$\mathcal{N}$=4 Multi-Particle Mechanics, WDVV Equation and Roots*

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Abstract. We review the relation of $\mathcal{N}$=4 superconformal multi-particle models on the real line to the WDVV equation and an associated linear equation for two prepotentials, F and U. The superspace treatment gives another variant of the integrability problem, which we also reformulate as a search for closed flat Yang–Mills connections. Three- and four-particle solutions are presented. The covector ansatz turns the WDVV equation into an algebraic condition, for which we give a formulation in terms of partial isometries. Three ideas for classifying WDVV solutions are developed: ortho-polytopes, hypergraphs, and matroids. Various examples and counterexamples are displayed.

Key words: superconformal mechanics; Calogero models; WDVV equation; deformed root systems

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1 Introduction

Over the past decade, there has been substantial progress in the construction of $\mathcal{N}$=4 superconformal multi-particle mechanics (in one space dimension) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In 2004 a deep connection between these physical systems and the so-called WDVV equation [11, 12] was discovered [4], relating Calogero-type models with $D(2,1;\alpha)$ superconformal symmetry to a branch of mathematics concerned with solving this equation [13, 14, 15, 16, 17, 18, 19]. Here, we describe physicists' attempts to take advantage of the mathematical literature on this subject and to develop it further towards constructing and classifying such multi-particle models.

There exist different versions of the WDVV equation in the literature, so let us be more specific. Originally, the WDVV equation appeared as a consistency relation in topological field theory, where the puncture operator singles out one of the coordinates, so that the associated Frobenius algebra is unital and carries a constant metric [11, 12, 13]. A few years later, a more general form of the WDVV equation appeared as a condition on the prepotential $\widetilde{F}$ of Seiberg–Witten theory (four-dimensional $\mathcal{N}$=2 super Yang–Mills theory) [20, 21, 22]. Here, the distinguished coordinate is absent, and so the Frobenius structure constants do not lead to a natural metric. This so-called generalized WDVV equation takes the form

$$\widetilde{\boldsymbol{A}}_i\widetilde{\boldsymbol{A}}_k^{-1}\widetilde{\boldsymbol{A}}_j=\widetilde{\boldsymbol{A}}_j\widetilde{\boldsymbol{A}}_k^{-1}\widetilde{\boldsymbol{A}}_i \qquad \forall\, i,j,k=1,\dots,n \tag{1}$$

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for a collection of $n \times n$ matrix functions $(\widetilde{\boldsymbol{A}}_i)_{\ell m} = -\partial_i \partial_\ell \partial_m \widetilde{F}$. However, any invertible linear combination of these matrices yields an admissible metric

$$\boldsymbol{\eta} = \sum_i \eta^i \widetilde{\boldsymbol{A}}_i \qquad \text{so that} \qquad \widetilde{\boldsymbol{A}}_i \boldsymbol{\eta}^{-1} \widetilde{\boldsymbol{A}}_j = \widetilde{\boldsymbol{A}}_j \boldsymbol{\eta}^{-1} \widetilde{\boldsymbol{A}}_i . \tag{2}$$

It is easy to see that this metric can be absorbed in a redefinition [14, 15],

$$\boldsymbol{A}_i := \boldsymbol{\eta}^{-1} \widetilde{\boldsymbol{A}}_i \quad \longrightarrow \quad \big[\boldsymbol{A}_i, \boldsymbol{A}_j\big] = 0 \qquad \text{with} \quad \textstyle\sum_i \eta^i \boldsymbol{A}_i = \mathbb{1},$$

giving a formulation equivalent to (1). For a constant metric, e.g. $\boldsymbol{\eta} = \widetilde{\boldsymbol{A}}_1 = \text{const}$, we fall back to the more special case which arose in topological field theory. Since 1999, Veselov and collaborators have been constructing particular solutions to (2), introducing so-called $\vee$-systems and featuring a *constant* metric $\boldsymbol{\eta} = \sum_i x^i \widetilde{\boldsymbol{A}}_i$ [15, 17, 18, 19].

In comparison, $\mathcal{N}$=4 supersymmetric multi-particle models are determined by two prepotentials, F and U, the first of which is subject to the generalized WDVV equation [4]. Here, the conformal invariance imposes a supplementary condition on our matrices, which amounts to choosing the *Euclidean* metric

$$\sum_i x^i \widetilde{\boldsymbol{A}}_i = \mathbb{1} \qquad \longrightarrow \qquad \widetilde{\boldsymbol{A}}_i = \boldsymbol{A}_i ,$$

and so one may drop the label 'generalized'. The map to Veselov's formulation is achieved by a linear coordinate change, $x^i \to x^j M_j{}^i$ with $\boldsymbol{\eta} = \boldsymbol{M}\boldsymbol{M}^\perp$ [23].

The goal of the paper is fourfold. First, we would like to review the appearance of the WDVV equation in the construction of one-dimensional multi-particle models with $su(1,1|2)$ symmetry [4, 5, 6, 7, 8, 9, 10]. In particular, we draw the attention of the mathematical readers to the *second* prepotential U, which enlarges the WDVV structure in a canonical fashion, and to the superspace formulation, which yields an alternative formulation of the integrability condition. Second, we plan to provide some explicit three- and four-particle examples for the physical model builders. Third, we intend to rewrite Veselov's notion of $\vee$-systems in a manner we hope is more accessible to physicists, using the notion of partial isometry and providing further examples. Fourth, we want to advertize some novel attempts to attack the classification problem for the WDVV equation. The standard ansatz for the propotential employs a collection of covectors, which are subject to intricate algebraic conditions. These relations may be visualized in terms of certain polytopes, or in terms of hypergraphs, or by a particular kind of matroid. Neither of these concepts is fully satisfactory; the classification problem remains open. However, for low dimension and a small number of covectors they can solve the problem.

2 Conformal quantum mechanics: Calogero system

As a warm-up, we introduce $n+1$ identical particles with unit mass, moving on the real line, with coordinates x^i and momenta p_i, where $i = 1, 2, \ldots, n+1$, and define their dynamics by the Hamiltonian[1]

$$H = \tfrac{1}{2} p_i p_i + V_B\big(x^1, \ldots, x^{n+1}\big). \tag{3}$$

For the quantum theory, we impose the canonical commutation relations ($\hbar = 1$)

$$[x^i, p_j] = \mathrm{i}\delta_j{}^i .$$

[1] Equivalently, it describes a single particle moving in $\mathbb{R}^{n+1}$ under the influence of the external potential V_B.

Together with the dilatation and conformal boost generators

$$D = -\tfrac{1}{4}\big(x^i p_i + p_i x^i\big) \qquad \text{and} \qquad K = \tfrac{1}{2} x^i x^i,$$

the Hamiltonian (3) spans the conformal algebra $so(2,1)$ in $1+0$ dimensions,

$$[D,H] = -\mathrm{i}H, \qquad [H,K] = 2\mathrm{i}D, \qquad [D,K] = \mathrm{i}K,$$

if and only if $(x^i\partial_i + 2)V_B = 0$, i.e. the potential is homogeneous of degree -2. If one further demands permutation and translation invariance and allows only two-body forces, one ends up with the Calogero model,

$$V_B = \sum_{i<j} \frac{g^2}{(x^i - x^j)^2}.$$

3 $\mathcal{N}$=4 superconformal extension: $su(1,1|2)$ algebra

Our goal is to $\mathcal{N}$=4 supersymmetrize conformal multi-particle mechanics. The most general $\mathcal{N}$=4 extension of $so(2,1)$ is the superalgebra $D(2,1;\alpha)$, but here we specialize to $D(2,1;0) \simeq su(1,1|2) \ni su(2)$. Further, we break the outer $su(2)$ to $u(1)$ by allowing for a central charge C. The set of generators then gets extended [24]

$$(H, D, K) \;\to\; (H, D, K, Q_\alpha, S_\alpha, J_a, C) \qquad \text{with} \qquad \alpha = 1,2 \quad \text{and} \quad a = 1,2,3$$

and hermiticity properties $(Q_\alpha)^\dagger = \bar{Q}^\alpha$ and $(S_\alpha)^\dagger = \bar{S}^\alpha$.

The nonvanishing (anti)commutators of $su(1,1|2)$ read

$$\begin{aligned}
&[D,H] = -\mathrm{i}H, && [H,K] = 2\mathrm{i}D,\\
&[D,K] = +\mathrm{i}K, && [J_a,J_b] = \mathrm{i}\epsilon_{abc}J_c,\\
&\{Q_\alpha,\bar{Q}^\beta\} = 2H\delta_\alpha{}^\beta, && \{Q_\alpha,\bar{S}^\beta\} = +2\mathrm{i}(\sigma_a)_\alpha{}^\beta J_a - 2D\delta_\alpha{}^\beta - \mathrm{i}C\delta_\alpha{}^\beta,\\
&\{S_\alpha,\bar{S}^\beta\} = 2K\delta_\alpha{}^\beta, && \{\bar{Q}^\alpha,S_\beta\} = -2\mathrm{i}(\sigma_a)_\beta{}^\alpha J_a - 2D\delta_\beta{}^\alpha + \mathrm{i}C\delta_\beta{}^\alpha,\\
&[D,Q_\alpha] = -\tfrac{\mathrm{i}}{2}Q_\alpha, && [D,S_\alpha] = +\tfrac{\mathrm{i}}{2}S_\alpha,\\
&[K,Q_\alpha] = +\mathrm{i}S_\alpha, && [H,S_\alpha] = -\mathrm{i}Q_\alpha,\\
&[J_a,Q_\alpha] = -\tfrac{1}{2}(\sigma_a)_\alpha{}^\beta Q_\beta, && [J_a,S_\alpha] = -\tfrac{1}{2}(\sigma_a)_\alpha{}^\beta S_\beta,\\
&[D,\bar{Q}^\alpha] = -\tfrac{\mathrm{i}}{2}\bar{Q}^\alpha, && [D,\bar{S}^\alpha] = +\tfrac{\mathrm{i}}{2}\bar{S}^\alpha,\\
&[K,\bar{Q}^\alpha] = +\mathrm{i}\bar{S}^\alpha, && [H,\bar{S}^\alpha] = -\mathrm{i}\bar{Q}^\alpha,\\
&[J_a,\bar{Q}^\alpha] = \tfrac{1}{2}\bar{Q}^\beta(\sigma_a)_\beta{}^\alpha, && [J_a,\bar{S}^\alpha] = \tfrac{1}{2}\bar{S}^\beta(\sigma_a)_\beta{}^\alpha.
\end{aligned}$$

To realize this algebra on the $(n+1)$-particle state space, we must enlarge the latter by adding Grassmann-odd degrees of freedom, ψ^i_α and $\bar{\psi}^{i\alpha} = \psi^{i\,\dagger}_\alpha$, with $i = 1,\ldots,n+1$ and $\alpha = 1,2$, and subject them to canonical anticommutation relations,

$$\{\psi^i_\alpha, \psi^j_\beta\} = 0, \qquad \{\bar{\psi}^{i\alpha}, \bar{\psi}^{j\beta}\} = 0 \qquad \text{and} \qquad \{\psi^i_\alpha, \bar{\psi}^{j\beta}\} = \delta_\alpha{}^\beta \delta^{ij}.$$

In the absence of a potential (subscript '0'), the generators are given by the bilinears

$$\begin{aligned}
&Q_{0\alpha} = p_i\psi^i_\alpha, \quad \bar{Q}^\alpha_0 = p_i\bar{\psi}^{i\alpha} \quad \text{and} \quad S_{0\alpha} = x^i\psi^i_\alpha, \quad \bar{S}^\alpha_0 = x^i\bar{\psi}^{i\alpha},\\
&H_0 = \tfrac{1}{2}p_ip_i, \quad D_0 = -\tfrac{1}{4}(x^ip_i + p_ix^i), \quad K_0 = \tfrac{1}{2}x^ix^i, \quad J_{0a} = \tfrac{1}{2}\bar{\psi}^{i\alpha}(\sigma_a)_\alpha{}^\beta\psi^i_\beta,
\end{aligned}$$

where σ_a denote the Pauli matrices. Surprisingly however, the free generators fail to obey the $su(1,1|2)$ algebra, and interactions are mandatory! The minimal deformation touches only the supercharge and the Hamiltonian,

$$Q_\alpha = Q_{0\alpha} - \mathrm{i}\,[S_{0\alpha}, V], \qquad \bar{Q}^\alpha = \bar{Q}_0^\alpha - \mathrm{i}[\bar{S}_0^\alpha, V] \qquad \text{and} \qquad H = H_0 + V,$$

keeping $S = S_0$, $\bar{S} = \bar{S}_0$, $D = D_0$, $K = K_0$ and $J = J_0$.

Being a Grassmann-even function of ψ, $\bar{\psi}$ and x, the potential V may be expanded in even powers of the fermionic variables. It turns out that we must go to fourth order for closing the algebra, i.e. [2, 4, 6]

$$V = V_{\mathrm{B}}(x) - U_{ij}(x)\langle\psi_\alpha^i \bar{\psi}^{j\alpha}\rangle + \tfrac{1}{4} F_{ijkl}(x)\langle\psi_\alpha^i \psi^{j\alpha} \bar{\psi}^{k\beta} \bar{\psi}_\beta^l\rangle, \tag{4}$$

where the angle brackets $\langle\cdots\rangle$ denote symmetric (or Weyl) ordering. The functions U_{ij} and F_{ijkl} are totally symmetric in their indices and homogeneous of degree -2 in $\{x^1, \ldots, x^{n+1}\}$. For completeness, we also give the interacting supercharge,

$$Q_\alpha = \big(p_j - \mathrm{i}x^i U_{ij}(x)\big)\psi_\alpha^j - \tfrac{\mathrm{i}}{2} x^i F_{ijkl}(x)\langle\psi_\beta^j \psi^{k\beta} \bar{\psi}_\alpha^l\rangle. \tag{5}$$

4 The structure equations for (F, U): WDVV, Killing, inhomogeneity

Inserting the minimal ansatz (4) for V into the $su(1,1|2)$ algebra and demanding closure, one finds that

$$U_{ij} = \partial_i \partial_j U \qquad \text{and} \qquad F_{ijkl} = \partial_i \partial_j \partial_k \partial_l F$$

are determined by two scalar prepotentials U and F, which are subject to so-called *structure equations* [2, 4, 6],

$$(\partial_i\partial_k\partial_p F)(\partial_p\partial_l\partial_j F) - (\partial_i\partial_l\partial_p F)(\partial_p\partial_k\partial_j F) = 0, \qquad x^i\partial_i\partial_j\partial_k F = -\delta_{jk}, \tag{6}$$

$$\partial_i\partial_j U - (\partial_i\partial_j\partial_k F)\partial_k U = 0, \qquad x^i\partial_i U = -C. \tag{7}$$

The left equations (6a) and (7a) are homogeneous quadratic in F (known as the WDVV equation) [11, 12] and homogeneous linear in U (a type of Killing equation). The right equations (6b) and (7b) introduce well-defined inhomogeneities, so that the prepotential must be of the form

$$F = -\tfrac{1}{2} x^2 \ln x + F_{\mathrm{hom}} \qquad \text{and} \qquad U = -C \ln x + U_{\mathrm{hom}} \tag{8}$$

with F_{hom} of degree -2 and U_{hom} of degree 0 in x. This also shows the redundancies

$$U \simeq U + \text{const} \qquad \text{and} \qquad F \simeq F + \text{quadratic polynomial},$$

which for F is also apparent in the twice-integrated form of (6b),

$$(x^i\partial_i - 2)F = -\tfrac{1}{2} x^i x^i.$$

It is convenient to separate the center-of-mass dynamics from the relative particle motion, since the two decouple in all equations. The center-of-mass motion is already nonlinear but explicitly solved by (8) without homogeneous terms (the central charge is additive). In new relative-motion coordinates, which again we name x^i but with $i = 1, 2, \ldots, n$, the configuration space is reduced to $\mathbb{R}^n$. The Killing-type equation (7a) implies, as its compatibility condition, the WDVV equation (6a) contracted with $\partial_j U$. Furthermore, the contraction of (6a) with x^i is

trivially valid, thanks to (6b). This effectively projects the WDVV equation to $n-1$ dimensions. Since its symmetry is that of the Riemann tensor, it comprises as many independent equations, namely $\frac{1}{12}n(n-1)^2(n-2)$ in number. In particular, (6a) is empty for up to three particles and a single condition for four particles.

The leading part of the potential is also determined by U and F,[2]

$$V_{\rm B} = \tfrac{1}{2}(\partial_i U)(\partial_i U) + \tfrac{\hbar^2}{8}(\partial_i\partial_j\partial_k F)(\partial_i\partial_j\partial_k F) > 0,$$

and the expressions in (5) simplify to

$$x^i F_{ijkl} = -\partial_j\partial_k\partial_l F \qquad \text{and} \qquad x^i U_{ij} = -\partial_j U.$$

Therefore, finding a pair (F,U) amounts to defining an $su(1,1|2)$ invariant $(n+1)$-particle model. For more than three particles, however, this is a difficult task, and very little is known about the space of solutions.

5 Superspace approach: inertial coordinates in $\mathbb{R}^{n+1}$

When analyzing supersymmetric systems, it is often a good idea to employ superspace methods. This is also possible for the case at hand, where the construction of a classical Lagrangian seems straightforward in $\mathcal{N}$=4 superspace [25, 26, 27, 28, 29].

For each particle, we introduce a standard untwisted $\mathcal{N}$=4 superfield

$$\mathbb{u}^A(t,\theta^a,\bar\theta_a) = u^A(t) + O(\theta,\bar\theta) \qquad \text{with} \quad A = 1,\ldots,n+1,$$

obeying the constraints[3]

$$D^2\mathbb{u}^A = 0 = \overline{D}^2\mathbb{u}^A \qquad \longrightarrow \qquad \partial_t[D^a,\overline{D}_a]\mathbb{u}^A = 0 \qquad \longrightarrow \qquad [D^a,\overline{D}_a]\mathbb{u}^A = 2g^A$$

with constants g^A, which will turn out to be the coupling parameters. The general $\mathcal{N}$=4 superconformal action for these fields takes the form[4]

$$S = -\int \mathrm{d}t\mathrm{d}^2\theta\mathrm{d}^2\bar\theta G(\mathbb{u}) = \tfrac{1}{2}\int \mathrm{d}t\,\big[G_{AB}(u)\dot u^A\dot u^B - G_{AB}(u)g^A g^B + \text{fermions}\big]$$

already written in [1], with a superpotential $G(u)$ subject to the conformal invariance condition

$$G - G_A u^A = \tfrac{1}{2}c_A u^A$$

for arbitrary constants c_A, so that it is of the form $G = -\frac{1}{2}cu\ln u + $ terms of degree one.

Generically, such sigma-model-type actions do not admit a multi-particle interpretation, however, unless the target space is flat. This requirement imposes a nontrivial condition on the target-space metric $G_{AB}(u)$ [9],

$$\text{Riemann}(G_{AB}) = 0 \qquad \longleftrightarrow \qquad G_{A[BX}G^{XY}G_{YC]D} = 0.$$

Equivalently, there must exist so-called *inertial coordinates* x^i, with $i = 1,2,\ldots,n+1$, such that

$$S = \int \mathrm{d}t\,\Big[\tfrac{1}{2}\delta_{ij}\dot x^i\dot x^j - V_{\rm B}^{\rm cl}(x) + \text{fermions}\Big]\,.$$

[2] Here and later, we sometimes reinstate $\hbar$ to ease the interpretation.

[3] The constants g^A can be SU(2)-rotated into the constraints, so that $D^2\mathbb{u}^A = \mathrm{i}g^a = -\overline{D}^2\mathbb{u}^A$ but $[D^a,\overline{D}_a]\mathbb{u}^A = 0$.

[4] Subscripts on G denote derivatives with respect to u, i.e. $G_A = \partial G/\partial u^A$ etc.

The goal is, therefore, to find admissible functions $u^A = u^A(x)$ and compute the corresponding G and $V_{\rm B}^{\rm cl}$. The above flatness requirement leads to a specific integrability condition for $u^A_i := \partial_i u^A$, namely

$$\frac{\partial x^i}{\partial u^A}\big(u(x)\big) \equiv \big((u^\bullet_\bullet)^{-1}\big)^i{}_A =: w_{A,i} \stackrel{!}{=} \partial_i w_A \equiv \frac{\partial w_A}{\partial x^i}(x), \tag{9}$$

which says that the transpose of the inverse Jacobian for $u \to x$ is again a Jacobian for a map $w \to x$. This defines a set of functions $w_A(x)$ dual to $u^a(x)$, in the sense that their Jacobians are inverses [9],

$$w_{A,i}u^B_i = \delta_A{}^B \qquad \longleftrightarrow \qquad w_{A,i}u^A_j = \delta_{ij}.$$

Equivalent versions of the integrability condition (9) are [9]

$$\begin{aligned} &u^{[A}{}_i \partial_j u^{B]}_i = 0 \qquad \longleftrightarrow \qquad w_{[A,i}\partial_j w_{B],i} = 0, \\ &f_{ijk} := -w_{A,i}\partial_k u^A_j \qquad \text{is totally symmetric,} \\ &f_{ijk} = \partial_i\partial_j\partial_k F \qquad \text{and} \qquad f_{im[k}f_{l]mj} = 0, \end{aligned} \tag{10}$$

which includes the WDVV equation for F. In contrast, there is no formulation purely in terms of U.

Conformal invariance restricts u^A to be homogeneous quadratic in x, hence w_A to be homogeneous of degree zero (including logarithms!), thus f_{ijk} is of degree -1. The second prepotential U is also determined by $u^A(x)$ via

$$U(x) = -g^A w_A(x) \qquad \text{so that} \qquad C = -x^i\partial_i U = c_A g^A,$$

and automatically fulfills the Killing-type equation (7a). The classical bosonic potential then reads

$$V_{\rm B}^{\rm cl} = \tfrac{1}{2}(\partial_i U)(\partial_i U) = \tfrac{1}{2}g^A g^B w_{A,i} w_{B,i}.$$

Finally, for the superpotential $G(u)$ the integrability condition becomes

$$u^A_i u^B_j G_{AB} = -\delta_{ij} \qquad \longleftrightarrow \qquad G_{AB} = -w_{A,i}w_{B,i} = -\partial_A w_B = -\partial_B w_A,$$

so that, up to an irrelevant u-linear shift of G,

$$w_A = -G_A \qquad \longleftrightarrow \qquad G = -u^A w_A,$$

and we have

$$U = g^A G_A \qquad \text{and} \qquad f_{ijk} = -\tfrac{1}{2}u^A_i u^B_j u^C_k G_{ABC} \qquad \longrightarrow \qquad V_{\rm B}^{\rm cl} = -\tfrac{1}{2}g^A g^B G_{AB}.$$

However, knowing the superpotential does not suffice: the relation between x^i and u^A is needed to determine $U(x)$ and $F(x)$. On the other hand, if a solution F to the WDVV equation can be found, this problem reduces to a linear one [9]:

$$u^A_{ij} + f_{ijk}u^A_k = 0 \qquad \text{or} \qquad w_{A,ij} - f_{ijk}w_{A,k} = 0, \qquad \text{with} \quad f_{ijk} = \partial_i\partial_j\partial_k F. \tag{11}$$

Finally, we remark again that the center-of-mass degree of freedom can be decoupled, so that all indices may run form 1 to n only.

6 Structural similarity to closed flat Yang–Mills connections

It is instructive to rewrite our integrability problem in terms of $n \times n$-matrix-valued differential forms, in a compact formulation closer to Yang–Mills theory. To this end, we define

$$\big(u^A_i\big) := \boldsymbol{u}, \qquad \big(-\partial_i\partial_j F\big) := \boldsymbol{f} \qquad \text{and} \qquad \big(-\partial_i\partial_j\partial_k F \mathrm{d}x^k\big) := \boldsymbol{A} \ = \boldsymbol{A}_k \mathrm{d}x^k. \tag{12}$$

Since $\boldsymbol{A}_k = \partial_k \boldsymbol{f}$ and $\partial_i\partial_j\partial_k F = w_{A,i}\partial_k u^A_j$, we have

$$\boldsymbol{A} = \mathrm{d}\boldsymbol{f} \quad \to \quad \mathrm{d}\boldsymbol{A} = 0 \qquad \text{and} \qquad \boldsymbol{A} = \boldsymbol{u}^{-1}\mathrm{d}\boldsymbol{u} \quad \to \quad \mathrm{d}\boldsymbol{A} + \boldsymbol{A}\wedge\boldsymbol{A} = 0,$$

from which we learn that

$$0 = \boldsymbol{A}\wedge\boldsymbol{A} = \tfrac{1}{2}\mathrm{d}[\boldsymbol{f}, \mathrm{d}\boldsymbol{f}] = -\mathrm{d}\boldsymbol{u}^{-1}\wedge\mathrm{d}\boldsymbol{u}, \tag{13}$$

which is nothing but the WDVV equation again. Hence, we are looking for connections $\boldsymbol{A}$ which are at the same time closed and flat. Dealing with a topologically trivial configuration space $\mathbb{R}^n$, it implies that $\boldsymbol{A}$ is simultaneously exact and pure gauge. The exactness is already part of the definition (12), and the pure-gauge property is what relates $\boldsymbol{A}$ with $\boldsymbol{u}$. We remark that $\boldsymbol{A}$ and $\boldsymbol{f}$ are symmetric matrices while $\boldsymbol{u}$ is not. Furthermore, the inhomogeneity (6b) demands that $x^i\partial_i\boldsymbol{f} = \mathbb{1}$. The task is to solve (13) for $\boldsymbol{f}$ and for $\boldsymbol{u}$, which then yield $\partial^3 F$ and $\vec{\nabla}U = -2\boldsymbol{u}^{-1}\vec{g}$.

Of course, we cannot 'solve' the WDVV equation by formal manipulations. But even given a solution $\boldsymbol{A}$ (and hence $\boldsymbol{f}$), it is nontrivial to construct an associated matrix function $\boldsymbol{u}$. For this, we must integrate the linear matrix differential equation (11),

$$\mathrm{d}\boldsymbol{u}^\top = \boldsymbol{A}\boldsymbol{u}^\top, \tag{14}$$

which qualifies $\boldsymbol{u}$ as covariantly constant in the WDVV background. The formal solution reads

$$\boldsymbol{u}^\top = \sum_{k=0}^{\infty} \boldsymbol{f}^{(k)} \qquad \text{with} \qquad \boldsymbol{f}^{(0)} = \mathbb{1}, \quad \boldsymbol{f}^{(1)} = \boldsymbol{f} \quad \text{and} \quad \mathrm{d}\boldsymbol{f}\boldsymbol{f}^{(k)} = \mathrm{d}\boldsymbol{f}^{(k+1)},$$

up to right multiplication with a constant matrix. The matrix functions $\boldsymbol{f}^{(k)}$ are local because

$$\mathrm{d}(\mathrm{d}\boldsymbol{f}\boldsymbol{f}^{(k)}) = -\mathrm{d}\boldsymbol{f}\wedge\mathrm{d}\boldsymbol{f}^{(k)} = -\mathrm{d}\boldsymbol{f}\wedge\mathrm{d}\boldsymbol{f}\boldsymbol{f}^{(k-1)} = -\boldsymbol{A}\wedge\boldsymbol{A}\boldsymbol{f}^{(k-1)} = 0$$

due to the WDVV equation. Likewise, one has

$$\boldsymbol{f}\mathrm{d}\boldsymbol{f}^{(k)} = \boldsymbol{f}\mathrm{d}\boldsymbol{f}\boldsymbol{f}^{(k-1)} = \mathrm{d}\big(\boldsymbol{f}\boldsymbol{f}^{(k)} - \boldsymbol{f}^{(k+1)}\big).$$

Note that the naive guess $\boldsymbol{u}^\top = \mathrm{e}^{\boldsymbol{f}}$ is wrong since $[\boldsymbol{f}, \mathrm{d}\boldsymbol{f}] = \mathrm{d}(\boldsymbol{f}^2 - 2\boldsymbol{f}^{(2)}) \neq 0$.

We provide two explicit examples for $n = 2$, with the notation

$$x^{i=1} =: x, \qquad x^{i=2} =: y \qquad \text{and} \qquad x^2 + y^2 =: r^2.$$

Starting from the B_2 solution with a radial term [6, 9]

$$F = -\tfrac{1}{2}x^2\ln x - \tfrac{1}{2}y^2\ln y - \tfrac{1}{4}(x+y)^2\ln(x+y) - \tfrac{1}{4}(x-y)^2\ln(x-y) + \tfrac{1}{2}r^2\ln r, \tag{15}$$

we have

$$\boldsymbol{f} = \frac{1}{2}\begin{pmatrix} \ln\big[(x^2-y^2)\frac{x^2}{r^2}\big] & \ln\frac{x+y}{x-y} \\ \ln\frac{x+y}{x-y} & \ln\big[(x^2-y^2)\frac{y^2}{r^2}\big] \end{pmatrix} - \frac{1}{r^2}\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$$

with $(x\partial_x + y\partial_y)\boldsymbol{f} = \mathbb{1}$ and, hence,

$$\boldsymbol{A} = \mathrm{d}\boldsymbol{f} = \frac{(x^2-y^2)^2}{xyr^4}\begin{pmatrix} y\mathrm{d}x & 0 \\ 0 & x\mathrm{d}y \end{pmatrix} + \frac{4x^2y^2}{(x^2-y^2)r^4}\begin{pmatrix} x\mathrm{d}x - y\mathrm{d}y & x\mathrm{d}y - y\mathrm{d}x \\ x\mathrm{d}y - y\mathrm{d}x & x\mathrm{d}x - y\mathrm{d}y \end{pmatrix}.$$

It is easy to check that indeed $\boldsymbol{A}\wedge\boldsymbol{A} = 0$ but $[\boldsymbol{A},\boldsymbol{f}] \neq 0$. The solution to (14) turns out to be

$$\boldsymbol{u} = \frac{\Gamma}{r^4}\begin{pmatrix} xr^4 & yr^4 \\ x\,y^4 & y\,x^4 \end{pmatrix} \qquad \xrightarrow{\Gamma=\mathbb{1}} \qquad \begin{cases} u^1 = \frac{1}{2}r^2, \\ u^2 = \frac{1}{2}x^2y^2/r^2 \end{cases}$$

with an arbitrary non-degenerate constant matrix Γ, as may be checked by inserting it into (14).

One may also begin with a purely radial WDVV solution [9],

$$F = -\tfrac{1}{2}r^2\ln r \qquad \longrightarrow \qquad \boldsymbol{f} = \tfrac{1}{2}(\ln r^2)\mathbb{1} + \frac{x^2-y^2}{2r^2}\sigma_3 + \frac{xy}{r^2}\sigma_1,$$

and find

$$\boldsymbol{u} = \Gamma\begin{pmatrix} 2x & 2y \\ 2x\arctan\frac{y}{x} - y & 2y\arctan\frac{y}{x} + x \end{pmatrix} \qquad \xrightarrow{\Gamma=\mathbb{1}} \qquad \begin{cases} u^1 = r^2, \\ u^2 = r^2\arctan\frac{y}{x}. \end{cases}$$

For more generic weight factors in (15), u^2 is expressed in terms of hypergeometric functions [9].

7 Three- and four-particle solutions

An alternative method for constructing solutions (F,U) attempts to find functions $u^A(x)$ satisfying (10). It is successful for $n+1=3$ since the WDVV equation is empty in this case. Imposing also permutation invariance, a natural choice for three homogeneous quadratic symmetric functions of $(x^i) = (x,y,z)$ is

$$\begin{aligned} u^1 &= (x+y+z)^2, \\ u^2 &= (x-y)^2 + (y-z)^2 + (z-x)^2, \\ u^3 &= [(2x-y-z)(2y-z-x)(2z-x-y)]^{2/3}h(s), \end{aligned} \tag{16}$$

where h is an (almost) arbitrary function of the ratio

$$s = \frac{[(2x-y-z)(2y-z-x)(2z-x-y)]^2}{[(x-y)^2+(y-z)^2+(z-x)^2]^3}.$$

Not surprisingly, (16) fulfils the integrability condition (10), so we are guaranteed to produce solutions. It is straightforward to compute the Jacobians u^A_i and $w_{A,i}$ and proceed to the prepotentials. Writing $(g^A) = (g_1,g_2,g_3)$, the bosonic potential comes out as

$$\begin{aligned} V_{\mathrm{B}}^{\mathrm{cl}} &= \frac{g_1^2/24}{(x+y+z)^2} + \frac{1}{324}\left[(1-2s)g_2^2 + 2s\frac{(hg_2 - g_3/\sqrt[3]{s})^2}{(h+3sh')^2}\right] \\ &\quad\times\left(\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2}\right) \\ &= \frac{g_1^2/24}{(x+y+z)^2} + \frac{g_2^2 - 4s^{\frac{2}{3}-\delta}g_2g_3 + 2s^{\frac{1}{3}-2\delta}g_3^2}{324(1+3\delta)^2}\left(\frac{1}{(x-y)^2} + \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2}\right) \\ &\quad + \frac{\delta(2+3\delta)}{8(1+3\delta)^2}\frac{g_2^2}{(x-y)^2+(y-z)^2+(z-x)^2}, \end{aligned}$$

where in the second equality we specialized to

$$h(s) = s^\delta \qquad \longleftrightarrow \qquad u^3 = \frac{[(2x-y-z)(2y-z-x)(2z-x-y)]^{2/3+2\delta}}{[(x-y)^2+(y-z)^2+(z-x)^2]^{3\delta}}.$$

Putting $g_3 = 0$ for simplicity, the corresponding prepotentials are

$$\begin{aligned}
U &= -\frac{g_1}{6}\ln(x+y+z) - \frac{g_2}{18(1+3\delta)}\ln(x-y)(y-z)(z-x) \\
&\quad - \frac{\delta g_2}{4(1+3\delta)}\ln\big[(x-y)^2+(y-z)^2+(z-x)^2\big], \\
F &= -\tfrac{1}{6}(x+y+z)^2\ln(x+y+z) - \tfrac{1}{4}\big[(x-y)^2\ln(x-y) + (y-z)^2\ln(y-z) \\
&\quad + (z-x)^2\ln(z-x)\big] + \tfrac{1-6\delta}{36}\big[(2x-y-z)^2\ln(2x-y-z) \\
&\quad + (2y-z-x)^2\ln(2y-z-x) + (2z-x-y)^2\ln(2z-x-y)\big] \\
&\quad + \tfrac{\delta}{4}\big[(x-y)^2+(y-z)^2+(z-x)^2\big]\ln[(x-y)^2+(y-z)^2+(z-x)^2\big].
\end{aligned}$$

We recognize the roots of G_2 plus a radial term in the coordinate differences. The potential simplifies in two special cases:

$$\begin{aligned}
&\delta = 0 \quad \Leftrightarrow \quad h = 1: \qquad V_{\rm B}^{\rm cl}(g_1 = g_3 = 0) \quad \text{is pure Calogero,} \\
&\delta = \tfrac{1}{6} \quad \Leftrightarrow \quad h = s^{1/6}: \qquad V_{\rm B}^{\rm cl}(g_1 = g_2 = 0) \quad \text{is pure Calogero.}
\end{aligned}$$

In the full quantum potential, $V_{\rm B} = V_{\rm B}^{\rm cl} + \frac{\hbar^2}{8}F'''F'''$, the couplings g^A receive quantum corrections.

Stepping up to four particles, i.e. $n+1 = 4$, it becomes much more difficult to construct solutions, since the integrability condition is no longer trivial. Our attempts to take a known WDVV solution and exploit the linear equations (11) for u_i^A have met with success only sporadically. In most cases, the hypergeometric function ${}_2F_1$ turns up in the expressions. A simple permutation-symmetric example uses the A_3 solution with a radial term,

$$F = -\tfrac{1}{8}\Big(\sum_i x^i\Big)^2\ln\sum_i x^i + \tfrac{1}{8}\sum_{i<j}(x^i-x^j)^2\ln(x^i-x^j) - \tfrac{1}{8}\Big(\sum_{i<j}(x^i-x^j)^2\Big)\ln\sum_{i<j}(x^i-x^j)^2,$$

for which we discovered [9]

$$\begin{aligned}
&u^1 = (x+y+z+w)^2, \\
&u^2 = (x-y)^2+(x-z)^2+(x-w)^2+(y-z)^2+(y-w)^2+(z-w)^2, \\
&u^3 = u^2 I\left(\frac{x+y-z-w}{pq}\right) \qquad \text{and} \qquad u^4 = u^2 I\left(\frac{p}{q}\right),
\end{aligned}$$

with

$$\begin{aligned}
&p^2 = (x-y+z-w) + 2\sqrt{(w-x)(y-z)}, \qquad q^2 = (x-y-z+w) + 2\sqrt{(w-y)(x-z)} \\
&\text{and} \qquad I(x) = \int_0^x \frac{{\rm d}t}{\sqrt{1-t^4}}.
\end{aligned}$$

The Jacobians and the bosonic potential are algebraic but not of Calogero type. It remains a challenge to find (u^2, u^3, u^4) for the A_3 WDVV solution *without* radial term,

$$F = -\tfrac{1}{8}\Big(\sum_i x^i\Big)^2\ln\sum_i x^i - \tfrac{1}{8}\sum_{i<j}(x^i-x^j)^2\ln(x^i-x^j).$$

8 Covector ansatz for prepotential F

For the rest of the presentation, we concentrate on the WDVV equation in $\mathbb{R}^n$,

$$(\partial_i\partial_k\partial_p F)(\partial_p\partial_l\partial_j F) - (\partial_i\partial_l\partial_p F)(\partial_p\partial_k\partial_j F) = 0 \qquad \text{with} \qquad (x^i\partial_i - 2)F = -\tfrac{1}{2}x^i x^i,$$

since, together with $U\equiv 0$, its solutions already produce genuine $\mathcal{N}=4$ superconformal mechanics models. Leaving aside a possible radial term

$$F_{\text{rad}} = -r^2 \ln r \qquad \text{with} \qquad r^2 := \sum_i (x^i)^2, \tag{17}$$

we employ the standard 'rank-one' or 'covector' ansatz [2]

$$F = -\tfrac{1}{2}\sum_\alpha (\alpha\cdot x)^2 \ln \alpha\cdot x$$

containing a set $\{\alpha\}$ of covectors

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \in (\mathbb{R}^n)^* \quad \text{or} \quad \in \mathrm{i}(\mathbb{R}^n)^* \qquad \longrightarrow \qquad \alpha(x) = \alpha\cdot x = \alpha_i x^i,$$

subject to the normalization

$$\sum_\alpha \alpha_i\alpha_j = \delta_{ij} \qquad \longleftrightarrow \qquad \sum_\alpha \alpha\otimes\alpha = \mathbb{1} \tag{18}$$

which takes care of the inhomogeneity in (17). The WDVV equation turns into an algebraic condition on the set of covectors [14, 15, 6],

$$\sum_{\alpha,\beta} \frac{\alpha\cdot\beta}{\alpha\cdot x\,\beta\cdot x}(\alpha_i\beta_j - \alpha_j\beta_i)(\alpha_k\beta_l - \alpha_l\beta_k) = 0 \qquad \text{with} \qquad \alpha\cdot\beta = \delta^{ij}\alpha_i\beta_j. \tag{19}$$

Apart from the normalization (18), the covectors are projective, so we may think of them as a bunch of rays. Let us denote their number (the cardinality of $\{\alpha\}$) by p. We may assume that no two covectors are collinear. Since an orthogonal pair of covectors does not contribute to the double sum, two mutually orthogonal subsets of covectors decouple in (19), and it suffices to consider indecomposable covector sets. In $n=2$ dimensions, (18) implies (19), but already for the lowest nontrivial dimension $n=3$ only partial results are known [2, 17, 4, 5, 18, 19, 6].

9 Partial isometry formulation of WDVV

Let us gain a geometric understanding of (19). Each of the $\frac{1}{2}p(p-1)$ pairs (α,β) in the double sum spans some plane $\pi \sim \alpha\wedge\beta \in \Lambda^2((\mathbb{R}^n)^*)$, but not all of these planes need be different. When we group the pairs according to these planes[5], the tensor structure $(\alpha\wedge\beta)^{\otimes 2}$ of (19) tells us that this equation must hold separately for the subset of coplanar covectors pertaining to each plane π,

$$\sum_{\alpha,\beta\in\pi} \frac{\alpha\cdot\beta\ |\alpha\wedge\beta|^2}{\alpha\cdot x\,\beta\cdot x} = 0 \qquad \forall\,\pi. \tag{20}$$

[5] A given covector may occur in different pairs, thus in different groups. Covectors are not grouped, only their pairs.

Depending on the number q of covectors contained in a given plane π, one of three cases occurs [15, 19]:

$$\begin{aligned}
&\text{case (a)} \quad \pi \text{ contains zero or one covector} \quad \longrightarrow \quad \text{equation trivial,}\\
&\text{case (b)} \quad \pi \text{ contains two covectors, } \pi \sim \alpha \wedge \beta \quad \longrightarrow \quad \text{orthogonality } \alpha \cdot \beta = 0,\\
&\text{case (c)} \quad \pi \text{ contains } q > 2 \text{ covectors} \quad \longrightarrow \quad \text{projector condition on } \pi :\\
&\sum_{\alpha \in \pi} \alpha \otimes \alpha = \lambda_\pi \mathbb{1}_\pi =: \lambda_\pi P_\pi \qquad \text{for} \quad \lambda_\pi \in \mathbb{R} \quad \text{and} \quad P_\pi^2 = P_\pi \quad \text{with} \quad \operatorname{rank}(P_\pi) = 2. \qquad (21)
\end{aligned}$$

The latter is the proper covector normalization for the planar subsystem, which implies the (trivial) WDVV equation on π to hold. Establishing the projector condition (21) simultaneously for all planes is a nontrivial problem, since covectors usually lie in more than one plane, which imposes conditions linking the planes.

For a more quantitative formulation, we express (21) in terms of partial isometries. After introducing a counting index $a = 1, \ldots, p$ for the covectors $\{\alpha\} = \{\alpha_1, \ldots, \alpha_p\}$, we collect their components in an $n \times p$ matrix A. This defines a map

$$A : \ \mathbb{R}^p \to \mathbb{R}^n \qquad \text{given by} \qquad A = \big(\alpha_{ia}\big)_{a=1,\ldots,p}^{i=1,\ldots,n} \quad \text{with} \quad A\,A^\top = \mathbb{1}_n,$$

encoding the total normalization (18). For each nontrivial plane π, we select all $\alpha_{a_s} \in \pi$, $s = 1, \ldots, q$, via

$$B_\pi : \ \mathbb{R}^p \to \mathbb{R}^q \qquad \text{by} \qquad \{\alpha_a\} \mapsto \{\alpha_{a_s}\}$$

and write the combination

$$A_\pi \ : \ R^q \to \mathbb{R}^n \qquad \text{by} \qquad A_\pi := A B_\pi^\top = \big(\alpha_{i a_s}\big)_{s=1,\ldots,q}^{i=1,\ldots,n}.$$

Our projector condition then reads

$$A_\pi\, A_\pi^\top = \lambda_\pi P_\pi \qquad \longleftrightarrow \qquad A_\pi^\top A_\pi \ = \lambda_\pi Q_\pi \qquad (22)$$

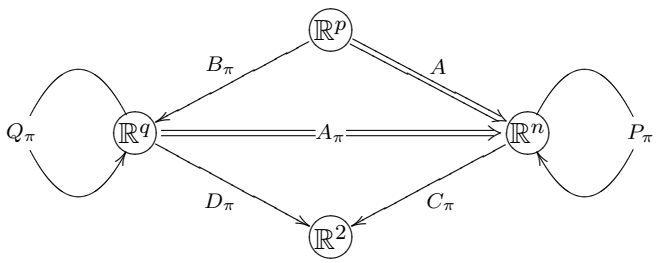
with projectors P_π on $\mathbb{R}^n$ and Q_π on $\mathbb{R}^q$ of rank two and multipliers λ_π, for any nontrivial plane π. Therefore, A is a WDVV solution iff $\frac{A_\pi}{\sqrt{\lambda_\pi}}$ is a rank-2 partial isometry (22) for each nontrivial plane π! An alternative version of (22) is

$$A_\pi\, A_\pi^\top A_\pi \ = \lambda_\pi A_\pi\,.$$

Note that $A \neq A_\pi\, B_\pi$. Since the projectors are of rank 2, we may split A_π over $\mathbb{R}^2$:

$$\exists\ D_\pi : \ \mathbb{R}^q \to \mathbb{R}^2 \qquad \text{and} \qquad C_\pi : \ \mathbb{R}^2 \leftarrow \mathbb{R}^n \qquad \text{such that} \qquad A_\pi \ = C_\pi^\top D_\pi\,.$$

The situation can be visualized in the following noncommutative diagram:



We illustrate the partial isometry formulation with the simplest nontrivial example, which occurs at $n = 3$ and $p = 6$, by providing a one-parameter family of covectors $\{\alpha, \beta, \gamma, \alpha', \beta', \gamma'\}(t)$ via

$$A = \frac{1}{6} \begin{array}{c} \begin{matrix} \alpha & \beta & \gamma & \alpha' & \beta' & \gamma' \end{matrix} \\ \begin{pmatrix} 6t & -3t & -3t & 0 & 3w & -3w \\ 0 & 3\sqrt{3}t & -3\sqrt{3}t & -2\sqrt{3}w & \sqrt{3}w & \sqrt{3}w \\ 0 & 0 & 0 & 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix} \end{array} \qquad \text{with} \quad w = \sqrt{2 - 3t^2}. \tag{23}$$

It is easily checked that $A\,A^\top = \mathbb{1}$. A quick analysis of linear dependence reveals that 12 of the 15 covector pairs are grouped into 4 planes of 3 pairs each, leaving 3 pairs ungrouped. 3 coplanar pairs imply 3 coplanar covectors, hence there are 4 nontrivial planes containing $q = 3$ covectors, namely

$$\langle \alpha\,\beta\,\gamma \rangle, \qquad \langle \alpha\,\beta'\gamma' \rangle, \qquad \langle \alpha'\beta\,\gamma' \rangle, \qquad \langle \alpha'\beta'\gamma \rangle,$$

and 3 planes containing just two covectors, which are indeed orthogonal,

$$\alpha \cdot \alpha' = \beta \cdot \beta' = \gamma \cdot \gamma' = 0.$$

Let us test the projector condition (22) for two of the planes:

$$A_{\langle \alpha\,\beta\,\gamma \rangle} = \frac{1}{2} \begin{pmatrix} 2t & -t & -t \\ 0 & \sqrt{3}\,t & -\sqrt{3}\,t \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad A_\pi\,A_\pi^\top = \tfrac{3}{2}t^2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tfrac{3}{2}t^2 \cdot P_\pi,$$

$$A_{\langle \alpha\,\beta'\gamma' \rangle} = \frac{1}{6} \begin{pmatrix} 6t & 3w & -3w \\ 0 & \sqrt{3}w & \sqrt{3}w \\ 0 & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix} \quad \Rightarrow \quad A_\pi\,A_\pi^\top = \frac{1 - \frac{1}{2}t^2}{6 - 3t^2} \begin{pmatrix} 6-3t^2 & 0 & 0 \\ 0 & 2-3t^2 & 2w \\ 0 & 2w & 4 \end{pmatrix},$$

where the matrices on the right are idempotent. Hence, in both cases, A_π is proportional to a partial isometry, with a (parameter-dependent) multiplier λ_π. The other two nontrivial planes work in the same way. We have proven that (23) produces a family of WDVV solutions. This scheme naturally extends to include imaginary covectors as well.

10 Deformed root systems and polytopes

It is known for some time [14, 15] that the set Φ^+ of positive roots of any simple Lie algebra (in fact, of any Coxeter system) is a good choice for the covectors. So let us take

$$\{\alpha\} = \Phi^\dagger = \Phi_{\rm L}^+ \cup \Phi_{\rm S}^+ \qquad \text{with} \qquad \alpha_{\rm L} \cdot \alpha_{\rm L} = 2 \quad \text{and} \quad \alpha_{\rm S} \cdot \alpha_{\rm S} = 1 \text{ or } \tfrac{2}{3}.$$

where the subscripts 'L' and 'S' pertain to long and short roots, respectively. Having fixed the root lengths, we must introduce scaling factors $\{f_\alpha\} = \{f_{\rm L}, f_{\rm S}\}$ in

$$F = -\tfrac{1}{2} \left(f_{\rm L} \sum_{\alpha \in \Phi_{\rm L}^+} + f_{\rm S} \sum_{\alpha \in \Phi_{\rm S}^+} \right) (\alpha \cdot x)^2 \ln |\alpha \cdot x|.$$

The normalization condition (18) has a one-parameter solution,

$$\left(f_{\rm L} \sum_{\alpha \in \Phi_{\rm L}^+} + f_{\rm S} \sum_{\alpha \in \Phi_{\rm S}^+} \right) \alpha \otimes \alpha = \mathbb{1} \qquad \longrightarrow \qquad \begin{cases} f_{\rm L} = \dfrac{1}{h^\vee} + (h - h^\vee)t, \\ f_{\rm S} = \dfrac{1}{h^\vee} + (h - r h^\vee)t, \end{cases} \qquad \text{with} \quad t \in \mathbb{R},$$

where h and $h^\vee$ are the Coxeter and dual Coxeter numbers of the Lie algebra, respectively. The roots define a family of over-complete partitions of unity. Amazingly, all simple Lie algebra root systems obey (20), and they do so separately for the pairs of long roots, for the pairs of short roots and for the mixed pairs, of any plane π. This leads to the freedom (t) to rescale the short versus the long roots and provides a one-parameter family of solutions to the WDVV equation [15, 16, 6]. (In the simply-laced case there is only one solution, of course.)

For illustration we give two examples. Let $\{e_i\}$ be an orthonormal basis in $\mathbb{R}^{n+1}$. For

$$A_n \oplus A_1 : \quad \{\alpha\} = \left\{ e_i - e_j,\ \sum_i e_i \;\middle|\; 1 \le i < j \le n+1 \right\} \qquad \text{we find}$$

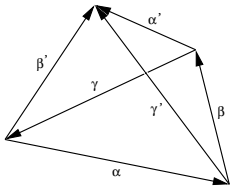
$$F_{A_n \oplus A_1} = -\frac{1/2}{n+1} \sum_{i<j} (x^i - x^j)^2 \ln(x^i - x^j) \ - \ \frac{1/2}{n+1} \left(\sum_i x^i \right)^2 \ln \left(\sum_i x^i \right)$$

with center-of-mass decoupling, while for the non-simply-laced case ($n = 2$, $p = 6$) without center of mass

$$G_2 : \ \{\alpha\} = \left\{ \tfrac{1}{\sqrt{3}}(e_i - e_j),\ \tfrac{1}{\sqrt{3}}(e_i + e_j - 2e_k) \;\middle|\; (i,j,k) \quad \text{cyclic} \right\} \qquad \text{one gets}$$

$$F_{G_2} = -\frac{1-24t}{24} \left(x^1 - x^2\right)^2 \ln \left(x^1 - x^2\right) - \frac{1+8t}{24} \left(x^1 + x^2 - 2x^3\right)^2 \ln \left(2x^1 - x^2 - x^3\right) + \text{cyclic.}$$

A natural question is whether one can deform the Lie algebraic root systems by changing the angles between covectors but keep (20) valid. So which deformations respect the WDVV equation? Based on a few examples, we conjecture that the (suitably rescaled and translated) covectors should form the edges of some polytope in $\mathbb{R}^n$. Non-concurrent pairs of edges then have no reason to be coplanar with other edges, thus better be orthogonal. Concurrent edge pairs, on the other hand, always belong to some polytope face, hence automatically combine with further coplanar edges to a nontrivial plane π. The hope is that the polytope's incidence relations take care of the WDVV equation, e.g. in the form of (22). For $p \ge \frac{1}{2}n(n+1)$, there is enough scaling freedom to finally arrange the normalization (18) with $\{f_\alpha\}$.



This expectation is actually bourne out in the case of the A_n root system, which, with $p = \frac{1}{2}n(n+1)$, is in fact the minimal irreducible system in each dimension n and uniquely fixes $\{f_\alpha\}$. Starting with an arbitrary bunch of $\frac{1}{2}n(n+1)$ rays in $\mathbb{R}^n$, we reduce the freedom in their directions by imposing firstly the n-simplex incidence relations and secondly the orthogonality conditions for skew edges. Let us do some counting of moduli (minus global translations, rotations and scaling):

	ray moduli	incidences	simplex moduli	orthogonality	final moduli
#	$\frac{1}{2}n^2(n-1)$	$-\frac{1}{2}(n-2)(n^2-1)$	$\frac{1}{2}(n-1)(n+2)$	$-\frac{1}{2}(n-2)(n+1)$	n
$n = 2,3,4$	2, 9, 24	$0, -4, -15$	2, 5, 9	$0, -2, -5$	2, 3, 4

We find that the moduli space $\mathcal{M}(A_n)$ of these so-called *orthocentric* n-simplices is just n-dimensional. It can be shown [23] that indeed it fits perfectly to a family of WDVV solutions found earlier [17, 18, 19], lending support to our polytope idea. We remark that the previous example (23) represents a one-parameter subset in $\mathcal{M}(A_3)$.

Let us make this observation more explicit in the case of $n = 4$. Using the recursive construction of orthocentric n-simplices presented in [6] for $n = 4$ and computing the corresponding

scaling factors is feasible but algebraically involved. Therefore, we just present a 'nice' one-parameter subfamily of solutions, with $t \in \mathbb{R}_+$ and $w^2 = t^2 - \frac{1}{4}$,

$$A = \frac{1}{2t}\begin{pmatrix} w\sqrt{2} & 0 & \frac{w}{2}\sqrt{2} & -\frac{w}{2}\sqrt{2} & -\frac{w}{2}\sqrt{2} & \frac{w}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & -\frac{w}{3}\sqrt{6} & \frac{w}{2}\sqrt{6} & \frac{w}{6}\sqrt{6} & \frac{w}{2}\sqrt{6} & \frac{w}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} & -\frac{1}{3}\sqrt{6} & 0 \\ 0 & \frac{2w}{3}\sqrt{3} & 0 & \frac{2w}{3}\sqrt{3} & 0 & \frac{2w}{3}\sqrt{3} & \frac{1}{6}\sqrt{3} & \frac{1}{6}\sqrt{3} & \frac{1}{6}\sqrt{3} & -\frac{1}{2}\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & t & t & t & t \end{pmatrix}.$$

For $t^2 = \frac{5}{4}$ we have the root system of A_4, at $t^2 = \frac{1}{4}$ the first six covectors disappear and leave A_1^4. When $0 < t^2 < \frac{1}{4}$, the first six covectors are imaginary, and in the singular limit $t^2 \to 0$ we obtain the A_3 roots and fundamental weights, but can no longer maintain our normalization.

A more familiar parametrization embeds the A_4 root system into $\mathbb{R}^5$, in the hyperplane orthogonal to the center-of-mass covector $\sum_i e_i$, with $s \in \mathbb{R}_+$ and $u^2 = 20s^2 - 10s + 1$,

$$A = \frac{1}{(1-4s)\sqrt{5}}\begin{pmatrix} u & 0 & u & 0 & u & 0 & 1-s & -s & -s & -s \\ -u & 0 & 0 & u & 0 & u & -s & 1-s & -s & -s \\ 0 & u & -u & 0 & 0 & -u & -s & -s & 1-s & -s \\ 0 & -u & 0 & -u & -u & 0 & -s & -s & -s & 1-s \\ 0 & 0 & 0 & 0 & 0 & 0 & 4s-1 & 4s-1 & 4s-1 & 4s-1 \end{pmatrix}.$$

Now $s = 0$ yields the roots of A_4, beyond $s = \frac{1}{4}(1 - \frac{1}{\sqrt{5}})$ the first six covectors turn imaginary, and the singular limit $s \to \frac{1}{4}$ ($u \to \frac{1}{2}$) gives the A_3 roots and fundamental weights, orthogonal also to $\sum_i e_i - 5e_5$. This pattern generalizes to an interpolation between the A_n roots and the A_{n-1} roots and fundamental weights.

What about deformations of other root or weight systems? We give two more prominent examples in $n = 3$ dimensions. First, consider the $p = 9$ positive roots of B_3 and observe that, from four copies of them, we can assemble the edges of a truncated cube. It is possible to deform the latter into a truncated cuboid while keeping the orthogonalities and producing a six-parameter family of covectors,

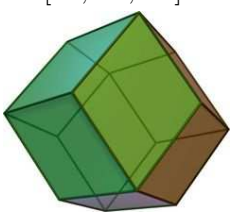
$$\{\alpha \cdot x\} = \{d_1x^1,\ d_2x^2,\ d_3x^3;\ c_3(c_2x^1 \pm c_1x^2),\ c_1(c_3x^2 \pm c_2x^3),\ c_2(c_1x^3 \pm c_3x^1)\}, \quad c_i, d_i \in \mathbb{R}.$$

The normalization $\sum_\alpha f_\alpha \alpha \otimes \alpha = \mathbb{1}$ can be achieved with

$$\{f_\alpha\} = \left\{ \frac{c_0^2 + c_1^2 - c_2^2 - c_3^2}{c^2\ d_1^2}, \frac{c_0^2 - c_1^2 + c_2^2 - c_3^2}{c^2\ d_1^2}, \frac{c_0^2 - c_1^2 - c_2^2 + c_3^2}{c^2\ d_1^2}; \frac{1}{c^2\ c_3^2}, \frac{1}{c^2\ c_1^2}, \frac{1}{c^2\ c_2^2} \right\},$$
$$c^2 = c_0^2 + c_1^2 + c_2^2 + c_3^2.$$

One sees that the relevant combinations $\sqrt{f_\alpha}\alpha$ depend only the three ratios $\frac{c_i}{c_0}$. It turns out that we have constructed a three-dimensional moduli space of WDVV solutions [17, 18, 19].

Second, again using A_3, it is possible to combine four copies of its three positive vector weights with six copies of its four positive spinor weights to the edge set of a rhombic dodecahedron, with each rhombic face being dissected into two triangles. There exists a three-parameter family of deformations in line with the orthogonalities, given by



$$\alpha \cdot x = d_1x^1, \qquad \beta \cdot x = d_2x^2, \qquad \gamma \cdot x = d_3x^3;$$
$$\frac{\alpha + \beta + \gamma}{2}, \qquad \frac{\alpha - \beta - \gamma}{2}, \qquad \frac{-\alpha + \beta - \gamma}{2}, \qquad \frac{-\alpha - \beta + \gamma}{2},$$

and re-normalization is achieved by

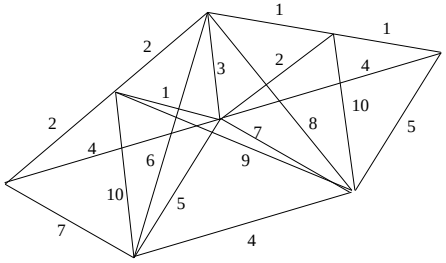
$$f_\alpha = \frac{-d_1^2+d_2^2+d_3^2}{d^2 d_1^2}, \qquad f_\beta = \frac{d_1^2-d_2^2+d_3^2}{d^2 d_2^2}, \qquad f_\gamma = \frac{d_1^2+d_2^2-d_3^2}{d^2 d_3^2};$$
$$f_{\text{spinor}} = \frac{2}{d^2}, \qquad d^2 = d_1^2+d_2^2+d_3^2.$$

In this case, the combinations $\sqrt{f_\alpha}\alpha$ depend only on the ratios $\frac{d_i}{d_j}$, and we again discover a two-dimensional family of WDVV solutions [18, 19]. It seems that indeed the polytope's incidence relations imply the WDVV equation, thus allowing us to construct solutions F purely geometrically, by guessing appropriate polytopes with certain edge multiplicities.

11 Hypergraphs

Sadly, our ortho-polytope concept fails, as may be seen from the first counterexample at $(n,p) = (3,10)$:

$$A = \frac{1}{4\sqrt{3}} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} \\ \begin{pmatrix} 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} & \sqrt{6} & -\sqrt{6} & 0 & 0 \\ 2\sqrt{2} & -2\sqrt{2} & 0 & 4 & \sqrt{3} & \sqrt{3} & -1 & -1 & -\sqrt{6} & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 3 & 3 & \sqrt{6} & 3\sqrt{2} \end{pmatrix} \end{array} \tag{24}$$

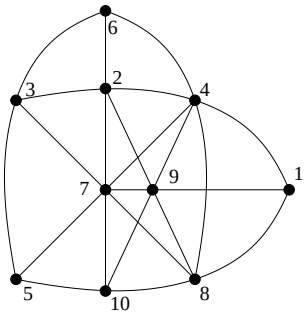


is properly normalized, $AA^\top = \mathbb{1}_3$, and may be checked to fulfil the partial-isometry conditions (22) for each nontrivial plane. It turns out, however, that there exists no polyhedron whose edges are built from (suitably rescaled copies of) all ten column vectors in (24). In the attempt shown to the left, one of the would-be edges (labelled '9') runs *inside* the convex hull created by the others.

This lesson demonstrates that it may be better to restrict ourselves to the essential feature of A, which is the coplanarity property of its columns α. Even though there is not always an ortho-polytope, we may still hope that each $n \times p$ matrix A with

$$AA^\top = \mathbb{1}_n \qquad \text{and} \qquad q(\alpha_a \wedge \alpha_b) = 2 \quad \Rightarrow \quad \alpha_a \cdot \alpha_b = 0 \qquad \forall\, a,b = 1,\ldots,p$$

already obeys the crucial conditions (22) for all $q > 2$ planes.



Suppose we have m nontrivial planes and label them by $\mu = 1,\ldots,m$. The $q_\mu > 2$ covectors in the plane π_μ are grouped in the subset $\{\alpha_{a_s^\mu}\} \subset \{\alpha_a\}$, with $s = 1,\ldots,q_\mu$. A shorter way of encoding this coplanarity information is by using only the *labels* rather than denoting the covectors. Thus, we combine the a_s^μ for each nontrivial plane π_μ in the subset $\{a_s^\mu | s = 1,\ldots,q_\mu\} =: \Pi_\mu \subset \{1,\ldots,p\}$, and then write down the collection $H(A) := \{\Pi_1, \Pi_2, \ldots, \Pi_m\} \subset \mathcal{P}(\{1,\ldots,p\})$ of these (overlapping) subsets. Such subset collections are known as *simple hypergraphs* [30]. They are graphically represented by writing a vertex for each covector label and then, for each μ, by connecting all vertices whose labels occur in Π_μ. The resulting graph has p vertices and m connections Π_μ, called *hyperedges*. Note that a vertex represents a covector, and a hyperedge stands for a (nontrivial) plane, thus gaining us one dimension in drawing[6]. As an example, the hypergraph for (24) reads

[6] Our simple hypergraphs contain only q_μ-vertex hyperedges with $q_\mu > 2$, hence no one- or two-vertex hyperedges.

{{1234}{1580}{2670}{179}{289}{356}{378}{457}{468}{490}} and is represented above (with '0'='10'). To the mathematically inclined reader, we note that our simple hypergraphs are not of the most general kind: they are also

- *linear*: the intersection of two hyperedges has at most one vertex (uniqueness of planes)
- *irreducible*: the hypergraph is connected (the covector set does not decompose)
- *complete*: when adding the $q_\mu = 2$ planes, each vertex pair is contained in a hyperedge
- *orthogonal*: a nonconnected vertex pair is 'orthogonal' (property of the $q_\mu = 2$ planes)

Of course, two hypergraphs related by a permutation of labels are equivalent. Thus, our program is to construct, for a given value of p, all orthogonal complete irreducible linear simple hypergraphs and check the partial-isometry conditions (22) for each plane π. Unfortunately, this is not so easy, because the orthogonality is not a natural hypergraph property but depends on the dimension n of a possible covector realization. In fact, it is not guaranteed that such a realization exists at all. Therefore, the classification of complete irreducible linear simple hypergraphs with p vertices has to be amended by the construction of the corresponding covector sets in $\mathbb{R}^n$, subject to the orthogonality condition.

12 Matroids

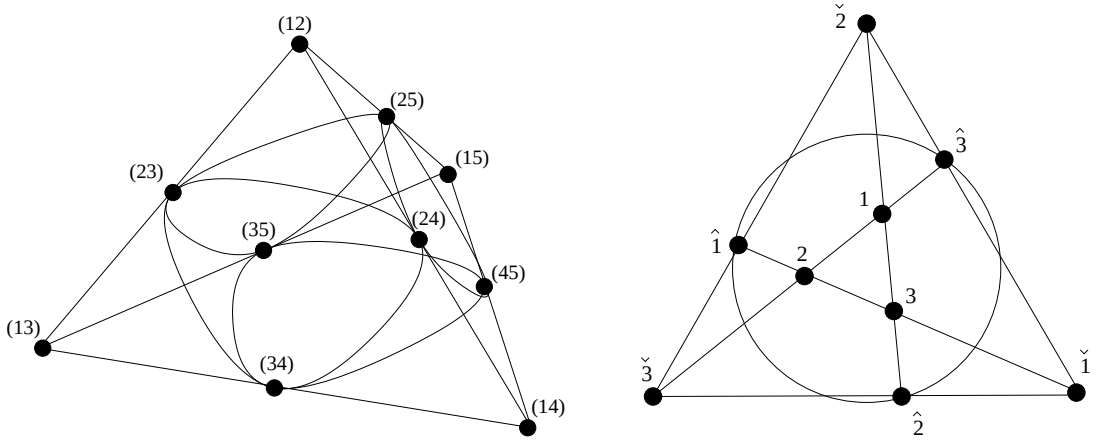
Luckily, there is another mathematical concept which abstractly captures the linear dependence in a subset of a power set, namely the notion of a *matroid* [31, 32, 33]. There exist several equivalent definitions of a matroid, for example as the collection $\{C_\mu\}$ of all *circuits* $C_\mu \subset \{1,\dots,p\}$, which are the minimal dependent subsets of our ground set $\{1,\dots,p\}$:

- The empty set is not a circuit.
- No circuit is contained in another circuit.
- If $C_1 \neq C_2$ share an element e, then $(C_1 \cup C_2)\backslash\{e\}$ is or contains another circuit.

Of course, we identify matroids related by permutations of the ground set.

The idea is that each circuit corresponds to a subset of linearly dependent covectors. Indeed, every $n \times p$ matrix A produces a matroid. However, the converse is false: not every matroid is representable in some $\mathbb{R}^n$. If so, it is called an $\mathbb{R}$-*vector* matroid, with rank $r \leq n$. The rank $r_\mu = |C_\mu| - 1$ of an individual circuit C_μ is the dimension of the vector space spanned by its covectors. Excluding one- and two-element circuits qualifies our matroids as *simple*. It may happen that two rank-d circuits span the same vector space, for example if they agree in d of their elements. Hence, it is useful to unite all rank-d circuits spanning the same d-dimensional subspace in a so-called *d-flat* F_d, with $2 \leq d < r$. We call such a d-flat *minimal* if it arises from a single circuit, i.e. $|F_d| = d + 1$. In this way, we may label the matroid more efficiently by listing all 2-flats, 3-flats etc., all the way up to $r - 1$. Needless to say, we are only interested in *connected* matroids, i.e. those which do not decompose as a direct sum. Also, for a given dimension n we study only $\mathbb{R}$-vector matroids of rank $r = n$ and ignore those of smaller rank, since they can already be represented in a smaller vector space. Finally, we need to implement the orthogonality property. So let us call a matroid *orthogonal*, if any pair of covectors which does not share a 2-flat is orthogonal. Note that further orthogonalities (inside 2-flats) may be enforced by the representation.

A matroid of rank r can be represented geometrically in $\mathbb{R}^{r-1}$ as follows. Mark a node for every element of the ground set (the covectors). Then, connect by a line all covectors in one 2-flat, for all 2-flats. Next, draw a two-surface containing all covectors in one 3-flat, for all 3-flats, and so on. We illustrate this method on two examples, the A_4 and the B_3 matroid:



The A_4 case has $r = 4$, and it is natural to label the ten covectors by pairs (ij), with $1 \le i < j \le 5$. Then,

$$\{C_\mu\} = \{\{(ij)(ik)(jk)\},\{(ij)(ik)(j\ell)(k\ell)\},\{(ij)(i\ell)(jk)(k\ell)\},\{(ik)(i\ell)(jk)(j\ell)\},$$
$$\{(1i)\tbinom{i}{j}\tbinom{j}{k}\tbinom{k}{\ell}(1\ell)\}\} \qquad \text{with} \quad \tbinom{i}{j} = (ij) \text{ or } (j\,i)$$

lists ten circuits of rank 2, fifteen circuits of rank 3 and twelve circuits of rank 4. The former represent ten 2-flats, the middle unite in triples to five 3-flats and the latter combine to the trivial 4-flat,

$$\{F_2\} = \{\{(ij)(ik)(jk)\}\},$$
$$\{F_3\} = \{\{(ij)(ik)(i\ell)(jk)(j\ell)(k\ell)\}\},$$
$$\{F_4\} = \{\{(12)(13)(14)(15)(23)(24)(25)(34)(35)(45)\}\}.$$

Orthogonality is required between pairs with fully distinct labels. The B_3 example is of rank three but less symmetric. We label the three short roots by i and the six long ones by $\hat{\imath}$ and $\check{\imath}$, with $i = 1, 2, 3$, and obtain sixteen rank-2 circuits grouping into seven 2-flats and thirty-nine rank-3 circuits combining into the unique 3-flat ($i \ne j \ne k \ne i$),

$$\{C_\mu\} = \{\{ij\hat{k}\},\{ij\check{k}\},\{i\hat{j}\check{j}\},\{\hat{i}\hat{j}\hat{k}\},\{\hat{i}\check{j}\check{k}\},$$
$$\{ij\,\hat{i}\hat{j}\},\{ij\,\check{i}\check{j}\},\{ij\,\hat{i}\check{j}\},\{i\,\hat{i}\,\check{i}\hat{j}\},\{i\,\hat{i}\,\check{i}\check{j}\},\{i\,\hat{i}\hat{j}\check{k}\},\{i\,\check{i}\hat{j}\hat{k}\},\{i\,\check{i}\check{j}\check{k}\},\{\hat{i}\,\check{i}\hat{j}\check{j}\}\},$$
$$\{F_2\} = \{\{1,2,\hat{3},\check{3}\},\{1,3,\hat{2},\check{2}\},\{2,3,\hat{1},\check{1}\},\{\hat{1},\hat{2},\hat{3}\},\{\hat{1},\check{2},\check{3}\},\{\check{1},\hat{2},\check{3}\},\{\check{1},\check{2},\hat{3}\}\},$$
$$\{F_3\} = \{\{1,2,3,\hat{1},\hat{2},\hat{3},\check{1},\check{2},\check{3}\}\}.$$

Here, we see that $i \perp \hat{\imath}$ and $i \perp \check{\imath}$, but the realization in $\mathbb{R}^3$ actually enforces $i \perp j$ as well.

The task then is to classify all connected simple orthogonal $\mathbb{R}$-vector matroids for given data (n, p). There are tables in the literature which, however, do not select for orthogonality. Another disadvantage is the fact that matroids capture linear dependencies of covector subsets at any rank up to r, while the WDVV equation sees only coplanarities. Therefore, it is enough to write down only the 2-flats, which brings us back to the complete irreducible linear simple hypergraphs again. Still, the advantage of matroids over hypergraphs is that they provide a natural setting for the orthogonality property and the partial-isometry condition (22). Once we have constructed a parametric representation of an $\mathbb{R}$-vector matroid as a family of $n \times p$ matrices A, we may implement the orthogonalities and directly test (22) for all nontrivial planes π. A *good matroid* is one which passes the test and thus yields a (family of) solution(s) to the WDVV equation.

Another bonus is the possibility to reduce a good matroid to a smaller good one by graphical methods. The two fundamental operations on a matroid M are the deletion and the contraction

of an element $a \in \{1, \ldots, p\}$ (corresponding to a covector). In the geometrical representation these look as follows

- *deletion* of a, denoted $M\backslash\{a\}$: remove the node a and all minimal d-flats it is part of
- *contraction* of a, denoted $M/\{a\}$: remove the node a and identify all nodes on a line with a, then remove the loops and identify the multiple lines created

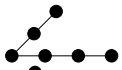
Both operations reduce p by one. Deletion keeps the rank while contraction lowers it by one. On the matrix A, the former means removing the column a (corresponding to the covector α_a) while the latter in addition projects orthogonal to α_a. Connectedness has to be rechecked after deletion, but simplicity and the $\mathbb{R}$-vector property are hereditary for both actions! Furthermore, contraction preserves the orthogonality, but deletion may produce a non-orthogonal matroid. Since the contraction of a good matroid corresponds precisely to the restriction of $\vee$-systems introduced by [18, 19], we are confident that it generates another good matroid. A similar statement holds for the multiple deletion which produces a $\vee$-subsystem in the language of [18, 19].

The first nontrivial dimension is $n = r = 3$, where simple matroids (determined by $\{F_2\}$) are identical with complete linear simple hypergraphs (given by $\{H_\mu\}$). Their number grows rapidly with the cardinality p:

number p of covectors	2	3	4	5	6	7	8	9	10	11	12
how many simple matroids?	0	1	2	4	9	23	68	383	5249	232928	28872972
of the above are connected	0	0	0	1	3	12	41	307	4844	227612	28639649
of the above are $\mathbb{R}$-vector	0	0	0	1	3	11	38	?	?	?	?
of the above are orthogonal	0	0	0	0	1	1	1	1	3	?	?

Below, we list all good (✓) and a few bad (↯) cases up to $p = 10$, with graphical/geometric representation and the name of the corresponding root system. Parameters s, t, u indicate continuous moduli.

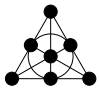
$\{\{123\},\{145\}\}$  $A_3\backslash\{6\}$ $\mathbb{R}$-vector but not orthogonal ↯

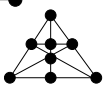
$\{\{123\},\{1456\}\}$  $B_3\backslash\{4,5,9\}$ $\mathbb{R}$-vector but not orthogonal ↯

$\{\{123\},\{145\},\{356\}\}$  $D(2,1;\alpha)\backslash\{7\}$ $\mathbb{R}$-vector but not orthogonal ↯

$\{\{123\},\{145\},\{356\},\{246\}\}$  =  $A_3(s,t,u)$ ✓

$\{\{123\},\{145\},\{356\},\{347\},\{257\},\{167\}\}$  $\underline{6} \oplus \underline{4}$ of A_3 = $D(2,1;\alpha)(s,t)$ ✓

$\{\{123\},\{145\},\{356\},\{347\},\{257\},\{167\},\{246\}\}$  Fano matroid – not $\mathbb{R}$-vector ↯

$\{\{123\},\{145\},\{356\},\{347\},\{257\},\{248\},\{1678\}\}$  $B_3\backslash\{9\}(s,t)$ ✓

$\{\{123\},\{145\},\{347\},\{257\},\{2489\},\{1678\},\{3569\}\}$  $B_3(s,t,u)$ ✓

$\{\{150\}\{167\}\{259\}\{268\}\{456\}\{479\}\{480\}\{1234\}\{3578\}\{3690\}\}$  $\subset AB(1,3)(t)$ ✓

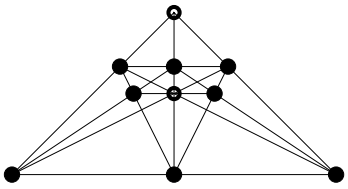
$\{\{179\}\{289\}\{356\}\{378\}\{457\}\{468\}\{490\}\{1234\}\{1580\}\{2670\}\}$  $\subset AB(1,3)(t)$ ✓

The last two lines (with $p = 10$) arise from restrictions of a one-parameter deformation of the $p = 18$ exceptional Lie superalgebra $AB(1,3)$ root system [18]. More precisely, the first of these

two cases is rigid and can also be obtained from the E_6 roots, while the second case retains the deformation parameter.

We have developed a Mathematica program which automatically generates all hypergraphs subject to the simplicity, linearity, completeness and irreducibility properties up to a given p. Furthermore, hypergraphs that admit no orthogonal covector realization are ruled out, thereby drastically reducing their number. For a generated hypergraph we then gradually build a parametrization of the most general admissible set of covectors whereby it turns out whether the hypergraph is representable. A major step forward would be to completely automate this process also; we are confident that this is feasible. Finally, on the surviving families $A(s,t,\ldots)$ of covector sets, the program tests the partial-isometry property (22) equivalent to the WDVV equation, for all nontrivial planes π.

A natural conjecture is that our class of hypergraphs or matroids always produces WDVV solutions, rendering this final test obsolete. However, running the program for a while reveals a counterexample at $(n,p)=(3,10)$, given by the hypergraph to the right. In this diagram, the hollow nodes indicate additional orthogonality inside a plane spanned by four covectors. We must conclude that a geometric construction of WDVV solutions is still missing.



Although the connected simple orthogonal $\mathbb{R}$-vector matroids are not classified and the WDVV property does not automatically follow from such a matroid, this approach is still useful in exhausting all covector solutions for a small number of covectors at low dimension, i.e. for a limited number of particles. In this way one of us has, in fact, proven [34, 35] that there are no other four-particle solutions ($n=3$) with $p \leq 10$ beyond those determined in [18, 19]. The matroid itself does not capture the moduli space of solutions with a given linear dependence structure, but its systematic realization by an iterative algorithm will do so (as it did for $n=3$). Around a given solution, the local moduli space may be probed by investigating the zero modes of the WDVV equation linearized around it.

13 Summary

We begin by listing the main points of this article:

- $\mathcal{N}$=4 superconformal n-particle mechanics in $d=1$ is governed by U and F
- U and F are subject to inhomogeneity, Killing-type and WDVV conditions
- a geometric interpretation via flat superpotentials gave new variants of the integrability
- there is a structural similarity to flat and exact Yang–Mills connections
- the general 3-particle system is constructed, with three couplings and one free function
- higher-particle systems exist, tedious to construct; hypergeometric functions appear
- the covector ansatz for F leads to partial isometry conditions with multipliers λ_π
- finite Coxeter root systems and certain deformations thereof yield WDVV solutions
- certain solution families admit an ortho-polytope interpretation
- hypergraphs and matroids are suitable concepts for a classification of WDVV solutions
- the generation of candidates can be computer programmed
- not all connected simple orthogonal $\mathbb{R}$-vector matroids are 'good'

There remain a lot of open questions. First, can our hypergraph/matroid construction program detect new WDVV solutions not already in the list of [18, 19]? Second, given a 'good'

matroid, can we generate its moduli space, e.g. by linearizing the WDVV equation around it? Third, the explicit Hamiltonian of the $\mathcal{N}$=4 four-particle Calogero system is still unknown. Fourth, can one construct $\boldsymbol{u}$ as a path-ordered exponential of $\mathrm{d}\boldsymbol{f}$ in a practical way? Fifth, what happens if we allow for twisted superfields in the superspace approach? We hope to come back to some of these issues in the future.

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