lagrangian. The first reduction is done w.r.t. the full symmetry group $SE(2)$, and the second reduction w.r.t. the Abelian normal subgroup $\mathbb{R}^2$.

Full reduction. Let $\mu = \mu_1 e^1 + \mu_2 e^2 + \mu_3 e^3$ be a generic element in $\mathfrak{g}^*$. An element $\xi = \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3$ of the isotropy algebra $\mathfrak{g}_\mu$ satisfies

$$\xi^3\mu_2 = 0, \qquad \xi^3\mu_1 = 0, \qquad \xi^1\mu_2 - \xi^2\mu_1 = 0.$$

So if we suppose that μ_1 and μ_2 do not both vanish – we will set $\mu_1 = 1$ from now on – then a typical element of $\mathfrak{g}_\mu$ is $\xi = \xi^1(e_1 + \mu_2 e_2)$. Since $\mathfrak{g}_\mu$ is 1-dimensional, G_μ is of course Abelian. A convenient way to describe the manifold $P/G_\mu = SE(2)/G_\mu$ locally is by considering a coordinate transformation (x', y', θ') in the group coordinates such that the vector field associated to an element in $\mathfrak{g}_\mu$ becomes a coordinate vector field: in the new coordinates, we should have $\partial_{x'} = \partial_x + \mu_2\partial_y$. This is obtained by the following transformation

$$x' = x, \qquad y' = y - \mu_2 x, \qquad \theta' = \theta.$$

Then clearly (y', θ) is a coordinate chart on $SE(2)/G_\mu$. And simultaneously, we have that (y', θ, ψ) is a coordinate chart on the reduced configuration manifold $P/G_\mu = (SE(2) \times S^1)/G_\mu$, and the fibration $P/G_\mu \to Q/G = S^1$ is locally represented by $(y', \theta, \psi) \mapsto (\psi)$. For a more systematic treatment on appropriate coordinate changes, we refer to [6].

We now compute the Routhian L_0 and the 2-form $\mathcal{B}_0$. Following [19], a convenient way to compute the (unreduced) Routhian for mechanical Lagrangians is by using $2(L_0 + V) = \big(-p_x\dot{x} - p_y\dot{y} - \dot{\theta}p_\theta + \dot{\psi}p_\psi\big)_{J_L^{-1}(\mu)}$, where p_i is the momentum in the ith coordinate. We have:

$$2(L_0 + V) = \Big(-m(\dot{x}^2 + \dot{y}^2) - I_1\dot{\theta}^2 - I_2(\dot{\theta} + \dot{\psi})\dot{\theta} + I_2(\dot{\theta} + \dot{\psi})\dot{\psi}\Big)_{J_L^{-1}(\mu)}$$

$$= \Big(-m(\dot{x}^2 + \dot{y}^2) - (I_1 + I_2)\dot{\theta}^2 + I_2\dot{\psi}^2\Big)_{J_L^{-1}(\mu)}$$

$$= -\frac{1}{m}(1 + \mu_2^2) - \frac{\big(\mu_3 - (x\mu_2 - y) - I_2\dot{\psi}\big)^2}{I_1 + I_2} + I_2\dot{\psi}^2$$

$$= \frac{I_1 I_2}{I_1 + I_2}\dot{\psi}^2 + 2I_2\frac{\mu_3 - (x\mu_2 - y)}{I_1 + I_2}\dot{\psi} - \frac{(\mu_3 - (x\mu_2 - y))^2}{I_1 + I_2}.$$

In the last step we have left out some constant terms. The reduced Lagrangian is then obtained by taking the quotient w.r.t. the action of G_μ. This is done by applying the coordinate transformation introduced above. We get:

$$L_0 = \frac{1}{2}\frac{I_1 I_2}{I_1 + I_2}\dot\psi^2 + I_2\frac{\mu_3 + y'}{I_1 + I_2}\dot\psi - \left(V(\psi) + \frac{1}{2}\frac{(\mu_3 + y')^2}{I_1 + I_2}\right),$$

which is clearly independent of x'. The 2-form $\mathcal{B}_0$ is obtained by reducing the 2-form

$$d\left((dx + yd\theta) + \mu_2(dy - xd\theta) + \mu_3 d\theta\right) = d(y - \mu_2 x) \wedge d\theta.$$

Using the coordinate change we get $\mathcal{B}_0 = dy' \wedge d\theta$.

In this example the Routhian L_0 depends on the velocity corresponding to the coordinate ψ on S^1, but is independent of the velocities corresponding to the two remaining coordinates (y', θ). With the above, the reduced Euler–Lagrange equations take the form

$$\dot y' = 0,$$
$$\dot\theta = \frac{1}{I_1 + I_2}(y' + \mu_3 - I_2\dot\psi),$$
$$\ddot\psi = -\frac{I_1 + I_2}{I_1 I_2}V' - \frac{1}{I_1}\dot y'.$$

Note that the second order equation in ψ decouples from the first order equations, and that these two first order equations are the momentum equations rewritten in normal form.

Abelian reduction. We now perform first Routh reduction w.r.t. the Abelian symmetry group $\mathbb{R}^2$ of translations in the x and y direction. Let us denote the symmetry group by $K = \mathbb{R}^2$ and study the quotient spaces. We will use the same notations as before: the Lie algebra elements e_1, e_2 denote a basis for the subalgebra $\mathfrak{K}$ of K in $\mathfrak{g}$. The momentum map for this action is now $J_K = i^* \circ J_L = m\dot x e^1 + m\dot y e^2$. We choose ν to be the projection of the momentum μ we had used in the full reduction: let $\nu = e^1 + \mu_2 e^2 \in \mathfrak{g}^*$. Since K is Abelian, $K_\nu = K$ and the quotient space is $SE(2) \times S^1/\mathbb{R}^2 = S^1 \times S^1$. If we choose $\mathcal{A}^1 = dxe_1 + dye_2$ to be the trivial connection, we simply get $\mathcal{B}_1 = 0$. The Routhian L_1 can now be obtained from

$$2(L_1 + V) = \left(-p_x\dot x - p_y\dot y + p_\theta\dot\theta + p_\psi\dot\psi\right)_{J_L^{-1}(\nu)}$$
$$= \left(-m\dot x^2 - m\dot y^2 + I_1\dot\theta^2 + I_2(\dot\theta + \dot\psi)^2\right)_{J_L^{-1}(\nu)} = I_1\dot\theta^2 + I_2(\dot\theta + \dot\psi)^2,$$

where we ignored again some constant terms. The Routh reduced system is now a standard Lagrangian system on $S^1 \times S^1$ with Lagrangian $L_1 = \frac{1}{2}I_1\dot\theta^2 + \frac{1}{2}I_2(\dot\theta + \dot\psi)^2 - V(\psi)$ (see the paragraphs on Abelian Routh reduction). Its equations of motion are

$$\ddot\theta = \frac{1}{I_1}V', \qquad \ddot\psi = -\frac{I_1 + I_2}{I_1 I_2}V'.$$

For this example there is actually no second stage: the group $\bar G_\nu = G_\nu/K_\nu$ is the trivial one $\{e\}$, and the vector space $\bar{\mathfrak{g}}_\nu = \mathfrak{g}_\nu/\mathfrak{K}_\nu$ is only the zero vector. So, there is no second momentum map to take into account, and there is no further symmetry to quotient out.

In the reduction by stages process we have not made use of μ_3. We now show that the two ways of reducing the system are equivalent.

Equivalence between direct reduction and reduction by stages. Let us compute the diffeomorphism F for this example. Here, it is a map $J_L^{-1}(\mu)/G_\mu \to J_K^{-1}(\nu)/K$ that is obtained by projection of the inclusion map $J_L^{-1}(\mu) \to J_K^{-1}(\nu)$. The latter equals, in coordinates

$$(x, y, \theta, \psi, \dot\psi) \mapsto \left(x, y, \theta, \psi, \dot\theta = \frac{1}{I_1 + I_2}(\mu_3 - (x\mu_2 - y) - I_2\dot\psi), \dot\psi\right).$$

The map is reducible, and after taking the quotient it becomes

$$F: \ J_L^{-1}(\mu)/G_\mu \to J_K^{-1}(\nu)/K; \qquad (y', \theta, \psi, \dot\psi) \mapsto \left(\theta, \psi, \dot\theta = \frac{1}{I_1 + I_2}(\mu_3 + y' - I_2\dot\psi), \dot\psi \right).$$

This diffeomorphism maps the G_μ-reduced system on the K_ν-reduced system, as is obvious from the respective equations of motion.

7.2 Rigid bodies on the Heisenberg group

As a second example of Routh reduction by stages, we discuss the dynamics of a rigid body immersed in a potential flow with circulation [20]. We assume that the body is circular, and in this case the equations of motion are given by

$$\frac{d}{dt} \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \Gamma \begin{bmatrix} -v_y \\ v_x \end{bmatrix}, \qquad \text{where} \qquad \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \mathbb{M} \begin{bmatrix} v_x \\ v_y \end{bmatrix}. \tag{7}$$

Here Γ represents the circulation and $\mathbb{M}$ is a (non-diagonal) mass matrix, which incorporates the inertia and added masses of the body. The right-hand side of the equations of motion represents the so-called *Kutta–Joukowski lift force*, a gyroscopic force due to circulation [9, 17].

While this system is extremely easy to integrate, it nevertheless exhibits all the interesting geometric characteristics of more complicated examples. As we show below, the configuration space for this system is the Heisenberg group, arguably the simplest non-trivial central extension group, and the procedure of reduction by stages demonstrated here can be applied equally well to more complicated central extensions, such as the oscillator group (describing the dynamics of rigid bodies of arbitrary cross section in circulatory flow) and the Bott–Virasoro group describing the KdV equation.

In this context, the Heisenberg group H is the Euclidian space $\mathbb{R}^3$, equipped with the multiplication

$$(x, y, s) \cdot (x', y', s') = \left(x + x', y + y', s + s' + \frac{1}{2}(xy' - yx') \right),$$

using the definition of H given in [18]. The coordinates (x, y) describe the center of the circular disc, while the coordinate s corresponds to the *flux* of the fluid around the body. The conjugate momentum corresponding to s will be the circulation of the fluid.

The Heisenberg group can alternatively be described as the central extension of $\mathbb{R}^2$ constructed by means of the cocycle $B((x, y), (x', y')) = \frac{1}{2}(xy' - yx')$. We now introduce the infinitesimal cocycle $C : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, given by

$$C((v_x, v_y), (v'_x, v'_y)) = v_x v'_y - v_y v'_x.$$

For the relation between B and C, see for instance [12]. The Lie algebra $\mathfrak{h}$ of H can then be identified with $\mathbb{R}^3$ with the bracket

$$[(v_x, v_y, v), (v'_x, v'_y, v')] = (0, 0, C((v_x, v_y), (v'_x, v'_y))) = (0, 0, v_x v'_y - v_y v'_x).$$

The dual Lie algebra $\mathfrak{h}^*$ can again be identified with $\mathbb{R}^3$, with coordinates (p_x, p_y, p) and duality pairing $\langle (p_x, p_y, p), (v_x, v_y, v) \rangle = p_x v_x + p_y v_y + pv$.

We now induce the following quadratic Lagrangian ℓ on $\mathfrak{h}$:

$$\ell(v_x, v_y, v) = \frac{1}{2} [v_x, v_y] \mathbb{M} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \frac{v^2}{2},$$

and we define L on H by left extension: $L(g, \dot{g}) = \ell(g^{-1}\dot{g})$, or explicitly

$$L(x, y, s; \dot{x}, \dot{y}, \dot{s}) = \frac{1}{2}[\dot{x}, \dot{y}]\,\mathbb{M}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} + \frac{1}{2}\left(\dot{s} - \frac{1}{2}(x\dot{y} - y\dot{x})\right)^2. \tag{8}$$

The Euler–Poincaré equations obtained from ℓ are given by

$$\frac{d}{dt}\begin{bmatrix}p_x \\ p_y \\ p\end{bmatrix} = p\begin{bmatrix}-v_y \\ v_x \\ 0\end{bmatrix}.$$

After setting $p = \Gamma$, we obtain the equations (7). These equations also coincide with the Euler–Lagrange equations obtained from the Lagrangian (8).

Reduction with respect to $\mathbb{R}$. The center of H is the normal subgroup isomorphic to $\mathbb{R}$ which consists of all elements of the form $(0, 0, s)$, where $s \in \mathbb{R}$. We first perform Routh reduction with respect to the left action of this subgroup on H. On the principal bundle $H \to H/\mathbb{R} \cong \mathbb{R}^2$ we consider the connection one-form given at the identity by $\mathcal{A}(e)(v_x, v_y, v) = v$, and extended to the whole of H by left translation. Explicitly, we have

$$\mathcal{A}(x, y, s) = ds - \frac{1}{2}(x\,dy - y\,dx).$$

Since the structure group $\mathbb{R}$ is Abelian, the curvature of $\mathcal{A}$ is given by $\mathcal{B} = d\mathcal{A} = -dx \wedge dy$.

Similarly, the momentum map $J_L : TH \to \mathbb{R}$ for the $\mathbb{R}$-action on the tangent bundle TH is given by $J_L(x, y, z; \dot{x}, \dot{y}, \dot{z}) = \dot{s} - (x\dot{y} - y\dot{x})/2$, so that J_L^{-1} consists of all points $(x, y, s; \dot{x}, \dot{y}, \dot{s})$ with $\dot{s} = \Gamma + (x\dot{y} - y\dot{x})/2$. The isotropy subgroup $\mathbb{R}_\Gamma = \mathbb{R}$ acts on this level set by translations in the s-direction, so that the reduced velocity space is given by

$$J^{-1}(\Gamma)/\mathbb{R}_\Gamma = T\mathbb{R}^2.$$

The symplectic form on the reduced space can easily be computed, and is explicitly given by

$$A\,d\dot{x} \wedge dx + B(d\dot{x} \wedge dy + d\dot{y} \wedge dx) + C\,d\dot{y} \wedge dy - \Gamma\,dx \wedge dy,$$

where A, B, C are the entries of the mass matrix $\mathbb{M}$. The last term of the symplectic form, $-\Gamma\,dx \wedge dy$, is the curvature term of the connection, paired with $\Gamma \in \mathbb{R}$. Finally, a quick computation shows that the reduced Lagrangian is just the kinetic energy Lagrangian on $T\mathbb{R}^2$:

$$L_1(x, y; \dot{x}, \dot{y}) = \frac{1}{2}[\dot{x}, \dot{y}]\,\mathbb{M}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix}, \tag{9}$$

up to constant terms.

Second reduction. We now perform reduction with respect to the remaining symmetry group, $H/\mathbb{R} \cong \mathbb{R}^2$, using the results from Section 5.3. We have a left invariant magnetic Lagrangian system on the group $\mathbb{R}^2$, with Lagrangian (9) and magnetic form $\mathcal{B}_\Gamma = -\Gamma\,dx \wedge dy$. The potential $\delta : \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the latter is given by

$$\delta(x, y) = \Gamma\begin{bmatrix}-y \\ x\end{bmatrix},$$

and the momentum map is therefore $J_2(x, y, \dot{x}, \dot{y}) = \mathbb{M}(\dot{x}, \dot{y})^T - \delta(x, y)$. The non-equivariance 2-cocycle of the momentum map is $\Sigma_\delta = \mathcal{B}_\Gamma$.

The affine action of $\mathbb{R}^2$ on itself is given by $(x, y) \cdot (p_x, p_y) = (p_x - \Gamma y, p_y + \Gamma x)$. If we fix a momentum value $(\lambda, \mu) \in \mathbb{R}^2$, the isotropy group $\mathbb{R}^2_{(\lambda,\mu)}$ of the affine action consists of just the zero element, and consequentially the twice-reduced space $J_2^{-1}(\lambda, \mu)/\mathbb{R}^2_{(\lambda,\mu)}$ is nothing but $\mathbb{R}^2$.

The reduced Euler–Lagrange equations (3) in the case of a left action, assume the following form

$$\frac{d}{dt}\begin{bmatrix} p_x \\ p_y \end{bmatrix} = -i_{(v_x,v_y)}\Sigma_\delta = \Gamma \begin{bmatrix} -v_y \\ v_x \end{bmatrix},$$

with $(p_x, p_y)^T = \mathbb{M}(v_x, v_y)^T$, and these are nothing but the equations (7).

Acknowledgements

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