On Darboux's Approach to R-Separability of Variables*

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Abstract. We discuss the problem of R-separability (separability of variables with a factor R) in the stationary Schrödinger equation on n-dimensional Riemann space. We follow the approach of Gaston Darboux who was the first to give the first general treatment of R-separability in PDE (Laplace equation on $\mathbb{E}^3$). According to Darboux R-separability amounts to two conditions: metric is isothermic (all its parametric surfaces are isothermic in the sense of both classical differential geometry and modern theory of solitons) and moreover when an isothermic metric is given their Lamé coefficients satisfy a single constraint which is either functional (when R is harmonic) or differential (in the opposite case). These two conditions are generalized to n-dimensional case. In particular we define n-dimensional isothermic metrics and distinguish an important subclass of isothermic metrics which we call binary metrics. The approach is illustrated by two standard examples and two less standard examples. In all cases the approach offers alternative and much simplified proofs or derivations. We formulate a systematic procedure to isolate R-separable metrics. This procedure is implemented in the case of 3-dimensional Laplace equation. Finally we discuss the class of Dupin-cyclidic metrics which are non-regularly R-separable in the Laplace equation on $\mathbb{E}^3$.

Key words: separation of variables; elliptic equations; diagonal n-dimensional metrics; isothermic surfaces; Dupin cyclides; Lamé equations

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1 Introduction

One of the highlights of Darboux's research on the whole is a memoir [11] devoted mainly to orthogonal coordinates in Euclidean spaces. The fundamental monograph [13] includes much of the material of [11]. The last fifty pages of the third and last part of the memoir [12] are nothing else but the first general treatment of the R-separability of variables (separability of variables with a factor R) in a PDE.

1.1 R-separability setting

Let

$$\Lambda\psi = 0 \tag{1.1}$$

be a linear PDE in variables $x = \left(x^1, x^2, \ldots, x^n\right)$ for an unknown (function) $\psi(x)$ and of order N.

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Definition 1. PDE (1.1) is R-separable (into ODEs) or x-variables are R-separable in equation (1.1) if there exist a non-zero function $R(x)$ and n linear ODEs

$$L_i\varphi_i = 0, \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

each of order $\nu_i \le N$ and for a function $\varphi_i(x^i)$ such that the following implication holds

$$\text{if} \quad L_i\varphi_i = 0, \quad i = 1, 2, \ldots, n, \qquad \text{then} \quad \psi(x) = R(x)\prod_i \varphi_i(x^i) \quad \text{solves (1.1)}. \tag{1.3}$$

Also we say that R-separation occurs in equation (1.1). Equations (1.2) are called separation equations.

Remark 1. Following Darboux we assume that coefficients of each equation (1.2) are just functions of variable x^i not necessarily dependent on extra variables (parameters). This freedom from the Stäckel imperative is essential. Hence if (1.3) holds then we have a family of solutions to (1.1) depending at least on $\sum_i \nu_i$ parameters.

Remark 2. If $R = 1$ or more generally if $R = \prod_i r_i(x^i)$, we replace the term "R-separability" by the term "separability".

1.2 R-separability in the Schrödinger equation

We assume that a Riemann space $\mathcal{R}^n$ admits local orthogonal coordinates $u = (u^1, \ldots, u^n)$ in which the metric has the following form

$$ds^2 = \sum_{i=1}^{n} H_i^2 \big(du^i\big)^2. \tag{1.4}$$

H.P. Robertson was the first to consider the stationary Schrödinger equation on $\mathcal{R}^n$ equipped with orthogonal coordinates

$$\Delta\psi + \big(k^2 - V\big)\psi = 0, \tag{1.5}$$

where

$$\Delta = h^{-1}\sum_{i=1}^{n} \frac{\partial}{\partial u^i}\frac{h}{H_i^2}\frac{\partial}{\partial u^i}, \qquad h = H_1 H_2 \cdots H_n$$

is the Laplace–Beltrami operator on $\mathcal{R}^n$, k is a scalar and $V = V(u)$ is a potential function [27].

We adapt the Definition 1 to the case of equation (1.5) as follows.

Definition 2. The Schrödinger equation is R-separable or metric (1.4) and potential V are R-separable in the Schrödinger equation if there exist $2n+1$ functions $R(u)$ and $p_i(u^i)$, $q_i(u^i)$ $(i = 1, 2, \ldots, n)$ such that the following implication holds

$$\varphi_i'' + p_i\varphi_i' + q_i\varphi_i = 0, \quad i = 1, 2, \ldots, n \quad \Rightarrow \quad \psi(u) = R(u)\prod_i \varphi_i\big(u^i\big) \quad \text{solves} \quad (1.5). \tag{1.6}$$

Particular cases of equation (1.5) are

a) n-dimensional Laplace equation ($k = 0$ and $V = 0$)

$$\Delta\psi = 0,$$

b) n-dimensional Helmholtz equation ($V = 0$)

$$\Delta\psi + k^2\psi = 0, \tag{1.7}$$

c) n-dimensional Schrödinger equation with $k = 0$

$$\Delta\psi = V\psi. \tag{1.8}$$

In the context of the R-separability in the Schrödinger equation the following problem seems to be fundamental.

***R*-separability problem.** Let $\mathcal{R}^n$ be a Riemann space with a metric $ds^2 = g_{ij}dx^i dx^j$, where (x^i) are local coordinates. We assume that $\mathcal{R}^n$ admits orthogonal coordinates and we are given a class of R-separable metrics (1.4). By R-separability problem, we mean the problem of isolating those metrics of the class which are equivalent to the metric ds^2.

Remark 3. As is well known a generic $\mathcal{R}^n$ for $n > 3$ does not admit orthogonal coordinates [1, p. 470]. Any analytic $\mathcal{R}^3$ always admits orthogonal coordinates [5] and even more any $\mathcal{R}^3$ of C^∞-class also admits orthogonal coordinates [15]. Also the problem when a given metric is diagonalizable seems to be very difficult [30].

Robertson proved in [27] that any n-dimensional Stäckel metrics satisfying the so called Robertson condition is separable in the Schrödinger equation (1.5) (see also (3.1) of this paper). The corresponding R-separability problem for n-dimensional Euclidean space has been solved by L.P. Eisenhart [16].

1.3 Darboux's R-separability problem

Gaston Darboux was interested in R-separability of variables in the Laplace equation on $\mathbb{E}^3$. His pioneering research in the field of R-separability [9, 10, 12, 13] has been almost completely forgotten. It can be interpreted as an advanced attempt to solve the following specific R-separability problem.

Here we do not use the original Darboux's notation dating back to Lamé writings. Instead, we apply the notation used in this paper.

Theorem 1. *The* 3*-dimensional diagonal metric*

$$ds^2 = H_1^2(u)\big(du^1\big)^2 + H_2^2(u)\big(du^2\big)^2 + H_3^2(u)\big(du^3\big)^2 \tag{1.9}$$

is R-separable in 3*-dimensional Laplace equation*

$$\left(\sum_{i=1}^{3} \partial_i \frac{H_1H_2H_3}{H_i^2}\partial_i\right)\psi = 0 \tag{1.10}$$

if and only if the following two conditions are satisfied

$$i)\quad H_1 = \frac{G_{(2)}G_{(3)}}{R^2 f_1}, \qquad H_2 = \frac{G_{(1)}G_{(3)}}{R^2 f_2}, \qquad H_3 = \frac{G_{(1)}G_{(2)}}{R^2 f_3}, \tag{1.11}$$

where $G_{(i)}$ does not depend on u^i and f_i depends only on u^i,

$$ii)\quad \sum_{i=1}^{3} G_{(i)}^2 f_i^2\left[\partial_i^2 R^{-1} + \frac{f_i'}{f_i}\partial_i R^{-1} + q_i R^{-1}\right] = 0 \tag{1.12}$$

for appropriately chosen functions $q_i(u^i)$.

Moreover, the resulting separation equations are

$$\varphi_i'' + \frac{f_i'}{f_i}\varphi_i' + q_i\varphi_i = 0, \qquad i = 1, 2, 3.$$

Remark 4. Theorem 1 has never been explicitly stated by Darboux. Actually he applied this theorem in many places of his research. E.g. (3) in Chapter IV of [13] is a special case of (1.11) while (69) in Chapter V of [13] is a special case of (1.12).

In view of the Theorem 1 the question of R-separability of variables in the Laplace equation on $\mathbb{E}^3$ amounts to the following R-separability problem: to isolate all the metrics with Lamé coefficients (1.11) which are flat and which satisfy (1.12). In other words, firstly, one has to find (classify) all solutions to the Lamé equations $(i, j, k = (1, 2, 3), (2, 3, 1), (3, 1, 2))$

$$H_{i,jk} - \frac{1}{H_j} H_{i,j} H_{j,k} - \frac{1}{H_k} H_{i,k} H_{k,j} = 0, \tag{1.13}$$

$$\left(\frac{1}{H_i} H_{j,i}\right)_{,i} + \left(\frac{1}{H_j} H_{i,j}\right)_{,j} + \frac{1}{H_k^2} H_{i,k} H_{j,k} = 0, \tag{1.14}$$

under the ansatz (1.11) and, secondly, to select among them those satisfying the constraint (1.12).

Darboux was successful in solving the Lamé equations under the ansatz (1.11). However as a rule he paid no closer attention to the question of separation equations and thus with one exception the constraint (1.12) was not the subject of his detailed analysis. This exceptional case not covered by the modern treatments of R-separability in the Laplace equation on $\mathbb{E}^3$ is one of the Dupin-cyclidic metrics [13, pp. 283–286]. Indeed, Dupin-cyclidic metrics are non-regularly R-separable in the Laplace equation on $\mathbb{E}^3$ and cannot be treated by the standard techniques discussed e.g. in [3]. For a discussion of regular and non-regular R-separability see [23].

Definition 3. A surface in $\mathbb{E}^3$ is isothermic if, away from umbilics, its curvature net can be conformally parametrized.

Another important Darboux's result is as follows.

Theorem 2. *If the metric (1.9) is R-separable in the Laplace equation on $\mathbb{E}^3$, then all the corresponding parametric surfaces are isothermic.*

The class of isothermic surfaces is conformally invariant and in particular includes

- planes and spheres,
- surfaces of revolution,
- quadrics,
- tori, cones, cylinders and their conformal images, i.e. Dupin cyclides,
- cyclides or better Darboux–Moutard cyclides,
- constant mean curvature surfaces and in particular minimal surfaces.

Apart from the fact that the Theorem 2 is a necessary condition for R-separability in the Laplace equation on $\mathbb{E}^3$, it is an interesting connection between the linear mathematical physics (separation of variables) and the non-linear mathematical physics (solitons). Indeed, the current interest in isothermic surfaces is mainly due to the fact that their geometry is an important example of the so called integrable or soliton geometry [8, 28, 4, 19].

Definition 4. The metric (1.9) with Lamé coefficients (1.11) is called isothermic.

1.4 Aims and results of the paper

In this paper we extend the original Darboux's approach to R-separability of variables in the Laplace equation on $\mathbb{E}^3$ to the case of the stationary Schrödinger equation on n-dimensional Riemann space $\mathcal{R}^n$ admitting orthogonal coordinates.

The Darboux's Theorem 1 is generalized as Theorem 3. Correspondingly 3-dimensional isothermic metrics (1.11) are generalized to n-dimensional isothermic metrics (2.3) while 3-dimensional constraint (1.12) is generalized to n-dimensional constraint (2.11) which we call R-equation.

We distinguish a subclass (2.9) of isothermic metrics which we call the binary metrics. A representative example of the binary metric is n-elliptic metric (2.10). In the case of a binary metric the R-equation assumes the simpler form (2.12).

The approach is illustrated by examples of the Section 3. Here we discuss two standard results and two less standard results. These are 1) Robertson paper revisited (Subsection 3.1), 2) the n-elliptic metric (Subsection 3.2) (standard results) and 3) remarkable example of Kalnins–Miller revisited (Subsection 3.3), 4) fixed energy R-separation revisited (Subsection 3.4) (less standard examples). In all cases the approach offers alternative and simplified proofs or derivations.

The main result of the paper encoded in Theorem 3 suggests the following procedure to identify a given n-dimensional diagonal metric (1.4) as R-separable in n-dimensional Schrödinger equation. The procedure in question consists of three steps.

Firstly, we have to prove or disprove that the metric is isothermic. If the metric is not isothermic, then it is not R-separable. Suppose it is isothermic. As a result this step predicts R-factor and p_i coefficients in the separation equations. Secondly, we set up the corresponding R-equation which we treat as an equation for q_i coefficients in the separation equations. Thirdly, we attempt to solve the R-equation. Any solution to the R-equation concludes the procedure: R-separability of the starting metric is proved and in particular the corresponding separation equations are explicitly constructed. Notice that unknowns q_i enter into R-equation linearly and this is the right place to introduce (linearly) extra parameters (separation constants) into the separation equations. In Subsection 2.3 we introduce remarkable algebraic identities (Bôcher–Ushveridze identities) which can be successfully applied in solving R-equation. This is a remarkably simple procedure and its implementation in the case of 3-dimensional Laplace equation is discussed in Subsection 4.1 together with the relevant examples.

Gaston Darboux found a class of Dupin-cyclidic metrics which are R-separable in the Laplace equation on $\mathbb{E}^3$. These are non-regularly R-separable and can not be covered by the modern standard approaches. In Subsection 4.3 we re-derive this remarkable result. The original Darboux's calculations are long and rather difficult to control. Here we simplify the derivation using the standard Riemannian tools (Ricci tensor and Cotton–York criterion of conformal flatness).

2 Isothermic metrics and R-equation

2.1 The main result

Here we extend Darboux's Theorem 1 valid for 3-dimensional Laplace equation (1.10) to the case of n-dimensional stationary Schrödinger equation (1.5). Correspondingly, we extend the Definition 4 of isothermic metrics to n-dimensional case.

Theorem 3. A. *The metric (1.4) and the potential V are R-separable in the Schrödinger equation (1.5) if and only if the following two conditions are satisfied*

- *first condition of R-separability*

$$\left[\ln\left(R^2\frac{h}{H_i^2}\right)\right]_{,ij} = 0, \qquad i \neq j, \tag{2.1}$$

- *second condition of R-separability (called R-equation)*

$$\Delta R + \left(k^2 - V - \sum_{i=1}^{n} \frac{1}{H_i^2} q_i\right) R = 0. \tag{2.2}$$

B. *The metric (1.4) satisfies the first condition of R-separability if and only if it can be cast into the form*

$$ds^2 = R^{4/(2-n)} G_{(1)}^{2/(n-2)} G_{(2)}^{2/(n-2)} \cdots G_{(n)}^{2/(n-2)} \sum_{i=1}^{n} G_{(i)}^{-2} \frac{1}{f_i^2} (du^i)^2, \tag{2.3}$$

where $G_{(i)}$ does not depend on u^i while f_i depends only on u^i.

C. *If conditions (2.3) and (2.2) are satisfied then the corresponding separation equations read*

$$\varphi_i'' + \frac{f_i'}{f_i} \varphi_i' + q_i \varphi_i = 0. \tag{2.4}$$

Proof. A. R-separability implies (2.1) and (2.2).

Indeed, we insert $\psi = R \prod_i \varphi_i$ into (1.5) and make use of (1.6). This results in

$$\sum_i \frac{1}{H_i^2} \left[\left(\ln R^2 \frac{h}{H_i^2} \right)_{,i} - p_i \right] \frac{\varphi_i'}{\varphi_i} + R^{-1} \Delta R - \sum_i \frac{1}{H_i^2} q_i + k^2 - V = 0 \tag{2.5}$$

for an arbitrary choice of solutions φ_i. Let $(\varphi_{i1}, \varphi_{i2})$ be a basis in the solution space of the corresponding equation. We put

$$\varphi_i = \lambda_i \varphi_{i1} + \mu_i \varphi_{i2}, \qquad \lambda_i, \mu_i = \text{const}.$$

Thus for each φ_i $(\lambda_i \neq 0)$ we have

$$\frac{\varphi_i'}{\varphi_i} = \frac{\varphi_{i1}' + \alpha_i \varphi_{i2}'}{\varphi_{i1} + \alpha_i \varphi_{i2}}, \tag{2.6}$$

where $\alpha_i = \mu_i / \lambda_i = \text{const}$. Since (2.5) with $\frac{\varphi_i'}{\varphi_i}$ replaced by r.h.s. of (2.6) is valid for arbitrary α_i we have

$$\left(\ln R^2 \frac{h}{H_i^2} \right)_{,i} = p_i, \qquad i = 1, 2, \dots, n \tag{2.7}$$

and from (2.7) both (2.1) and (2.2) follow.

Conditions (2.1) and (2.2) imply R-separability.

Indeed, we form the equations of (1.6) with $p_i = \left(\ln R^2 \frac{h}{H_i^2} \right)_{,i}$ and q_i given by (2.2). Then the implication (1.6) in Definition 2 is satisfied.

B. Indeed, the metric (1.4) satisfies (2.1) if and only if there exist $2n$ functions $f_i(u^i)$ and $F_{(i)}(u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^n)$ such that

$$\frac{h}{H_i^2} = \frac{1}{R^2} f_i F_{(i)}. \tag{2.8}$$

Certainly, without loss of generality we can replace $F_{(i)}$ by $\big(\prod_{k \neq i} f_k \big)^{-1} G_{(i)}^2$, where $G_{(i)} = G_{(i)}(u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^n)$. Now (2.8) implies (2.3) and vice-versa. ∎

Remark 5. Willard Miller Jr. derived (2.1) in [23]. See (3.23) of [23] and notice that his R is our $\ln R$.

Notice that (2.3) for $n = 3$ gives the isothermic metric of Definition 4.

Definition 5. The metric (2.3) is called isothermic.

We introduce now an important sub-class of isothermic metrics. Given $\binom{n}{2}$ functions $G_{ij} = G_{ij}(u^i, u^j)$ $(i < j)$. We select $G_{(i)}$ as follows

$$G_{(i)} = \prod_{p \neq i \neq q} G_{pq}.$$

Then (2.3) assumes the form

$$ds^2 = R^{4/(2-n)} \sum_{i=1}^{n} \frac{\prod\limits_{i<q} G_{iq}^2 \prod\limits_{p<i} G_{pi}^2}{f_i^2} \left(du^i\right)^2. \tag{2.9}$$

Definition 6. The metric (2.9) is called binary.

Example. The n-elliptic coordinates on $\mathbb{E}^n$ [20, 22, 17, 33]. We choose n real numbers b_i such that $b_1 > b_2 > \cdots > b_n$. The n-elliptic coordinates $\lambda = \left(\lambda^1, \lambda^2, \ldots, \lambda^n\right)$ satisfy inequalities

$$\lambda^1 > b_1 > \lambda^2 > \cdots > b_{n-1} > \lambda^n > b_n.$$

The following formulae give rise to a diffeomorphism onto any of 2^n open n-hyper-octants of $\mathbb{E}^n$ equipped with the standard Cartesian coordinates $x = (x^1, x^2, \ldots, x^n)$

$$\left(x^i\right)^2 = \frac{\prod\limits_{j=1}^{n} \left(\lambda^j - b_i\right)}{\prod\limits_{j \neq i} \left(b_j - b_i\right)}, \qquad i = 1, 2, \ldots, n.$$

The corresponding n-elliptic metric is

$$ds^2 = \sum_{i=1}^{n} \frac{\prod\limits_{j \neq i} (\lambda^i - \lambda^j)}{4 \prod\limits_{k=1}^{n} (\lambda^i - b_k)} (d\lambda^i)^2. \tag{2.10}$$

The n-elliptic metric is binary ($R = 1$, $G_{ij} = \sqrt{\lambda^i - \lambda^j}$ and $f_i^2 = 4(-1)^{i-1} \prod\limits_{k=1}^{n} (\lambda^i - b_k)$) and thus isothermic.

2.2 R-equation

Having found the general formulae (2.3) and (2.9) for isothermic metrics which – ex definitione – satisfy the 1st condition of R-separability, we are in a position to claim that the various questions of R-separability amount to the 2nd condition of R-separability (2.2) which we call R-equation.

Remark 6. (2.2) is not the Schrödinger equation since it is either a functional equation (when R is harmonic) or Δ involves R.

Theorem 4. **A.** *The metric (2.3) is R-separable in the Schrödinger equation if and only if*

$$\sum_{i=1}^{n} G_{(i)}^2 f_i^2 \left[\left(\frac{1}{R}\right)_{,ii} + \frac{f_i'}{f_i}\left(\frac{1}{R}\right)_{,i} + q_i \frac{1}{R} \right] = R^{(n+2)/(2-n)} G_{(1)}^{2/(n-2)} \cdots G_{(n)}^{2/(n-2)} \left(k^2 - V\right). \tag{2.11}$$

B. *The binary metric (2.9) is R-separable in the Schrödinger equation if and only if*

$$\sum_{i=1}^{n} \frac{f_i^2}{\prod_{i<q} G_{iq}^2 \prod_{p<i} G_{pi}^2} \left[\left(\frac{1}{R}\right)_{,ii} + \frac{f_i'}{f_i}\left(\frac{1}{R}\right)_{,i} + q_i \frac{1}{R} \right] = R^{(n+2)/(2-n)} \left(k^2 - V\right). \tag{2.12}$$

Proof. Indeed, both (2.11) and (2.12) are R-equations rewritten in terms of the corresponding metric. ■

Remark 7. Notice that the linear operators acting on R^{-1} in (2.11) and (2.12) also define the separation equations (2.4).

2.3 Bôcher–Ushveridze identities

Gaston Darboux was the first to discuss the so called triply conjugate coordinates in $\mathbb{E}^3$ [12]. These constitute a projective generalization of orthogonal coordinates in $\mathbb{E}^3$. In this context he introduced the following system of three equations for a single unknown $M(x_1, x_2, x_3)$

$$\begin{aligned} &(x_1 - x_2)M_{,12} - M_{,1} + M_{,2} = 0, \\ &(x_1 - x_3)M_{,13} - M_{,1} + M_{,3} = 0, \\ &(x_2 - x_3)M_{,23} - M_{,2} + M_{,3} = 0. \end{aligned} \tag{2.13}$$

and gave a general solution to it in the form

$$M = \frac{m_1(x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{m_2(x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{m_3(x_3)}{(x_3 - x_1)(x_3 - x_2)}, \tag{2.14}$$

where $m_i(x_i)$ are arbitrary functions (see formulae (40), (41) and (42) in [12]). A generalization of (2.13) and (2.14) is straightforward.

Consider in $\mathbb{R}^n$ the following system of $\binom{n}{2}$ PDEs for a single unknown $M(x_1, x_2, \ldots, x_n)$

$$(x_i - x_j)M_{,ij} - M_{,i} + M_{,j} = 0, \qquad i < j. \tag{2.15}$$

This is the overdetermined system of PDEs which is an example of the so called linear Darboux–Manakov–Zakharov system [34]. Fortunately (2.15) is involutive (see Proposition 1 in [34]). Its general solution reads

$$M = \sum_{i=1}^{n} \frac{m_i(x_i)}{\prod\limits_{j \neq i} (x_i - x_j)},$$

where $m_i(x_i)$ are arbitrary functions.

On the other hand each single equation of the system (2.15) is a particular case of the Euler–Poisson–Darboux equation [14, p. 54] provided we ignore variables not explicitly involved in the equation. For simplicity (2.15) will be called the Euler–Poisson–Darboux system.

Remark 8. Interestingly, in modern times the Euler–Poisson–Darboux system and its various modifications have been studied in the context of the so called integrable hydrodynamic type systems [31, 32, 25].

Notice particular solutions to (2.15): $M = 0$, $M = 1$ and $M = \sum_{i=1}^{n} x_i$. The obvious question arises as to what functions $m_i(x_i)$ correspond to them.

Maxime Bôcher in his monograph on R-separability in the Laplace equation on $\mathbb{E}^n$ published without proof a series of remarkable algebraic identities [2, p. 250]. These can be written in a compact form as follows

$$\sum_{i=1}^{n} \frac{x_i^{p-1}}{\prod_{j\neq i}(x_i - x_j)} = \delta_{pn}, \qquad p = 1, 2, \dots, n. \tag{2.16}$$

A.G. Ushveridze generalized the identities (2.16) [33]. We put $m = 0, 1, 2, \dots$, $n = 2, 3, \dots$, $d = m + 1 - n$ and

$$f_d^{(n)}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \frac{x_i^m}{\prod_{j\neq i}(x_i - x_j)}. \tag{2.17}$$

Then

$$f_d^{(n)} = \begin{cases} 0 & \text{for} \quad 0 \le m < n-1, \\ 1 & \text{for} \quad m = n-1, \\ \text{homogeneous polynomial} & \\ \text{of degree and homogeneity} = d & \text{for} \quad m \ge n. \end{cases} \tag{2.18}$$

I.e. for $m \ge n$

$$f_d^{(n)} = \sum_{1l_1 + 2l_2 + \dots + dl_d = d} f_{l_1 l_2 \dots l_d}\, \sigma_1^{l_1} \sigma_2^{l_2} \cdots \sigma_d^{l_d},$$

where σ_i are elementary symmetric polynomials: $\sigma_1 = \sum_{i=1}^{n} x_i$, $\sigma_2 = \sum_{i<j} x_i x_j$, $\dots$ and $f_{l_1 l_2 \dots l_d}$ are constants defined uniquely by r.h.s. of (2.17). In particular

$$f_1^{(n)} = \sigma_1 = \sum_{i=1}^{n} x_i, \qquad f_2^{(n)} = \sigma_1^2 - \sigma_2, \qquad f_3^{(n)} = \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3. \tag{2.19}$$

The identities (2.18) and in particular the identities (2.16) we call the Bôcher–Ushveridze identities. Certainly, both sides of any Bôcher–Ushveridze identity is a particular solution to the Euler–Poisson–Darboux system. Notice also that functions $m_i(x_i)$ are not defined by M uniquely. As we shall see both the Euler–Poisson–Darboux system and the Bôcher–Ushveridze identities can be applied in discussing R-equation.

3 Examples

In this section we discuss two standard results and two less standard results within the developed approach. In all cases the approach offers alternative and much simplified proofs or derivations.

3.1 Robertson paper revisited

Here we present the essence of Howard Percy Robertson fundamental paper [27]. Our aim is to re-derive the basic formulae (A), (B), (C) and (9) of the paper using earlier stated results.

In (1) of [27] we put $k = 1$ and replace E by k^2. Notice that e.g. Robertson's h_i is our H_i^{-2}. The paper deals with the case $R = 1$. R-equation (2.2) is now the functional constraint which is bilinear in H_i^{-2} and q_i

$$\sum_{i=1}^{n} \frac{1}{H_i^2} q_i = k^2 - V. \tag{3.1}$$

We decompose q_i as follows

$$q_i\big(u^i\big) = k^2 q_{i1}\big(u^i\big) - v_i(u^i) + Q_i\big(u^i\big), \tag{3.2}$$

where v_i are arbitrary. Inserting (3.2) into (3.1) gives

$$\sum_{i=1}^{n} \frac{1}{H_i^2} q_{i1} = 1, \tag{3.3}$$

$$\sum_{i=1}^{n} \frac{1}{H_i^2} Q_i = 0, \tag{3.4}$$

$$\sum_{i=1}^{n} \frac{1}{H_i^2} v_i = V. \tag{3.5}$$

Formally (3.4) means that vector (Q_i) belongs to $(n-1)$-dimensional orthogonal complement of the vector (H_i^{-2}). Select a basis (q_{ij}) $(j = 2, 3, \ldots, n)$ of the orthogonal complement

$$\sum_{i=1}^{n} \frac{1}{H_i^2} q_{ij}\big(u^i\big) = 0, \qquad j = 2, 3, \ldots, n \tag{3.6}$$

and decompose (Q_i) in this basis as follows

$$Q_i\big(u^i\big) = \sum_{j=2}^{n} k_j q_{ij}\big(u^i\big), \tag{3.7}$$

where the coefficients of the decomposition are arbitrary constants.

Remark 9. (3.3) and (3.6) introduce (non-uniquely!) an $n \times n$ matrix $q = [q_{ij}(u^i)]$. We assume that q is non-singular everywhere. It is called the Stäckel matrix. Notice that the co-factor Q_{ij} of q_{ij} does not depend on u^i.

We collect (3.3) and (3.6) as

$$\sum_{i=1}^{n} \frac{1}{H_i^2} q_{ij} = \delta_{1j}. \tag{3.8}$$

Inverting of (3.8) yields

$$\frac{1}{H_i^2} = \big(q^{-1}\big)_{1i} = \frac{Q_{i1}}{\det q}. \tag{3.9}$$

It is clear that the metric

$$ds^2 = \det q \sum_{i=1}^{n} \frac{(du^i)^2}{Q_{i1}} \tag{3.10}$$

satisfies (3.1) or the 2-nd condition of R-separability (2.2). Finally we demand the metric (3.10) has to satisfy the first condition of R-separability (2.1)

$$\left(\ln \frac{h}{H_i^2}\right)_{,ij} = \left(\ln \frac{h}{\det g} Q_{i1}\right)_{,ij} = \left(\ln \frac{h}{\det q}\right)_{,ij} = 0, \qquad i \neq j,$$

which implies

$$\frac{h}{\det q} = \prod_{i=1}^{n} f_i(u^i) \tag{3.11}$$

and thus (2.8) is

$$\frac{h}{H_i^2} = f_i(u^i) Q_{i1} \prod_{j \neq i} f_j(u^j), \tag{3.12}$$

which means that (3.12) exactly conforms to (2.8). As a result of (3.12), (3.2) and (3.7) the separation equations are

$$\varphi_i'' + \frac{f_i'}{f_i}\varphi_i' + \left[k^2 q_{i1} + \sum_{j=2}^{n} k_j q_{ij} - v_i\right]\varphi_i = 0. \tag{3.13}$$

To conclude we arrive at the following identifications: (A), (B), (C) and (9) of [27] are now (3.9), (3.5), (3.11) and (3.13) respectively.

Definition 7. (3.10) is called the Stäckel metric and (3.11) is called the Robertson condition.

3.2 The n-elliptic metric

It is well known that n-elliptic metric (2.10) is separable ($R = 1$) in the Schrödinger equation with an appropriately chosen potential function. An indirect proof consists in showing that (2.10) is the Stäckel metric (in this case the Robertson condition is satisfied) and Eisenhart stated it without proof in [16, p. 302].

Theorem 5. *The n-elliptic metric* (2.10) *is separable in the Schrödinger equation with a potential function*

$$V(\lambda) = \sum_{i=1}^{n} \frac{v_i(\lambda^i)}{\prod_{j\neq i}(\lambda^i - \lambda^j)},$$

where $v_i(\lambda^i)$ are arbitrary functions, i.e. $V(\lambda)$ is an arbitrary solution to the Euler–Poisson–Darboux system (2.15). *The corresponding separation equations are*

$$\varphi_i'' + \frac{1}{2}\frac{a_i'}{a_i}\varphi_i' + \frac{1}{a_i}\left[\sum_{m=0}^{n-2} k_m(\lambda^i)^m + k^2(\lambda^i)^{n-1} - v_i(\lambda^i)\right]\varphi_i = 0, \qquad i = 1, 2, \dots, n,$$

where $a_i = 4\prod_{k=1}^{n}(\lambda^i - b_k)$ and k_m are arbitrary constants $(m = 0, 1, \dots, n-2)$.

Proof. Indeed, from example (Subsection 2.1) we know that the metric (2.10) is isothermic. Again R-equation is reducible to the functional constraint

$$\sum_{i=1}^{n} \frac{a_i(\lambda^i) q_i(\lambda^i)}{\prod_{j\neq i} (\lambda^i - \lambda^j)} = k^2 - V(\lambda).$$

We put

$$a_i(\lambda^i) q_i(\lambda^i) = \sum_{m=0}^{n-2} k_m (\lambda^i)^m + k^2 (\lambda^i)^{n-1} - v_i(\lambda^i), \qquad i = 1, 2, \ldots, n,$$

where $k_m = \mathrm{const}$ and $v_i(\lambda^i)$ are arbitrary functions. Now the Bôcher–Ushveridze identity (2.16) implies the statement. ■

3.3 Remarkable example of Kalnins–Miller revisited

Our setting can be easily extended to the pseudo-Riemannian case. Consider the following metric

$$\begin{aligned} d\sigma^2 = {} & \left(\lambda^1 - \lambda^2\right)\left(\lambda^1 - \lambda^3\right)(d\lambda^1)^2 + \left(\lambda^2 - \lambda^1\right)\left(\lambda^2 - \lambda^3\right)(d\lambda^2)^2 \\ & + \left(\lambda^3 - \lambda^1\right)\left(\lambda^3 - \lambda^2\right)(d\lambda^3)^2, \end{aligned} \tag{3.14}$$

where $\lambda^1 > \lambda^2 > \lambda^3 > 0$. It is 3-dimensional Minkowski metric. Indeed, on replacing λ^i by t, x and y

$$\begin{aligned} t &= \frac{1}{9}\left(\lambda^1 + \lambda^2 + \lambda^3\right) - \frac{9}{16}\left(\lambda^1 + \lambda^2 - \lambda^3\right)\left(\lambda^1 - \lambda^2 + \lambda^3\right)\left(\lambda^1 - \lambda^2 - \lambda^3\right), \\ x &= \frac{1}{9}\left(\lambda^1 + \lambda^2 + \lambda^3\right) + \frac{9}{16}\left(\lambda^1 + \lambda^2 - \lambda^3\right)\left(\lambda^1 - \lambda^2 + \lambda^3\right)\left(\lambda^1 - \lambda^2 - \lambda^3\right), \\ y &= \frac{1}{4}\left(\lambda^1 + \lambda^2 - \lambda^3\right)^2 - \lambda^1\lambda^2, \end{aligned}$$

we arrive at

$$d\sigma^2 = -dt^2 + dx^2 + dy^2.$$

Certainly, any metric conformally equivalent to (3.14) is an isothermic metric and thus satisfies the 1st condition of R-separability (2.1). Kalnins and Miller proved that the metric

$$ds^2 = \left(\lambda^1 + \lambda^2 + \lambda^3\right) d\sigma^2 \tag{3.15}$$

is R-separable in the Helmholtz equation (1.7) [21, p. 472]. We re-derive this remarkable result within our approach.

First of all it is easy to predict R-factor (see (2.9)) and the form of the separation equations ($f_1^2 = f_3^2 = 1$, $f_2^2 = -1$)

$$\begin{aligned} & R(\lambda) = \left(\lambda^1 + \lambda^2 + \lambda^3\right)^{-1/4}, \\ & \varphi_i'' + q_i \varphi_i = 0, \qquad i = 1, 2, 3. \end{aligned} \tag{3.16}$$

The point is that (3.16) is harmonic with respect of (3.15). Again R-equation is reducible to the functional constraint

$$\left(\lambda^1 + \lambda^2 + \lambda^3\right) k^2 = \frac{q_1(\lambda^1)}{(\lambda^1 - \lambda^2)(\lambda^1 - \lambda^3)} + \frac{q_2(\lambda^2)}{(\lambda^2 - \lambda^1)(\lambda^2 - \lambda^3)} + \frac{q_3(\lambda^3)}{(\lambda^3 - \lambda^1)(\lambda^3 - \lambda^2)}.$$

Then from the Bôcher–Ushveridze identities (2.16) and (2.19) we have immediately

$$q_i(\lambda^i) = k^2 (\lambda^i)^3 + k_1 (\lambda^i) + k_0,$$

where k_0, k_1 are arbitrary constants.

3.4 Fixed energy R-separation revisited

In order to treat R-separability in the Schrödinger equation (1.8) a pretty complicated formalism was proposed in [7]. Presumably some part of the formalism of [7] can be simplified according to the following result.

Proposition 1. *Any isothermic metric which is R-separable in n-dimensional Laplace equation is R-separable in n-dimensional Schrödinger equation with $k = 0$ for an appropriately chosen potential function.*

Proof. Consider the isothermic metric given by (2.3). R-separability of (2.3) in n-dimensional Laplace equation implies that R-equation simplifies to

$$R^{-1}\Delta R - \sum_{i=1}^{n} \frac{1}{H_i^2} q_i = 0. \tag{3.17}$$

We put

$$V(u) = \sum_{i=1}^{n} \frac{1}{H_i^2} v_i(u^i), \tag{3.18}$$

where $v_i(u^i)$ are arbitrary functions and define $\bar{q}_i = q_i - v_i$. Then (3.17) can be rewritten as

$$R^{-1}\Delta R - \sum_{i=1}^{n} \frac{1}{H_i^2} v_i - \sum_{i=1}^{n} \frac{1}{H_i^2} \bar{q}_i = 0,$$

which is R-equation for n-dimensional Schrödinger equation (1.8) with the potential function (3.18). ■

4 R-separability in 3-dimensional case

4.1 Procedure to detect R-separable metrics

Here we describe a simple procedure to identify a given 3-dimensional diagonal metric as R-separable in 3-dimensional Laplace equation.

Proposition 2. *In the 3-dimensional case any isothermic metric is binary.*

Proof. Indeed, we put $n = 3$ in (2.3) and hence we deduce the following expressions for Lamé coefficients H_i

$$H_i = \frac{1}{R^2} G_{(1)} G_{(2)} G_{(3)} G_{(i)}^{-1} \frac{1}{f_i}, \qquad i = 1, 2, 3,$$

or more explicitly

$$\begin{aligned} H_1 &= \frac{G_{(2)}G_{(3)}}{R^2 f_1} = \frac{G_{12}G_{13}}{R^2 f_1}, \qquad H_2 = \frac{G_{(1)}G_{(3)}}{R^2 f_2} = \frac{G_{12}G_{23}}{R^2 f_2}, \\ H_3 &= \frac{G_{(1)}G_{(2)}}{R^2 f_3} = \frac{G_{13}G_{23}}{R^2 f_3}. \end{aligned} \tag{4.1}$$

Now see (2.9). ■

To simplify notation we rewrite (4.1) as

$$H_1 = \frac{G_2G_3}{Mf_1}, \qquad H_2 = \frac{G_1G_3}{Mf_2}, \qquad H_3 = \frac{G_1G_2}{Mf_3}.$$

In other words G_i does not depend on u^i, f_i depends on u^i and $R = \sqrt{M}$. Finally we arrive at the following general form of the isothermic metric in 3-dimensional case

$$ds^2 = \frac{1}{M^2}\left[\frac{G_2^2G_3^2}{f_1^2}\left(du^1\right)^2 + \frac{G_1^2G_3^2}{f_2^2}\left(du^2\right)^2 + \frac{G_1^2G_2^2}{f_3^2}\left(du^3\right)^2\right]. \tag{4.2}$$

The procedure in question consists of three steps. Suppose we are given any 3-dimensional diagonal metric

$$ds^2 = H_1^2(u)\left(du^1\right)^2 + H_2^2(u)\left(du^2\right)^2 + H_3^2(u)\left(du^3\right)^2. \tag{4.3}$$

In the first step we attempt to identify (4.3) as an isothermic metric (4.2). Suppose it is the case. This step provides us with (predicts) possible forms of R-factor and coefficients p_i in the separation equations.

In the second step we form the R-equation (2.2) for 3-dimensional Laplace equation either in terms of (4.3) as

$$R^{-1}\Delta R - \sum_{i=1}^{3}\frac{1}{H_i^2}q_i = 0, \tag{4.4}$$

or in terms of (4.2) as

$$\Delta R - \left(\frac{s_1}{G_2^2G_3^2} + \frac{s_2}{G_1^2G_3^2} + \frac{s_3}{G_1^2G_2^2}\right)R^5 = 0, \tag{4.5}$$

where $s_i = f_i^2 q_i$ or as

$$\sum_{i=1}^{3} G_i^2 f_i^2\left[\partial_i^2 R^{-1} + \frac{f_i'}{f_i}\partial_i R^{-1} + q_i R^{-1}\right] = 0. \tag{4.6}$$

In the third step we treat (4.4) and (4.6) as equations for unknowns $q_i(u^i)$ and (4.5) as equation for unknowns $s_i(u^i)$. Any solution to (4.4), (4.5) or (4.6) provides us with a coefficient q_i in the separation equations. If the third step is successful, then the starting metric (4.3) is R-separable in 3-dimensional Laplace equation and the separation equations are constructed explicitly.

If R is harmonic with respect of (4.2) (this case includes separability), then e.g. (4.4) becomes a linear in q_i constraint

$$\sum_{i=1}^{3}\frac{1}{H_i^2}q_i = 0 \tag{4.7}$$

and the corresponding solution space is at most 2-dimensional.

If R is not harmonic with respect of (4.2) and if e.g. (4.4) admits a special solution q_{i0}, then a general solution to (4.4) is

$$q_i = q_{i0} + q_{i1},$$

where q_{i1} is a solution to (4.7).

4.2 Examples

Here we present five examples proving efficiency of our procedure.

4.2.1 Spherical metric

The spherical metric

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

is isothermic. It is easily seen that

$$R = 1, \qquad G_1 = \sin\theta, \qquad G_2 = r, \qquad G_3 = r, \qquad f_1 = r^2, \qquad f_2 = \sin\theta, \qquad f_3 = 1$$

in this case. Equation (4.7) reads

$$q_1 + \frac{1}{r^2} q_2 + \frac{1}{r^2 \sin^2\theta} q_3 = 0$$

and can be easily solved

$$q_1 = -\frac{\alpha}{r^2}, \qquad q_2 = \alpha - \frac{\beta}{\sin^2\theta}, \qquad q_3 = \beta, \qquad \alpha, \beta = \text{const}.$$

The resulting separation equations read

$$\begin{aligned}
&\varphi_1'' + \frac{2}{r}\varphi_1' - \frac{\alpha}{r^2}\varphi_1 = 0, \\
&\varphi_2'' + \cot\theta\ \varphi_2' + \left(\alpha - \frac{\beta}{\sin^2\theta}\right)\varphi_2 = 0, \\
&\varphi_3'' + \beta\varphi_3 = 0.
\end{aligned}$$

4.2.2 Toroidal metric I

The so called toroidal metric

$$ds^2 = (\cosh\eta - \cos\theta)^{-2}\left(d\eta^2 + d\theta^2 + \sinh^2\eta\, d\phi^2\right), \tag{4.8}$$

discussed in e.g. [24, 7], is isothermic and

$$\begin{aligned}
&R = \sqrt{\cosh\eta - \cos\theta}, \qquad G_1 = G_3 = 1, \qquad G_2 = \sinh\eta, \\
&f_1 = \sinh\eta, \qquad f_2 = f_3 = 1.
\end{aligned}$$

We easily verify the equality

$$\Delta R - \frac{1}{4}R^5 = 0. \tag{4.9}$$

Hence equation (4.5) is satisfied if and only if

$$\frac{1}{\sinh^2\eta} s_1 + s_2 + \frac{1}{\sinh^2\eta} s_3 = \frac{1}{4}. \tag{4.10}$$

A general solution to (4.10) is

$$s_1 = f_1^2 q_1 = \left(\frac{1}{4} - \alpha_1\right)\sinh^2\eta - \alpha_2,$$

$$s_2 = f_2^2 q_2 = \alpha_1, \qquad s_3 = f_3^2 q_3 = \alpha_2, \qquad \alpha_1, \alpha_2 = \mathrm{const}.$$

The resulting separation equations read

$$\begin{aligned}
&\varphi_1'' + \coth\eta\ \varphi_1' + \left(\frac{1}{4} - \alpha_1 - \frac{1}{\sinh^2\eta}\alpha_2\right)\varphi_1 = 0,\\
&\varphi_2'' + \alpha_1\varphi_2 = 0,\\
&\varphi_3'' + \alpha_2\varphi_3 = 0.
\end{aligned}$$

4.2.3 Toroidal metric II

Interestingly, the metric (4.8) can be identified as isothermic in two ways. It was shown implicitly in [7]. Indeed, we rewrite (4.8) as follows

$$ds^2 = \frac{\sinh^2\eta}{(\cosh\eta - \cos\theta)^2}\left(\frac{d\eta^2 + d\theta^2}{\sinh^2\eta} + d\phi^2\right). \tag{4.11}$$

Metric (4.11) suggests the following identifications

$$R = \sqrt{\coth\eta - \frac{\cos\theta}{\sinh\eta}}, \qquad G_1 = G_2 = 1, \qquad G_3 = \frac{1}{\sinh\eta}, \qquad f_1 = f_2 = f_3 = 1.$$

Again (4.9) holds. Hence equation (4.5) is satisfied if and only if

$$s_1 \sinh^2\eta + s_2 \sinh^2\eta + s_3 = \frac{1}{4}. \tag{4.12}$$

A general solution to (4.12) is

$$s_1 = \left(\frac{1}{4} - \alpha_2\right)\frac{1}{\sinh^2\eta} - \alpha_1, \qquad s_2 = \alpha_1, \qquad s_3 = \alpha_2, \qquad \alpha_1, \alpha_2 = \mathrm{const}.$$

The resulting separation equations read

$$\begin{aligned}
&\varphi_1'' + \left[\left(\frac{1}{4} - \alpha_2\right)\frac{1}{\sinh^2\eta} - \alpha_1\right]\varphi_1 = 0,\\
&\varphi_2'' + \alpha_1\varphi_2 = 0,\\
&\varphi_3'' + \alpha_2\varphi_3 = 0.
\end{aligned}$$

4.2.4 Cyclidic metric

Consider the following metric

$$\begin{aligned}
ds^2 = \left(1 + p\sqrt{\lambda^1\lambda^2\lambda^3}\right)^{-2}\Bigg[&\frac{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)(d\lambda^1)^2}{\varphi(\lambda^1)} + \frac{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)(d\lambda^2)^2}{\varphi(\lambda^2)}\\
&+ \frac{(\lambda^3-\lambda^1)(\lambda^3-\lambda^2)(d\lambda^3)^2}{\varphi(\lambda^3)}\Bigg],
\end{aligned} \tag{4.13}$$

where $\varphi(x) = (x-a)(x-b)(x-c)(x-d)$ and p, a, b, c, d are constants. In general it is not flat.

Proposition 3. *The off-diagonal components of Ricci tensor of (4.13) vanish, i.e. part of Lamé equations (1.13) is satisfied. The diagonal components of Ricci tensor of (4.13) vanish, i.e. the other part of Lamé equations (1.14) is satisfied, if and only if*

$$pabcd = 0 \qquad \textit{and} \qquad p^2(abc + abd + acd + bcd) = 1.$$

We select $d = 0$ and $p = 1/\sqrt{abc}$. Hence the metric

$$ds^2 = \left(1 + \sqrt{\frac{\lambda^1\lambda^2\lambda^3}{abc}}\right)^{-2} \left[\frac{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)(d\lambda^1)^2}{\varphi(\lambda^1)} + \frac{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)(d\lambda^2)^2}{\varphi(\lambda^2)} + \frac{(\lambda^3-\lambda^1)(\lambda^3-\lambda^2)(d\lambda^3)^2}{\varphi(\lambda^3)}\right] \tag{4.14}$$

with $\varphi(x) = x(x-a)(x-b)(x-c)$ is flat. It is isothermic and

$$R^2 = \left(1 + \sqrt{\frac{\lambda^1\lambda^2\lambda^3}{abc}}\right), \qquad G_1^2 = \lambda^2 - \lambda^3, \qquad G_2^2 = \lambda^1 - \lambda^3, \qquad G_3^2 = \lambda^1 - \lambda^2,$$
$$f_1^2 = \varphi(\lambda^1), \qquad f_2^2 = -\varphi(\lambda^2), \qquad f_3^2 = \varphi(\lambda^3).$$

We readily check the equality

$$\Delta R - \frac{3}{16}R^5 = 0.$$

Hence equation (4.5) is satisfied if and only if

$$\frac{\varphi(\lambda^1)q_1}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)} + \frac{\varphi(\lambda^2)q_2}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)} + \frac{\varphi(\lambda^3)q_3}{(\lambda^3-\lambda^1)(\lambda^3-\lambda^2)} = \frac{3}{16}. \tag{4.15}$$

From Bôcher–Ushveridze identities (2.16) ($n = 3$) we deduce immediately a general solution to (4.15)

$$q_i = \frac{1}{\varphi(\lambda^i)}\left(\alpha_1 + \alpha_2\lambda^i + \frac{3}{16}\left(\lambda^i\right)^2\right), \qquad i = 1, 2, 3,$$

where $\alpha_1, \alpha_2 = \mathrm{const}$. The resulting separation equations read

$$\varphi_i'' + \frac{1}{2}\frac{\varphi'(\lambda^i)}{\varphi(\lambda^i)}\varphi_i' + \frac{1}{\varphi(\lambda^i)}\left[\alpha_1 + \alpha_2\lambda^i + \frac{3}{16}\left(\lambda^i\right)^2\right]\varphi_i = 0.$$

Definition 8. A diagonal 3-dimensional flat metric all whose parametric surfaces are cyclides (Dupin cyclides) is called cyclidic (Dupin-cyclidic).

General cyclides are discussed in [29]. For Dupin cyclides see Section 4.3 of the paper. Metric (4.14) is cyclidic but not Dupin-cyclidic.

4.2.5 Dupin-cyclidic metric

The metric

$$ds^2 = \frac{b^2(w - a\cosh v)^2}{(a\cosh v - c\cos u)^2}du^2 + \frac{b^2(w - c\cos u)^2}{(a\cosh v - c\cos u)^2}dv^2 + dw^2 \tag{4.16}$$

is Dupin-cyclidic [26]. It is R-separable in the Helmholtz equation (1.7) on $\mathbb{E}^3$ (see Theorem 1 in [26]). Here we give an alternative and remarkably simple proof of this result. Metric (4.16) is isothermic and

$$R = (a\cosh v - w)^{-1/2}(w - c\cos u)^{-1/2}, \qquad G_1 = (a\cosh v - w)^{-1},$$
$$G_2 = (w - c\cos u)^{-1}, \qquad G_3 = (a\cosh v - c\cos u)^{-1}, \qquad f_1 = f_2 = b^{-1}, \qquad f_3 = 1.$$

It is easily to verify the equality

$$R^{-1}\Delta R - \frac{1}{4}\left(H_1^{-2} - H_2^{-2}\right) = 0, \tag{4.17}$$

which is equation (4.4) in this case. Taking into account $H_3 = 1$ we rewrite (4.17)

$$R^{-1}\Delta R + k^2 - \frac{1}{4}\left(H_1^{-2} - H_2^{-2}\right) - k^2 H_3^{-2} = 0. \tag{4.18}$$

Certainly, (4.18) is R-equation (2.2) for 3-dimensional Helmholtz equation (1.7). The corresponding separation equations are

$$\varphi_1'' + \frac{1}{4}\varphi_1 = 0,$$
$$\varphi_2'' - \frac{1}{4}\varphi_2 = 0,$$
$$\varphi_3'' + k^2\varphi_3 = 0.$$

4.3 Dupin-cyclidic metrics

Gaston Darboux found a broad class of Dupin-cyclidic metrics which are R-separable in 3-dimensional Laplace equation [13, Section 162, p. 286]. Here we give an alternative and simplified proof of this remarkable result. The metric (4.16) belongs to this class.

There are many definitions (not necessarily equivalent) of Dupin cyclides (see [6, p. 148]). We select the following one.

Definition 9. A Dupin cyclide is a regular parametric surface in $\mathbb{E}^3$ whose both principal curvatures are constant along their curvature lines.

Let us recall the celebrated theorem of Dupin (see [18, p. 609]).

Theorem 6. *Let $u = (u^1, u^2, u^3)$ be orthogonal coordinates in $\mathbb{E}^3$. Two arbitrary parametric surfaces $u^i = \mathrm{const}$ and $u^j = \mathrm{const}$ $(i \neq j)$ intersect in a curvature line of each.*

Proposition 4. *The metric (4.3) is Dupin-cyclidic (see Definition 8) if and only if it is flat and its Lamé coefficients satisfy the following six PDEs*

$$\frac{\partial}{\partial u^j} H_i^{-1} \frac{\partial}{\partial u^i} \ln H_j = 0 \qquad i,j = 1,2,3, \quad i \neq j. \tag{4.19}$$

Proof. Indeed, $k_{ij} = -H_i^{-1}\frac{\partial}{\partial u^i}\ln H_j$ is a principal curvature on a parametric surface $u^i = \mathrm{const}$ in the direction of a curvature line u^j-variable [18, p. 608]. ■

A natural question arises as to when the isothermic metric (4.2) satisfies (4.19)? With no difficulty we prove the following result.

Proposition 5. *The isothermic metric (4.2) satisfies (4.19) if and only if*

$$\left(\frac{M}{G_1}\right)_{,23} = \left(\frac{M}{G_2}\right)_{,13} = \left(\frac{M}{G_3}\right)_{,12} = 0. \tag{4.20}$$

Certainly, one solution to (4.20) is provided by the metric (4.16). On performing re-scaling in (4.16)

$$u^1 = c\cos u, \qquad u^2 = a\cosh v, \qquad u^3 = w$$

we arrive at

$$ds^2 = H_1^2(u)\big(du^1\big)^2 + H_2^2(u)\big(du^2\big)^2 + H_3^2(u)\big(du^3\big)^2 \tag{4.21}$$

with

$$H_1 = \big(u^3 - u^1\big)\big(u^3 - u^2\big)\big(u^1 - u^2\big)^{-1}\big(u^1 - u^3\big)^{-1}\left[-\frac{(u^1)^2}{b^2} + \frac{c^2}{b^2}\right]^{-1/2}, \tag{4.22}$$

$$H_2 = \big(u^3 - u^1\big)\big(u^3 - u^2\big)\big(u^2 - u^1\big)^{-1}\big(u^2 - u^3\big)^{-1}\left[\frac{(u^2)^2}{b^2} - \frac{a^2}{b^2}\right]^{-1/2}, \tag{4.23}$$

$$H_3 = \big(u^3 - u^1\big)\big(u^3 - u^2\big)\big(u^3 - u^1\big)^{-1}\big(u^3 - u^2\big)^{-1}. \tag{4.24}$$

Obviously, (4.21) is isothermic and from (4.22)–(4.24) we have

$$G_1 = \big(u^2 - u^3\big)^{-1}, \qquad G_2 = \big(u^1 - u^3\big)^{-1}, \qquad G_3 = \big(u^1 - u^2\big)^{-1}. \tag{4.25}$$

Let us insert (4.25) into (4.20). The resulting system of equations

$$\big(u^i - u^j\big)M_{,ij} - M_{,i} + M_{,j} = 0 \qquad i, j = 1, 2, 3, \quad i < j$$

is exactly the Euler–Poisson–Darboux system (2.15) for $n = 3$. Hence M is of the form

$$M = \frac{b_1(u^1)}{(u^1 - u^2)(u^1 - u^3)} + \frac{b_2(u^2)}{(u^2 - u^1)(u^2 - u^3)} + \frac{b_3(u^3)}{(u^3 - u^1)(u^3 - u^2)}, \tag{4.26}$$

where $b_i(u^i)$ are arbitrary functions of a single variable. Thus we have the following result.

Lemma 1. *Any metric (4.21) with*

$$H_i = \frac{1}{M}\big(u^i - u^j\big)^{-1}\big(u^i - u^k\big)^{-1} a_i^{-1/2}, \qquad i, j, k \ \text{ are different}, \tag{4.27}$$

where M is given by (4.26) while $b_i(u^i)$ and $a_i(u^i)$ are arbitrary functions of a single variable is isothermic and satisfies (4.19).

Theorem 7. *Suppose a 3-dimensional Riemann space $\mathcal{R}^3$ admits the metric described in Lemma 1. Then*

1) *the off-diagonal components of the Ricci tensor vanish,*
2) *$\mathcal{R}^3$ is conformally flat if and only if*

$$a_i\big(u^i\big) = m_i\big(u^i\big)^2 + 2n_i u^i + p_i, \qquad m_i, n_i, p_i = \text{const}, \tag{4.28}$$

where

$$\sum_{i=1}^3 m_i = \sum_{i=1}^3 n_i = \sum_{i=1}^3 p_i = 0, \tag{4.29}$$

3) *$\mathcal{R}^3$ is flat if and only if it is conformally flat and the following identities hold*

$$\big(n_i^2 - m_i p_i\big)b_i^2 + 2[(\beta_i m_i - \alpha_i n_i)u^i + \beta_i n_i - \alpha_i p_i]b_i + \big(\alpha_i u^i + \beta_i\big)^2 + \gamma_i a_i = 0, \tag{4.30}$$

where the constants α_i, β_i and γ_i satisfy identities

$$\sum_{i=1}^3 \alpha_i = \sum_{i=1}^3 \beta_i = \sum_{i=1}^3 \gamma_i = 0. \tag{4.31}$$

Proof. 1) Suppose we are given the metric (4.21) whose Lamé coefficients are (4.27) and M is an arbitrary function. Then off-diagonal components of its Ricci tensor are

$$R_{ij} = \frac{1}{M}\left(u^i - u^j\right)^{-1}\left[\left(u^i - u^j\right)M_{,ij} + M_{,j} - M_{,i}\right], \qquad i < j.$$

2) The Cotton–York of the metric vanishes if and only if (4.28) and (4.29) hold.

3) Suppose (4.28), (4.29) are valid, then the diagonal components of the Ricci tensor vanish if and only if (4.30) and (4.31) hold. ■

Remark 10. The result 3) of Theorem 7 is essentially due to Gaston Darboux. See the formulae in [12, p. 335].

Lemma 2. *Suppose $\mathbb{E}^3$ is equipped with the metric described in* 3) *of Theorem* 7*. Then $R = \sqrt{M}$ satisfies the equation*

$$\begin{aligned}
&\left(u^2 - u^3\right)^{-2}\sqrt{a_1}\left[\partial_1\sqrt{a_1}\partial_1 R^{-1} - \frac{m_1}{4\sqrt{a_1}}R^{-1}\right] \\
&\quad + \left(u^1 - u^3\right)^{-2}\sqrt{a_2}\left[\partial_2\sqrt{a_2}\partial_2 R^{-1} - \frac{m_2}{4\sqrt{a_2}}R^{-1}\right] \\
&\quad + \left(u^1 - u^2\right)^{-2}\sqrt{a_3}\left[\partial_3\sqrt{a_3}\partial_3 R^{-1} - \frac{m_3}{4\sqrt{a_3}}R^{-1}\right] = 0.
\end{aligned} \tag{4.32}$$

We identify (4.32) as R-equation (4.6) for the Laplace equation on $\mathbb{E}^3$.

Theorem 8. *Any Dupin-cyclidic metric described in* 3) *of Theorem* 7 *is R-separable in the Laplace equation on $\mathbb{E}^3$. $R = \sqrt{M}$ and M is given by* (4.26)*. The separation equations are*

$$\varphi_i'' + \frac{1}{2}\frac{a_i'}{a_i}\varphi_i' - \frac{m_i}{4a_i}\varphi_i = 0, \qquad i = 1, 2, 3.$$

Remark 11. Theorem 8 is due to Gaston Darboux as well. See his remarkable result pointed out in [13, Section 162, p. 286].

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