# Toeplitz Quantization and Asymptotic Expansions: Geometric Construction*

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**Abstract.** For a real symmetric domain $G_{\mathbb{R}}/K_{\mathbb{R}}$, with complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$, we introduce the concept of "star-restriction" (a real analogue of the "star-products" for quantization of Kähler manifolds) and give a geometric construction of the $G_{\mathbb{R}}$-invariant differential operators yielding its asymptotic expansion.

*Key words:* bounded symmetric domain; Toeplitz operator; star product; covariant quantization

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## 1 Introduction

Geometric quantization of (complex) Kähler manifolds is of particular interest for *symmetric* manifolds $B = G/K$ (of compact or non-compact type). In this case the Hilbert state space $H$ carries an irreducible representation of $G$, whereas the various star products (Weyl calculus, Toeplitz–Berezin calculus) describe the (associative) product of *observables* (operators on $H$) as an asymptotic series of $G$-invariant bi-differential operators on $B$.

In this paper we introduce and study similar concepts for *real* symmetric manifolds (of flat or non-compact type), emphasizing the interplay between the real symmetric space and its "hermitification" which is a complex hermitian space (of flat or non-compact type). In general, for a real-analytic manifold $B_{\mathbb{R}}$ of dimension $n$, a complexification $B_{\mathbb{C}}$ is a complex manifold of (complex) dimension $n$, with $B_{\mathbb{R}}$ embedded (real-analytically) as a totally real submanifold [1, 16, 29]. If $B_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$ is a symmetric space, for a real (reductive) Lie group $G_{\mathbb{R}}$ with maximal compact subgroup $K_{\mathbb{R}}$, we write its hermitification as $B_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, where $G_{\mathbb{C}}$ denotes the (real, semi-simple) biholomorphic isometry group and $K_{\mathbb{C}}$ is the maximal compact subgroup. Thus, contrary to the usual notational conventions, $G_{\mathbb{C}}$ is *not* the complexification of $G_{\mathbb{R}}$ but the real Lie group "in the complex setting". For example, if $G_{\mathbb{R}} = SU(1,1)$ then $G_{\mathbb{C}}$ is given by $SU(1,1) \times SU(1,1)$ instead of $SL(2,\mathbb{C})$; similarly, for $G_{\mathbb{R}} = SO(1,1)$ we have $G_{\mathbb{C}} = SU(1,1)$.

On the level of states, the interplay between a real symmetric space $B_{\mathbb{R}} = G_{\mathbb{R}}/K_{\mathbb{R}}$ and its hermitification $B_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ corresponds to a "real-wave" realization of $H_{\mathbb{C}}$ via a Segal–Bargmann transformation [37], which is invariant under the subgroup $G_{\mathbb{R}} \subset G_{\mathbb{C}}$. On the other hand, the real analogue of the star-product is not so obvious. In this paper (and its companion paper [22]) we introduce such a concept, called "star-restriction" for real symmetric domains

*This paper is a contribution to the Special Issue on Deformation Quantization. The full collection is available at http://www.emis.de/journals/SIGMA/Deformation_Quantization.html of non-compact type and study its asymptotic expansion as a series of $G_{\mathbb{R}}$-invariant differential operators. Whereas the paper [22] establishes existence and uniqueness of the asymptotic expansion, closely related to spectral theory and harmonic analysis (spherical functions), the current paper gives a "geometric construction" of the differential operators involved, based on a $G_{\mathbb{R}}$-invariant retraction $\pi : B_{\mathbb{C}} \to B_{\mathbb{R}}$.

We emphasize that our $*$-restriction operator is a $G_{\mathbb{R}}$-equivariant map

$$\mathcal{C}^\infty(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$$

instead of a map $\mathcal{C}^\infty(B_{\mathbb{R}}) \otimes \mathcal{C}^\infty(B_{\mathbb{R}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$ analogous to the usual $*$-products. Thus we do not propose a quantization method for general real symmetric domains (which may not be symplectic nor even dimensional) but instead consider invariant operators which somewhat resemble boundary restriction operators such as Szegö or Poisson kernel integrals. In case $B_{\mathbb{R}}$ is the underlying real manifold of a complex hermitian domain $B$, then both concepts coincide and indeed yield the well-known covariant quantization methods applied to the Kähler manifold $B$.

In order to illustrate the two concepts, consider the simplest non-flat case of the open unit disk $B \subset \mathbb{C}$ and its real form $B_{\mathbb{R}} = (-1, 1) \subset \mathbb{R}$. The complexification $B_{\mathbb{R}}^{\mathbb{C}}$ coincides with $B$, and we have a restriction operator $\rho$, mapping a smooth function $f$ on $B = B_{\mathbb{R}}^{\mathbb{C}}$ to its restriction $\rho f$ on $B_{\mathbb{R}}$. A star-restriction is a *deformation* of the operator $\rho$, obtained by adding smooth, but non-holomorphic, differential operators on $B$ as higher order terms. In the context of symmetric domains, these differential operators should be invariant under the subgroup $G_{\mathbb{R}}$ of the holomorphic automorphism group $G$ of $B$ which leaves $B_{\mathbb{R}}$ invariant.

Now consider instead the (usual) complex situation. Here $B$ is regarded as a *real* (symplectic) manifold, denoted by $B^{\mathbb{R}}$, whose complexification $B_{\mathbb{C}}^{\mathbb{R}}$ is the product of $B$ and its complex conjugate $\overline{B}$, with $B^{\mathbb{R}}$ embedded as the diagonal. Then a star-product, regarded as a bilinear operator acting on $f \otimes g$ (with $f$, $g$ smooth functions on $B$), is precisely a deformation of the usual product $f \cdot g$ by ($G$-invariant) bi-differential operators on $B$ or, equivalently, differential operators on $B_{\mathbb{C}}^{\mathbb{R}} = B \times \overline{B}$. Since $f \cdot g$ is nothing but the restriction of $f \otimes g$ to the diagonal $B^{\mathbb{R}} \subset B_{\mathbb{C}}^{\mathbb{R}}$, we see that the concept of star-restriction yields in fact the star-product for the special case where the complexified domain is of product type. The higher-dimensional case is analogous.

In order to state our main result concerning the asymptotic expansion (in the deformation parameter $\nu$) of a $*$-restriction operator as above, we first note that for the basic Toeplitz–Berezin calculus (the only case considered in detail here) the $*$-restriction operator is trivial for anti-holomorphic functions so that we may concentrate on the holomorphic part, which is a $G_{\mathbb{R}}$-covariant map

$$\rho_\nu : \ \mathcal{O}(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}}).$$

Using deep facts from representation theory (of the compact Lie groups $K_{\mathbb{R}}$ and $K_{\mathbb{C}}$), we construct a family of differential operators

$$\rho^{\mathbf{m}} : \ \mathcal{O}(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$$

indexed by integer partitions $m_1 \geq \cdots \geq m_r \geq 0$ (cf. Definition 3.1), and (in Theorem 3.1) express $\rho_\nu$ as an asymptotic series

$$\rho_\nu \sim \sum_{\mathbf{m}} \frac{1}{[\nu]_{\mathbf{m}}} \, \rho^{\mathbf{m}}, \tag{1.1}$$

where the constants $[\nu]_{\mathbf{m}}$ are generalized Pochhammer symbols.

Using the Fourier–Helgason transform on $B_{\mathbb{R}}$, it is conceivable (see [22] for the details) that $\rho_\nu$ can also be expressed as an oscillatory integral

$$\rho_\nu F(x) \sim \int_{B_{\mathbb{C}}} F(z)\, a_\nu(z,x)\, e^{\nu S(z,x)}\, dz, \tag{1.2}$$

where $a_\nu$ is a suitable power series in $\frac{1}{\nu}$, and the "phase" $S$ is a function on $B_{\mathbb{C}} \times B_{\mathbb{R}}$ invariant under the diagonal action of $G_{\mathbb{R}}$. This is reminiscent of the WKB-quantization programme of Karasev, Weinstein and Zakrzewski [35], studied extensively in the context of symplectic (i.e. not necessarily Hermitian, or even Riemannian) symmetric spaces by Bieliavsky, Pevzner, Gutt, and other authors, see e.g. [11, 10, 12, 13]. As has already been pointed out above, real symmetric domains need not be symplectic (in fact, they can even be of odd dimension), so neither of the two approaches contains the other, and the situations where they both apply include the original Kähler case of an Hermitian symmetric space. A thorough comparison of both methods is, however, beyond the scope of this paper. For the flat cases of $B_{\mathbb{R}} = \mathbb{R}^d$ and $B_{\mathbb{R}} = \mathbb{C}^d$, the expansions (1.2) were obtained quite explicitly, and for a whole one-parameter class of calculi which includes the Toeplitz calculus, by Arazy and the second author [5].

In Section 4 the asymptotic series (1.1) are computed for the simplest cases of (real or complex) dimension 1. In general, finding explicit formulas may be quite difficult, but there is some hope that at least all symmetric domains of rank 1 (i.e., hyperbolic spaces in $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ and the Cayley plane) can be treated in a unified and explicit way.

## 2 Preliminaries

One of the most inspiring examples of deformation quantization is the Berezin quantization [7, 8] using the Berezin transform and Toeplitz operators (originally called co- and contra-variant symbols, respectively). Although it has subsequently been generalized and extended to various classes of compact and noncompact Kähler manifolds [15, 19, 28, 32], the theory is still richest in its original setting of complex symmetric spaces, or bounded symmetric domains, in $\mathbb{C}^d$ [9], due to the powerful machinery available from Lie groups and their representation theory on the one hand [26, 34], and from the theory of Jordan triple systems on the other [30].

More specifically, let $B = G/K$ be an irreducible bounded symmetric domain in $\mathbb{C}^d$ in the Harish-Chandra realization, with $G$ the identity connected component of the group of all biholomorphic self-maps of $B$ and $K$ the stabilizer of the origin. For $\nu > p-1$, $p$ being the genus of $B$, let $H^2_\nu(B)$ denote the standard weighted Bergman space on $B$, i.e. the subspace of all holomorphic functions in $L^2(B, d\mu_\nu)$, with

$$d\mu_\nu(z) = c_\nu\, K(z,z)^{1-\nu/p}\, dz,$$

where $dz$ stands for the Lebesgue measure, $K(z,w)$ is the ordinary (unweighted) Bergman kernel of $B$, and $c_\nu$ is a normalizing constant to make $d\mu_\nu$ a probability measure. The space $H^2_\nu(B)$ carries the unitary representation $U^{(\nu)}$ of $G$ given by

$$U^{(\nu)}_g f(z) = f(g^{-1}(z)) \cdot J_{g^{-1}}(z)^{\nu/p}, \qquad g \in G,\ f \in H^2_\nu(B),$$

where $J_g$ denotes the complex Jacobian of the mapping $g$. (In general, if $\nu/p$ is not an integer, then $U^{(\nu)}$ is only a projective representation due to the ambiguity in the choice of the power $J_{g^{-1}}(z)^{\nu/p}$.)

By a *covariant operator calculus*, or *covariant quantization*, on $B$ one understands a mapping $\mathcal{A} : f \mapsto \mathcal{A}_f$ from functions on $B$ into operators on $H^2_\nu(B)$ which is $G$-covariant in the sense that

$$\mathcal{A}_{f\circ g} = U^{(\nu)*}_g \mathcal{A}_f U^{(\nu)}_g, \qquad \forall\, g \in G.$$

In most cases, such calculi can be built by the recipe

$$\mathcal{A}_f = \int_B f(\zeta)\mathcal{A}_\zeta \, d\mu_0(\zeta)$$

where $d\mu_0$ is a $G$-invariant measure on $B$, and $\mathcal{A}_\zeta$ is a family of operators on $H^2_\nu(B)$ labelled by $\zeta \in B$ such that

$$\mathcal{A}_{g(\zeta)} = U_g^{(\nu)} \mathcal{A}_\zeta U_g^{(\nu)*}, \qquad \forall\, g \in G.$$

(One calls such a family a *covariant operator field* on $B$. One also usually normalizes the measure $d\mu_0$ so that $\mathcal{A}_{\mathbf{1}}$ is the identity operator.) Note that in view of the transitivity of the action of $G$ on $B$, any covariant operator field is uniquely determined by its value $\mathcal{A}_0$ at the origin $\zeta = 0$.

The best known examples of such calculi are the *Toeplitz calculus* $\mathcal{T}$ and the *Weyl calculus* $\mathcal{W}$, corresponding to $\mathcal{T}_0 = (\cdot|\mathbf{1})\mathbf{1}$ (the projection onto the constants) and $\mathcal{W}_0 f(z) = f(-z)$ (the reflection operator), respectively.

In addition to bounded symmetric domains, we will also consider the *complex flat* case of a Hermitian vector space $B = Z \approx \mathbb{C}^d$, with $B = G/K$ for $G$ the group of all orientation-preserving rigid motions of $Z$, and $K = U(Z) \approx U_d(\mathbb{C})$ the stabilizer of the origin in $G$; the spaces $H^2_\nu(Z)$ will then be the Segal–Bargmann spaces of all entire functions which are square-integrable with respect to the Gaussian measure

$$d\mu_\nu(z) = \left(\frac{\nu}{\pi}\right)^d e^{-\nu\|z\|^2}\, dz,$$

and $U^{(\nu)}$ will be the usual Schrödinger representation. In this setting, the Weyl calculus $\mathcal{W}$ above is just the well-known Weyl calculus from the theory of pseudodifferential operators [25].

Given a covariant operator calculus $\mathcal{A}$, the associated *star product* $*$ on functions on $B$ is defined by

$$\mathcal{A}_{f*g} = \mathcal{A}_f \mathcal{A}_g. \tag{2.1}$$

It follows from the construction that the star-product is $G$-*invariant* in the sense that

$$(f \circ \phi) * (g \circ \phi) = (f * g) \circ \phi \qquad \forall\, \phi \in G. \tag{2.2}$$

While $f * g$ is a well-defined object for some calculi (e.g. for $\mathcal{A} = \mathcal{W}$, at least on $\mathbb{C}^d$ and rank one symmetric domains, see [5]), in most cases (e.g. for $\mathcal{A} = \mathcal{T}$, the Toeplitz calculus), it makes sense only for very special functions $f$, $g$ and (2.1) is then usually understood as an equality of asymptotic expansions as the Wallach parameter $\nu$ tends to infinity. For instance, for $\mathcal{A} = \mathcal{T}$, it was shown in [14] that for any $f, g \in \mathcal{C}^\infty(B)$ with compact support,

$$\|\mathcal{T}_f \mathcal{T}_g - \mathcal{T}_{\sum_{j=0}^N \nu^{-j} C_j(f,g)}\| = O\big(\nu^{-N-1}\big)$$

as $\nu \to \infty$, for some bilinear differential operators $C_j$ (not depending on $f$, $g$ and $\nu$). (The assumption of compact support can be relaxed [18].) We can thus define $f * g$ as the formal power series

$$f * g := \sum_{j=0}^\infty \nu^{-j}\, C_j(f,g).$$

Interpreting $\nu$ as the reciprocal of the Planck constant, we recover the Berezin–Toeplitz star product [33], which is the dual to Berezin's original star-product mentioned above [19]. (This approach to the Berezin and Berezin–Toeplitz star-products, i.e. using covariant operator calculi and the definition (2.1), is not the traditional way of constructing the $G$-invariant Berezin quantization, however, for the case of bounded symmetric domains these two are equivalent [20].)

Viewing the Planck parameter $\nu$ as fixed for the moment, the formula (2.2) means that one can view $*$ as a mapping from the tensor product

$$*:\ \mathcal{C}^\infty(B\times B)\cong\mathcal{C}^\infty(B)\otimes\mathcal{C}^\infty(B)\to\mathcal{C}^\infty(B),\qquad f\otimes g\to f*g,$$

which intertwines the $G$-action $f\mapsto f\circ\phi$, $\phi\in G$, on $\mathcal{C}^\infty(B)$ with the diagonal $G$-action $f\otimes g\mapsto(f\circ\phi)\otimes(g\circ\phi)$ of $G$ on $\mathcal{C}^\infty(B\times B)$. This observation can be used as a starting point for extending the whole quantization procedure also to *real* bounded symmetric domains $B_\mathbb{R}\subset\mathbb{R}^d$, as follows.

Suppose $Z_\mathbb{C}$ is an *irreducible* hermitian Jordan triple [30, 34] endowed with a (conjugate-linear) involution

$$z\mapsto z^\#$$

which preserves the Jordan triple product and therefore the unit ball $B_\mathbb{C}$ of $Z_\mathbb{C}$, i.e. $(B_\mathbb{C})^\#=B_\mathbb{C}$. Define the real forms

$$\begin{aligned}Z_\mathbb{R}&:=\{z\in Z_\mathbb{C}:z^\#=z\},\\ B_\mathbb{R}&:=\{z\in B_\mathbb{C}:z^\#=z\}=Z\cap B_\mathbb{C}.\end{aligned}$$

For the groups $G_\mathbb{C}:=\mathrm{Aut}(B_\mathbb{C})$, $K_\mathbb{C}:=\mathrm{Aut}(Z_\mathbb{C})$ we have the subgroups

$$\begin{aligned}G_\mathbb{R}&:=\{g\in G_\mathbb{C}:g(z^\#)=g(z)^\#\},\\ K_\mathbb{R}&:=\{k\in K_\mathbb{C}:kz^\#=(kz)^\#\}=G_\mathbb{R}\cap K_\mathbb{C}\end{aligned}$$

acting on $B_\mathbb{R}$ and $Z_\mathbb{R}$, respectively. In this situation $Z_\mathbb{R}$ is an irreducible real Jordan triple, $G_\mathbb{R}$ is a reductive Lie group (it may have a nontrivial center), and

$$B_\mathbb{R}=G_\mathbb{R}/K_\mathbb{R}$$

is an irreducible real bounded symmetric domain. Up to a few exceptions, all non-hermitian Riemannian symmetric spaces of non-compact type arise in this way [30, Chapter 11].

A *covariant quantization* (or *covariant extension*) on the real bounded symmetric domain $B_\mathbb{R}$ is a map $f\mapsto\mathcal{A}_f$ from $\mathcal{C}^\infty(B_\mathbb{R})$ into $H^2_\nu(B_\mathbb{C})$ such that

$$\mathcal{A}_{f\circ g}=U_g^{(\nu)*}\mathcal{A}_f$$

for all $g\in G_\mathbb{R}$. The counterpart of the star product, associated to a covariant quantization $\mathcal{A}$ on $B_\mathbb{R}$ and a covariant quantization $\mathcal{A}^\mathbb{C}$ on $B_\mathbb{C}$, is the *star restriction*

$$\rho=\rho_\nu:\ \mathcal{C}^\infty(B_\mathbb{C})\to\mathcal{C}^\infty(B_\mathbb{R})$$

defined by

$$\mathcal{A}_{\rho F}=\mathcal{A}^\mathbb{C}_F I,\tag{2.3}$$

where

$$I(z)=I_\nu(z)=K^{(\nu)}(z,z^\#)^{1/2}$$

is the unique $G_\mathbb{R}$-invariant holomorphic function on $B_\mathbb{C}$ satisfying $I(0)=1$. In addition, we will again consider the above construction also in the case of the Segal–Bargmann spaces for an involutive Hermitian vector space $Z_{\mathbb{C}} \approx \mathbb{C}^d$, with the ordinary complex conjugation as the involution $z \mapsto z^{\#}$; thus $B = Z_{\mathbb{R}} \approx \mathbb{R}^d$.

In most cases, covariant extensions can again be constructed by the recipe

$$\mathcal{A}_f = \int_{B_{\mathbb{R}}} f(\zeta)\mathcal{A}_\zeta \, d\mu_0(\zeta),$$

where $d\mu_0$ is the $G_{\mathbb{R}}$-invariant measure in $B_{\mathbb{R}}$, and $\mathcal{A}_\zeta$ is a family of holomorphic functions (not necessarily belonging to $H^2_\nu(B_{\mathbb{C}})$) labelled by $\zeta \in B_{\mathbb{R}}$ which is covariant in the sense that

$$\mathcal{A}_{g(\zeta)} = U_g^{(\nu)}\mathcal{A}_\zeta, \qquad \forall\, g \in G_{\mathbb{R}},\, \zeta \in B_{\mathbb{R}}.$$

As before, one usually normalizes $d\mu_0$ so that $\mathcal{A}_{\mathbf{1}} = I$. The prime example is now the *real Toeplitz calculus* $\mathcal{A} = \mathcal{T}$ corresponding to $\mathcal{A}_0 = \mathbf{1}$ (the function constant one) [36, 31, 17, 6, 3]; there is also a notion of *real Weyl calculus*, but it is more complicated [4].

Here is how the complex hermitian case of a bounded symmetric domain $B \subset \mathbb{C}^d$ from the beginning of this section can be recovered within the more general real framework. Identify $B$ with the "diagonal" domain

$$B^{\mathbb{R}} := \{(z, \overline{z}) : z \in B\} \subset Z^{\mathbb{R}} := \{(z, \overline{z}) : z \in Z\},$$

where the bar indicates that we consider the "conjugate" complex structure for the second component. The complexifications

$$\begin{aligned} B^{\mathbb{R}}_{\mathbb{C}} &= \{(z, \overline{w}) : z, w \in B\} = B \times \overline{B}, \\ Z^{\mathbb{R}}_{\mathbb{C}} &= \{(z, \overline{w}) : z, w \in Z\} = Z \times \overline{Z} \end{aligned}$$

are endowed with the flip involution

$$(z, \overline{w})^{\#} := (w, \overline{z})$$

having fixed points $B^{\mathbb{R}}$ and $Z^{\mathbb{R}}$, respectively. Similarly we can identify $G$, $K$ with the groups

$$\begin{aligned} G^{\mathbb{R}} &:= \{(g, \overline{g}) : g \in G\}, \\ K^{\mathbb{R}} &:= \{(k, \overline{k}) : k \in K\} \end{aligned}$$

which act "diagonally" on $B^{\mathbb{R}}$ and $Z^{\mathbb{R}}$, respectively, and whose complexifications

$$\begin{aligned} G_{\mathbb{C}} &:= \{(g_1, \overline{g}_2) : g_1, g_2 \in G\} = G \times \overline{G}, \\ K_{\mathbb{C}} &:= \{(k_1, \overline{k}_2) : k_1, k_2 \in K\} = K \times \overline{K} \end{aligned}$$

act on $B_{\mathbb{C}}$ and $Z_{\mathbb{C}}$, with $B_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$.

Since $H^2_\nu(B)$ is a reproducing kernel space (with reproducing kernel $K^{(\nu)}(x, y) = h(x, y)^{-\nu}$, where $h(x, y) = [K(x, y)/c_p]^{-1/p}$ is the Jordan determinant polynomial), any bounded linear operator on $H^2_\nu(B)$ is automatically an integral operator: namely,

$$Tf(z) = \int_B f(w)\widetilde{T}(z, w)\, d\mu_\nu(w),$$

with

$$\widetilde{T}(z, w) = \overline{(T^* K^{(\nu)}(\cdot, z))(w)} = (TK^{(\nu)}(\cdot, w)|K^{(\nu)}(\cdot, z)).$$

This follows from the identity $Tf(z) = (Tf|K^{(\nu)}(\cdot, z)) = (f|T^* K^{(\nu)}(\cdot, z))$. In this way, we may identify operators on $H^2_\nu(B)$ with (some) functions on $B \times B$, holomorphic in the first and anti-holomorphic in the second variable; that is, with holomorphic functions on $B_\mathbb{C}$. Upon this identification, the covariant quantization rule $f \mapsto \mathcal{A}_f$ becomes simply a (densely defined) operator $f \mapsto \widetilde{\mathcal{A}}_f$ from $\mathcal{C}^\infty(B_\mathbb{R})$ into the Hilbert space

$$H^2_\nu(B_\mathbb{C}) \approx H^2_\nu(B) \otimes \overline{H^2_\nu(B)}$$

corresponding to the Hilbert–Schmidt operators, and the covariance condition means that $\widetilde{\mathcal{A}}$ is equivariant under $G_\mathbb{R} \approx G$, i.e. intertwines the $G$-action on the former with the diagonal $G$-action on the latter:

$$\widetilde{\mathcal{A}}_{f\circ g} = (U_g^{(\nu)*} \otimes \overline{U_g^{(\nu)*}})\widetilde{\mathcal{A}}_f.$$

Similarly, upon taking $\mathcal{A}^\mathbb{C} = \mathcal{A}\otimes\mathcal{A}$, and identifying pairs $f, g$ of functions on $B$ with the function $F(x,y) = f(x)\overline{g(y)}$ on $B_\mathbb{C}$, (2.3) reduces just to (2.1). Note, however, that the complexified domain $B_\mathbb{C}$ is now no longer irreducible, but of "product type". We will henceforth refer to this situation, i.e. of $B_\mathbb{R} = B$, $B_\mathbb{C} = B \times \overline{B}$ with a bounded symmetric domain $B \subset \mathbb{C}^d$, as the "complex" case.

To each covariant extension (or quantization) $\mathcal{A}$ we can consider its adjoint $\mathcal{A}^*$ from $H^2_\nu(B_\mathbb{C})$ into functions on $B_\mathbb{R}$, defined with respect to the inner products in $H^2_\nu(B_\mathbb{C})$ and $L^2(B_\mathbb{R}, d\mu_0)$. That is,

$$(\mathcal{A}^* f|\phi)_{L^2} = (f|\mathcal{A}_\phi)_\nu, \qquad \forall\, \phi \in L^2(B_\mathbb{R}, d\mu_0), \quad \forall\, f \in H^2_\nu(B_\mathbb{C}). \tag{2.4}$$

One sometimes calls $\mathcal{A}^*$ a *covariant restriction*; this should not be confused with the star-restriction $\rho$, which is a map from $\mathcal{C}^\infty(B_\mathbb{C})$ into functions on $B_\mathbb{R}$.

One can also consider the associated *link transform*, which is the composition $\mathcal{A}^*\mathcal{A}$, a $G_\mathbb{R}$-invariant operator on functions on $B_\mathbb{R}$. In particular, for the Toeplitz calculus $\mathcal{A} = \mathcal{T}$, the link transform

$$\mathcal{T}^*\mathcal{T} =: \mathcal{B}_\nu$$

is the *Berezin transform*, introduced for the 'complex'" case in Berezin's original papers (cf. Section 4 below).

A crucial role in the analysis on complex bounded symmetric domains is played by the Peter–Weyl decomposition of holomorphic functions on $B$ under the composition action $f \mapsto f \circ k$ of the (compact) group $K$. Namely, the vector space $\mathcal{P}$ of all holomorphic polynomials on $\mathbb{C}^d$ decomposes under this action into non-equivalent irreducible components

$$\mathcal{P} = \sum_\mathbf{m} \mathcal{P}_\mathbf{m}$$

labelled by *partitions* (or *signatures*) $\mathbf{m} \in \mathbb{N}^r_+$, that is, by $r$-tuples of integers $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$, where $r$ is the rank of $B$. With respect to the Fock inner product

$$(p|q)_F := \int_{\mathbb{C}^d} p(z)\overline{q(z)}e^{-|z|^2}\,dz = p(\partial)q^*(0), \qquad q^*(z) := \overline{q(z\#)},$$

each Peter–Weyl space $\mathcal{P}_\mathbf{m}$ possesses a reproducing kernel $K_\mathbf{m}(z,w)$, $z, w \in Z$. It was shown by Arazy and Ørsted [2] that the Berezin transform $\mathcal{B}_\nu$ admits the asymptotic expansion

$$\mathcal{B}_\nu = \sum_\mathbf{m} \frac{\mathcal{E}_\mathbf{m}}{(\nu)_\mathbf{m}} \qquad \text{as} \quad \nu \to +\infty,$$

where $\mathcal{E}_{\mathbf{m}}$ is the $G$-invariant differential operator on $B$ determined (uniquely) by the requirement that

$$\mathcal{E}_{\mathbf{m}} f(0) = K_{\mathbf{m}}(\partial, \partial) f(0), \qquad \forall\, f \in \mathcal{C}^\infty(B);$$

while $(\nu)_{\mathbf{m}}$ is the multi-Pochhammer symbol

$$(\nu)_{\mathbf{m}} = \prod_{j=1}^{r} \frac{\Gamma(\nu - \frac{a}{2}(j-1) + m_j)}{\Gamma(\nu - \frac{a}{2}(j-1))},$$

$a$ being the so-called *characteristic multiplicity* of $B$. Analogously, it was shown in [18] that the star-product (2.1) arising from the Toeplitz calculus $\mathcal{A} = \mathcal{T}$ admits an expansion

$$f * g = \sum_{\mathbf{m}} \frac{A_{\mathbf{m}}(f, g)}{(\nu)_{\mathbf{m}}}, \tag{2.5}$$

where $A_{\mathbf{m}}$ are certain (rather complicated) $G$-invariant (cf. (2.2)) bi-differential operators.

The main purpose of the present paper is an extension of the last formula to real symmetric domains. That is, to obtain a decomposition of the star restriction operator

$$\rho_\nu = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{[\nu]_{\mathbf{m}}} \tag{2.6}$$

with some $G_{\mathbb{R}}$-invariant differential operators $\rho^{\mathbf{m}} : \mathcal{C}^\infty(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$ (independent of $\nu$) and generalized “Pochhammer symbols” $[\nu]_{\mathbf{m}}$.

A prominent role in our analysis is played by holomorphic polynomials on $Z_{\mathbb{C}}$ which are invariant under the group $K_{\mathbb{R}}$. In the Peter–Weyl decomposition under $K_{\mathbb{C}}$ mentioned above, partitions $\mathbf{n}$ for which $\mathcal{P}_{\mathbf{n}}$ contains a nonzero $K_{\mathbb{R}}$-invariant vector are called “even”, and are in one-to-one correspondence with partitions $\mathbf{m}$ of length $r_{\mathbb{R}} = \operatorname{rank} B_{\mathbb{R}}$; furthermore, for each “even” Peter–Weyl space the subspace of $K_{\mathbb{R}}$-invariant vectors is one dimensional, consisting only of multiples of a certain polynomial which (under an appropriate normalization) we denote by $E^{\mathbf{m}}$. For more details, including the description of $E^{\mathbf{m}}$, bibliographic references, etc., as well as for the various preliminaries and notation not introduced here, we refer to [36, 22].

The construction of the decomposition (2.6) for general real symmetric domains is carried out in Section 3. In Section 4 it is shown that the decomposition obtained indeed reduces to (2.5) for the “complex” case. The final Section 5 contains a few examples with more or less explicit formulas for $\rho^{\mathbf{m}}$ and $[\nu]_{\mathbf{m}}$. For the reader's convenience, we are also attaching a table of all real bounded symmetric domains and their various parameters.

In some sense, our results can be perceived as a step towards building a version of Berezin's quantization for real (as opposed to Kähler) manifolds as phase spaces.

## 3 Invariant retractions and Moyal restrictions

As a first step towards a geometric construction of asymptotic expansions for the Moyal type restriction, we obtain an integral representation for the Moyal restriction operator, defined in terms of $G_{\mathbb{R}}$-invariant retractions

$$\pi : \; B_{\mathbb{C}} \to B_{\mathbb{R}}.$$

Here $B_{\mathbb{R}}$ is an irreducible real symmetric domain of rank $r$, in its bounded realization (real Cartan domain) and $B_{\mathbb{C}}$ is the open unit ball of the complexification $Z_{\mathbb{C}}$, which is a complex hermitian bounded symmetric domain, not necessarily irreducible [30, 24, 34, 27].

We will assume that the preimage $\pi^{-1}(0)$ of the origin $0 \in B_{\mathbb{R}}$ has the form

$$\pi^{-1}(0) = B_{\mathbb{C}} \cap Y = \Lambda B_{\mathbb{R}} \tag{3.1}$$

for some real vector subspace $Y \subset B_{\mathbb{C}}$ and real-linear $K_{\mathbb{R}}$-invariant map $\Lambda : Z_{\mathbb{R}} \to Z_{\mathbb{C}}$. Our construction in fact works even without these assumptions (cf. Remark 3.1 below), but all situations studied in this paper will be of the above form.

Let $h_{\mathbb{C}} : Z_{\mathbb{C}} \times \overline{Z}_{\mathbb{C}} \to \mathbb{C}$ denote the Jordan triple determinant (cf. [30]) of $Z_{\mathbb{C}}$ and define the *Berezin kernel*

$$\mathcal{B}_\nu : \ B_{\mathbb{C}} \to \mathbb{C}$$

by

$$\mathcal{B}_\nu(z) := h_{\mathbb{C}}(z,z)^\nu / |h_{\mathbb{C}}(z,z^\sharp)|^\nu, \tag{3.2}$$

where $z \mapsto z^\sharp$ is the involution with real form $B_{\mathbb{R}}$. Note that $h_{\mathbb{C}}(z,w) \neq 0$ for all $z, w \in B_{\mathbb{C}}$.

**Proposition 3.1.** *The Berezin kernel $\mathcal{B}_\nu$ is $G_{\mathbb{R}}$-invariant, i.e.,*

$$\mathcal{B}_\nu(gz) = \mathcal{B}_\nu(z)$$

*for all $g \in G_{\mathbb{R}}$ and $z \in B_{\mathbb{C}}$.*

**Proof.** Since $G_{\mathbb{R}} \subset G_{\mathbb{C}}$ we have

$$h_{\mathbb{C}}(gz, gw)^\nu = j_\nu(g,z)\ h_{\mathbb{C}}(z,w)^\nu\, \overline{j_\nu(g,w)}$$

for all $z, w \in B_{\mathbb{C}}$, where

$$j_\nu(g,z) = [\det g'(z)]^{\nu/p}$$

and $p$ is the (complex) genus of $B_{\mathbb{C}}$. For $g \in G_{\mathbb{R}}$ we have

$$g(z)^\sharp = g(z^\sharp)$$

and

$$\overline{j_\nu(g,z)} = j_\nu(g,z^\sharp)$$

since these relations are anti-holomorphic in $z \in B_{\mathbb{C}}$ and hold for $z = z^\sharp$. It follows that

$$\begin{aligned}
\frac{h_{\mathbb{C}}(gz,gz)^\nu}{|h_{\mathbb{C}}(gz,(gz)^\sharp)|^\nu} &= \frac{h_{\mathbb{C}}(gz,gz)^\nu}{h_{\mathbb{C}}(gz,g(z^\sharp))^{\nu/2}\, h_{\mathbb{C}}(g(z^\sharp),gz)^{\nu/2}} \\
&= \frac{j_\nu(g,z)\ h_{\mathbb{C}}(z,z)^\nu\ \overline{j_\nu(g,z)}}{j_{\nu/2}(g,z)\ h_{\mathbb{C}}(z,z^\sharp)^{\nu/2}\ \overline{j_{\nu/2}(g,z^\sharp)}\ j_{\nu/2}(g,z^\sharp)\ h_{\mathbb{C}}(z^\sharp,z)^{\nu/2}\ \overline{j_{\nu/2}(g,z)}} \\
&= \frac{h_{\mathbb{C}}(z,z)^\nu}{|h_{\mathbb{C}}(z,z^\sharp)|^\nu}\ \frac{j_{\nu/2}(g,z)\ \overline{j_{\nu/2}(g,z)}}{\overline{j_{\nu/2}(g,z^\sharp)}\ j_{\nu/2}(g,z^\sharp)} = \frac{h_{\mathbb{C}}(z,z)^\nu}{|h_{\mathbb{C}}(z,z^\sharp)|^\nu}.
\end{aligned}$$

Another proof can be given by observing that, using the familiar transformation rule for $h_{\mathbb{C}}$,

$$\frac{h_{\mathbb{C}}(gz,gz)}{|h_{\mathbb{C}}(gz,gz^\#)|} = \frac{\dfrac{h_{\mathbb{C}}(z,z)h_{\mathbb{C}}(a,a)}{|h_{\mathbb{C}}(z,a)|^2}}{\left|\dfrac{h_{\mathbb{C}}(z,z^\#)h_{\mathbb{C}}(a,a)}{h_{\mathbb{C}}(z,a)h_{\mathbb{C}}(a,z^\#)}\right|} = \frac{h_{\mathbb{C}}(z,z)}{|h_{\mathbb{C}}(z,z^\#)|}\left|\frac{h_{\mathbb{C}}(a,z)}{h_{\mathbb{C}}(a,z^\#)}\right.,$$

where $g \in G_{\mathbb{C}}$ and $a = g^{-1}(0)$. If $g \in G_{\mathbb{R}}$, then $gz^\# = (gz)^\#$, while $h_{\mathbb{C}}(a,z^\#) = \overline{h_{\mathbb{C}}(a^\#,z)} = \overline{h_{\mathbb{C}}(a,z)}$ (as $a^\# = a$) whence $|h_{\mathbb{C}}(a,z^\#)| = |h_{\mathbb{C}}(a,z)|$. Thus $\mathcal{B}_\nu(gz) = \mathcal{B}_\nu(z)$. ∎

The relationship between the Moyal restriction operator

$$\rho_\nu : \ \mathcal{C}^\infty(B_\mathbb{C}) \to \mathcal{C}^\infty(B_\mathbb{R})$$

and the Berezin kernel $\mathcal{B}_\nu$ is given by the following result.

**Proposition 3.2.** *For $G \in \mathcal{O}(B_\mathbb{C})$ and $F \in \mathcal{C}^\infty(B_\mathbb{C})$ we have, if $\nu$ is large enough,*

$$\int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{\frac{\nu-p}{2}} \overline{G(x)} (\rho_\nu F)(x) = \int_{B_\mathbb{C}} dz\, h_\mathbb{C}(z,z)^{-p} \mathcal{B}_\nu(z) \overline{(G/I_\nu)(z)} F(z),$$

*where*

$$I_\nu(z) = h_\mathbb{C}(z, z^\sharp)^{-\nu/2}.$$

**Proof.** The Toeplitz restriction map $\mathcal{T}_\mathbb{R}^*$ satisfies

$$(\mathcal{T}_\mathbb{R}^* G)(x) = h_\mathbb{C}(x,x)^{\nu/2}\, G(x) = (G/I_\nu)(x)$$

for all $x \in B_\mathbb{R}$ [36, 6]. Using the duality relation (2.4) and the definition (2.3) of $\rho_\nu$ we obtain

$$\begin{aligned}
\int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{\frac{\nu-p}{2}} \overline{G(x)}\ (\rho_\nu F)(x) &= \int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2} \overline{(G/I_\nu)(x)} (\rho_\nu F)(x) \\
&= \int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2} \overline{(\mathcal{T}_\mathbb{R}^* G)(x)} (\rho_\nu F)(x) = (\mathcal{T}_\mathbb{R}^* G | \rho_\nu F)_{B_\mathbb{R}} = (G | \mathcal{T}_\mathbb{R}\, \rho_\nu F)_\nu \\
&= (G | \mathcal{T}_\mathbb{C}(F)\, I_\nu)_\nu = (G | F \cdot I_\nu)_\nu = \int_{B_\mathbb{C}} dz\, h_\mathbb{C}(z,z)^{\nu-p} \overline{G(z)} F(z) I_\nu(z) \\
&= \int_{B_\mathbb{C}} dz\, h_\mathbb{C}(z,z)^{\nu-p} \overline{(G/I_\nu)(z)} F(z) |I_\nu(z)|^2.
\end{aligned}$$

Since

$$h_\mathbb{C}(z,z)^\nu |I_\nu(z)|^2 = \mathcal{B}_\nu(z)$$

the assertion follows. ∎

**Corollary 3.1.** *For $G \in \mathcal{O}(B_\mathbb{C})$ and $F \in \mathcal{C}^\infty(B_\mathbb{C})$, we have $\rho_\nu(\overline{G}\, F) = \overline{G}\ \rho_\nu F$.*

It follows from (3.1) that $Y \subset Z_\mathbb{C}$ is a $K_\mathbb{R}$-invariant subspace such that

$$Z_\mathbb{C} = Z_\mathbb{R} \oplus Y$$

(direct sum of real vector spaces). For $x \in B_\mathbb{R}$, let $\gamma_x \in G_\mathbb{R}$ be the "transvection" sending 0 to $x$, explicitly given by

$$\gamma_x(y) = x + B(x,x)^{1/2}(y^{-x}),$$

where $B$ is the Bergman operator and

$$y^x = B(y,x)^{-1}(y - Q_y x)$$

is the so-called quasi-inverse [30].

**Lemma 3.1.** *The mapping* $\Phi : B_{\mathbb{R}} \times (Y \cap B_{\mathbb{C}}) \to B_{\mathbb{C}}$ *defined by*

$$\Phi(x,y) = \gamma_x(y) \qquad (x \in B_{\mathbb{R}}, \quad y \in Y \cap B_{\mathbb{C}})$$

*is a real-analytic isomorphism, whose derivative at* $(0,y)$ *is given by*

$$\Phi'(0,y)(\xi,\eta) = \xi + \eta - \{y\xi y\}$$

*for all* $\xi \in Z_{\mathbb{R}} = T_x(B_{\mathbb{R}})$, $\eta \in Y = T_y(Y \cap B_{\mathbb{C}})$.

**Proof.** For $z \in B_{\mathbb{C}}$, set $x := \pi z$ and $y = \gamma_{-x} z$ $(= \gamma_x^{-1} z)$. Then $x \in B_{\mathbb{R}}$ while, by the $G_{\mathbb{R}}$-invariance of $\pi$,

$$\pi y = \gamma_{-x} \pi z = \gamma_{-x} x = 0,$$

so $y \in Y \cap B_{\mathbb{C}}$. This proves that $\Phi$ is surjective. Similarly, if $\Phi(x,y) = \Phi(x',y')$ for some $x,x' \in B_{\mathbb{R}}$ and $y,y' \in Y \cap B_{\mathbb{C}}$, then $x = \gamma_x 0 = \gamma_x \pi y = \pi\Phi(x,y) = \pi\Phi(x',y') = x'$ and $y = \gamma_{-x}\Phi(x,y) = \gamma_{-x'}\Phi(x',y') = y'$, showing that $\Phi$ is injective. It remains to prove the formula for the derivative. For this, we will use some of the formulas collected in [30, Appendix A1–A3]. For the quasi-inverse

$$\Psi(x,y) = x^y$$

we obtain, by definition,

$$\Psi(\xi,y) = B(\xi,y)^{-1}(\xi - Q_\xi\, y)$$

and hence

$$(\partial_1 \Psi)(0,y)\,\xi = \xi.$$

Using the *symmetry formula* [30, A3] we obtain

$$\Psi(x,\eta) = x^\eta = x + Q_x(\eta^x) = x + Q_x\ B(\eta,x)^{-1}(\eta - Q_\eta\, x)$$

and hence

$$(\partial_2\Psi)(x,0)\eta = Q_x\eta.$$

Now the *addition formulas* [30, A3] yield

$$(x+\xi)^y = x^y + B(x,y)^{-1}(\xi^{(y^x)})$$

and hence

$$(\partial_1\Psi)(x,y)\xi = B(x,y)^{-1}(\partial_1\Psi)(0,y^x)\xi = B(x,y)^{-1}\xi.$$

Similarly, we have

$$x^{(y+\eta)} = (x^y)^\eta$$

and hence, with (JP28) from [30, A2],

$$(\partial_2\Psi)(x,y)\eta = (\partial_2\Psi)(x^y,0)\eta = Q_{x^y}\eta = B(x,y)^{-1}Q_x\eta.$$

It follows that

$$\Psi'(x,y)(\xi,\eta) = B(x,y)^{-1}(\xi + Q_x\eta).$$

Since $B(x,x)^{1/2}$ is an even function of $x$, its derivative at $x=0$ vanishes and we obtain for

$$\Phi(x,y)=\gamma_x(y)=x+B(x,x)^{1/2}y^{-x}=x+B(x,x)^{1/2}\Psi(y,-x)$$

the derivatives

$$\partial_1\Phi(0,y)\xi=\xi-\partial_2\Psi(y,0)\xi=\xi-Q_y\xi$$

and

$$\partial_2\Phi(0,y)\eta=\partial_1\Psi(y,0)\eta=B(y,0)^{-1}\eta=\eta.$$

Therefore

$$\Phi'(0,y)(\xi,\eta)=(\partial_1\Phi)(0,y)\xi+(\partial_2\Phi)(0,y)\eta=\xi+\eta-Q_y\xi. \qquad \blacksquare$$

**Corollary 3.2.** *For all $y\in Y\cap B_{\mathbb{C}}$ we have $\det\Phi'(0,y)=\det_{Z_{\mathbb{R}}}(I-Q_y)$.*

Define $\mathcal{P}_\nu:\mathcal{C}^\infty(B_{\mathbb{C}})\to\mathcal{C}^\infty(B_{\mathbb{R}})$ by

$$\begin{aligned}(\mathcal{P}_\nu F)(x)&:=h_{\mathbb{C}}(x,x)^{p/2}\int_{Y\cap B_{\mathbb{C}}}dy\,F(\gamma_x\,y)|\det\Phi'(x,y)|\cdot h_{\mathbb{C}}(\gamma_x y,\gamma_x y)^{-p}\mathcal{B}_\nu(y)\\&=h_{\mathbb{C}}(x,x)^{p/2}\int_{Y\cap B_{\mathbb{C}}}dy\,F(\gamma_x\,y)|\det\Phi'(x,y)|h_{\mathbb{C}}(\gamma_x y,\gamma_x y)^{\nu-p}|h_{\mathbb{C}}(\gamma_x y,(\gamma_x y)^\sharp)|^{-\nu}\end{aligned}$$

for all $F\in\mathcal{C}^\infty(B_{\mathbb{C}})$ and $x\in B_{\mathbb{R}}$. Here $\Phi'(x,y)$ is the derivative of $\Phi$ at $(x,y)\in B_{\mathbb{R}}\times(Y\cap B_{\mathbb{C}})$. If $f\in\mathcal{C}^\infty(B_{\mathbb{R}})$, then $f\circ\pi\in\mathcal{C}^\infty(B_{\mathbb{C}})$ and

$$(f\circ\pi)(\gamma_x y)=f(\gamma_x\pi(y))=f(\gamma_x 0)=f(x).$$

It follows that

$$\mathcal{P}_\nu((f\circ\pi)F)=f\cdot(\mathcal{P}_\nu F), \tag{3.3}$$

i.e. $\mathcal{P}_\nu$ behaves like a "conditional" expectation.

**Proposition 3.3.** *For $F\in\mathcal{C}^\infty(B_{\mathbb{C}})$ we have*

$$\int_{B_{\mathbb{R}}}dx\,h_{\mathbb{C}}(x,x)^{-p/2}(\mathcal{P}_\nu F)(x)=\int_{B_{\mathbb{C}}}dz\,h_{\mathbb{C}}(z,z)^{-p}\mathcal{B}_\nu(z)F(z).$$

**Proof.** The change of variables $z=\gamma_x(y)=\Phi(x,y)$ yields in view of the invariance of $\mathcal{B}_\nu$

$$\begin{aligned}\int_{B_{\mathbb{C}}}dz\,h_{\mathbb{C}}(z,z)^{-p}\mathcal{B}_\nu(z)F(z)&=\int_{B_{\mathbb{R}}}dx\int_{Y\cap B_{\mathbb{C}}}dy\,|\det\Phi'(x,y)|h_{\mathbb{C}}(\gamma_x\,y,\gamma_x y)^{-p}\mathcal{B}_\nu(y)F(\gamma_x y)\\&=\int_{B_{\mathbb{R}}}dx\,h_{\mathbb{C}}(x,x)^{-p/2}(\mathcal{P}_\nu F)(x).\end{aligned} \qquad \blacksquare$$

**Corollary 3.3.** *The operator $\mathcal{P}_\nu$ is $G_{\mathbb{R}}$-invariant, i.e., we have*

$$\mathcal{P}_\nu(F\circ g)=(\mathcal{P}_\nu F)\circ g$$

*for all $F\in\mathcal{C}^\infty(B_{\mathbb{C}})$ and $g\in G_{\mathbb{R}}$.*

**Proof.** Let $f \in \mathcal{C}^\infty(B_\mathbb{R})$ be arbitrary. Using (3.3) and the $G_\mathbb{R}$-invariance of $\mathcal{B}_\nu$ and $\pi$, we obtain

$$\begin{aligned}
\int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2} f(gx)(\mathcal{P}_\nu F)(gx) &= \int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2}\mathcal{P}_\nu((f\circ\pi)F)(gx)\\
&= \int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2}\mathcal{P}_\nu((f\circ\pi)F)(x) = \int_{B_\mathbb{C}} dz\, h_\mathbb{C}(z,z)^{-p}\mathcal{B}_\nu(z)(f\circ\pi)(z)F(z)\\
&= \int_{B_\mathbb{C}} dz\, h_\mathbb{C}(z,z)^{-p}\mathcal{B}_\nu(gz)(f\circ\pi)(gz)F(gz)\\
&= \int_{B_\mathbb{C}} dz\, h_\mathbb{C}(z,z)^{-p}\mathcal{B}_\nu(z)(f\circ g)(\pi(z))(F\circ g)(z)\\
&= \int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2}\mathcal{P}_\nu(((f\circ g)\circ\pi)(F\circ g))(x)\\
&= \int_{B_\mathbb{R}} dx\, h_\mathbb{C}(x,x)^{-p/2}(f\circ g)(x)\mathcal{P}_\nu(F\circ g)(x). \qquad \blacksquare
\end{aligned}$$

For $x = 0 \in B_\mathbb{R}$ we have in particular

$$(\mathcal{P}_\nu F)(0) = \int_{Y\cap B_\mathbb{C}} dy\,|\det \Phi'(0,y)|\mathcal{B}_\nu(y)F(y)h_\mathbb{C}(y,y)^{-p}. \tag{3.4}$$

Our next goal is to obtain an asymptotic expansion of (3.4), as $\nu \to \infty$, using the method of stationary phase but also the more refined "$K_\mathbb{R}$-invariant" Taylor expansion of $F$ at $0 \in Y$. As a first step we recall that

$$Y = \Lambda Z_\mathbb{R} = \{\Lambda x :\ x \in Z_\mathbb{R}\},$$

for an $\mathbb{R}$-linear (but not necessarily $\mathbb{C}$-linear) isomorphism $\Lambda : Z_\mathbb{C} \to Z_\mathbb{C}$ which commutes with $K_\mathbb{R}$. For $f \in \mathcal{C}^\infty(B_\mathbb{R})$ we have $f\circ\Lambda^{-1} \in \mathcal{C}^\infty(Y \cap B_\mathbb{C})$. Consider the distribution

$$f \mapsto \mathcal{P}_\nu\big(f\circ\Lambda^{-1}\big)(0) \tag{3.5}$$

on $B_\mathbb{R}$, which by construction is $K_\mathbb{R}$-invariant. For any partition $\mathbf{m} \in \mathbb{N}_+^r$ let $E_\mathbb{R}^\mathbf{m}$ be the $K_\mathbb{R}$-invariant constant coefficient differential operator on $Z_\mathbb{R}$ corresponding to the polynomial $E^\mathbf{m}$ introduced in Section 2. Using multi-indices $\varkappa \in \mathbb{N}^d$ we may write

$$E^\mathbf{m}(x) = \sum_\varkappa c_\varkappa^\mathbf{m} x^\varkappa, \qquad E_\mathbb{R}^\mathbf{m} = \sum_\varkappa c_\varkappa^\mathbf{m}\partial_\mathbb{R}^\varkappa,$$

where $\partial_\mathbb{R}^\varkappa$ is the "real" partial derivative operator on $Z_\mathbb{R}$ associated with $\varkappa$ and $|\varkappa| \le |\mathbf{m}|$. Expressing $\partial_\mathbb{R}^\varkappa$ in terms of Wirtinger type derivatives $\partial_\mathbb{C}^\sigma$, $\overline{\partial}_\mathbb{C}^\tau$ on $Z_\mathbb{C}$, for multi-indices $\sigma,\tau \in \mathbb{N}^d$, such that $|\sigma| \le |\mathbf{m}| \ge |\tau|$, $E_\mathbb{R}^\mathbf{m}$ determines a complexified constant coefficient differential operator

$$E_\mathbb{C}^\mathbf{m} = \sum_{\sigma,\tau} c_{\sigma,\tau}^\mathbf{m}\partial_\mathbb{C}^\sigma\overline{\partial}_\mathbb{C}^\tau$$

for suitable constants $c_{\sigma,\tau}^\mathbf{m} \in \mathbb{C}$. Pulling back by the (real-linear) map $\Lambda$ we get

$$\partial_\mathbb{C}^\sigma\overline{\partial}_\mathbb{C}^\tau(F\circ\Lambda) = \sum_{\alpha,\beta}\Lambda_{\alpha,\beta}^{\sigma,\tau}(\partial_\mathbb{C}^\alpha\overline{\partial}_\mathbb{C}^\beta F)\circ\Lambda$$

for suitable constants $\Lambda_{\alpha,\beta}^{\sigma,\tau} \in \mathbb{C}$, and hence

$$E_\mathbb{C}^\mathbf{m}(F\circ\Lambda) = \sum_{\sigma,\tau} c_{\sigma,\tau}^\mathbf{m}\partial_\mathbb{C}^\sigma\overline{\partial}_\mathbb{C}^\tau(F\circ\Lambda) = \sum_{\sigma,\tau} c_{\sigma,\tau}^\mathbf{m}\sum_{\alpha,\beta}\Lambda_{\alpha,\beta}^{\sigma,\tau}(\partial_\mathbb{C}^\alpha\overline{\partial}_\mathbb{C}^\beta F)\circ\Lambda = \sum_{\alpha,\beta}P_{\alpha,\beta}^\mathbf{m}(\partial_\mathbb{C}^\alpha\overline{\partial}_\mathbb{C}^\beta F)\circ\Lambda,$$

where

$$P^{\mathbf{m}}_{\alpha,\beta} = \sum_{\sigma,\tau} c^{\mathbf{m}}_{\sigma,\tau}\ \Lambda^{\sigma,\tau}_{\alpha,\beta}. \tag{3.6}$$

Returning to the distribution (3.5) on $B_{\mathbb{R}}$, one has

**Proposition 3.4.** *There exist unique constants $[\nu]_{\mathbf{m}}$, for $\mathbf{m} \in \mathbb{N}^r_+$, such that for all $F \in \mathcal{C}^\infty(B_{\mathbb{C}})$*

$$(\mathcal{P}_\nu F)(0) \sim \sum_{\mathbf{m}} \frac{1}{[\nu]_{\mathbf{m}}} E^{\mathbf{m}}_{\mathbb{C}}(F \circ \Lambda)(0) \tag{3.7}$$

*as an asymptotic expansion.*

**Proof.** By the definition of $Y$, the real-linear operator $y \mapsto y^\#$ from $Y$ into $Z_{\mathbb{C}}$ is injective, and thus bounded below. It follows that also the ($G_{\mathbb{C}}$-invariant) pseudohyperbolic distance

$$\rho(y, y^\#) := \|\gamma_y(y^\#)\|, \qquad y \in B_{\mathbb{C}},$$

is bounded below by a multiple of $\|y - y^\#\|$ if $y \in Y$. Since, by the familiar transformation rule for the Jordan determinant $h_{\mathbb{C}}$,

$$\mathcal{B}_\nu(z)^2 = \frac{h(z,z)^\nu h(z^\#, z^\#)^\nu}{|h(z, z^\#)|^{2\nu}} = h(\gamma_z z^\#, \gamma_z z^\#)^\nu$$

and $h(w,w) \le 1$ on the closure of $B_{\mathbb{C}}$, with equality if and only if $w = 0$, it follows that $\mathcal{B}_\nu$ has a global maximum on $Y$ at $y = 0$, which also dominates the boundary values of $\mathcal{B}_\nu$ in the sense that $\mathcal{B}_\nu(y_k) \to 1$, $y_k \in Y$, implies that $y_k \to 0$. We may therefore apply the method of stationary phase exactly as in Section 3 of [22] to conclude that for any $F \in \mathcal{C}^\infty(B_{\mathbb{C}})$, for which the right-hand side exists for some $\nu > p - 1$, the integral

$$\begin{aligned}\mathcal{P}_\nu F(0) &= \int_Y F(y) |\det \Phi'(0, y)| \mathcal{B}_\nu(y) h_{\mathbb{C}}(y, y)^{-p/2}\, dy \\ &= |\det \Lambda| \int_{B_{\mathbb{R}}} F(\Lambda x) |\det \Phi'(0, \Lambda x)| \mathcal{B}_\nu(\Lambda x) h_{\mathbb{C}}(\Lambda x, \Lambda x)^{-p/2}\, dx\end{aligned}$$

has an asymptotic expansion as $\nu \to +\infty$

$$\mathcal{P}_\nu F(0) \sim \nu^{-d/2} \sum_{k \ge 0} S_k(\partial_{\mathbb{R}})(F \circ \Lambda)(0) \nu^{-k}$$

for some constant coefficient differential operators $S_k(\partial_{\mathbb{R}})$, with $S_k$ polynomials on $Z_{\mathbb{R}}$. Since $\mathcal{P}_\nu$ is $K_{\mathbb{R}}$-invariant, so must be the $S_k$; thus they admit a decomposition

$$S_k = \sum_{|\mathbf{m}| \le k} q_{k\mathbf{m}} E^{\mathbf{m}}, \qquad q_{k\mathbf{m}} \in \mathbb{C},$$

into the "even" Peter–Weyl components $E^{\mathbf{m}}$. Interchanging the two summations and setting

$$\frac{1}{[\nu]_{\mathbf{m}}} := \nu^{-d/2} \sum_k q_{k\mathbf{m}} \nu^{-k},$$

the claim follows. ∎

Using the transvections $\gamma_x \in G_{\mathbb{R}}$, for $x \in B_{\mathbb{R}}$, we define a $G_{\mathbb{R}}$-invariant differential operator

$$\mathcal{P}^{\mathbf{m}} :\ \mathcal{C}^\infty(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$$

by putting

$$\mathcal{P}^{\mathbf{m}}(F)(x) := E^{\mathbf{m}}_{\mathbb{C}}(F \circ \gamma_x \circ \Lambda)(0) = \sum_{\alpha,\beta} P^{\mathbf{m}}_{\alpha,\beta}\partial^\alpha_{\mathbb{C}}\overline{\partial}^\beta_{\mathbb{C}}(F \circ \gamma_x)(0). \tag{3.8}$$

Since $\gamma_x : B_{\mathbb{C}} \to B_{\mathbb{C}}$ is holomorphic, there exist smooth functions $\gamma^\alpha_\iota : B_{\mathbb{R}} \to \mathbb{C}$, with $|\iota| \le |\alpha|$, such that

$$\partial^\alpha_{\mathbb{C}}(H \circ \gamma_x)(0) = \sum_\iota \gamma^\alpha_\iota(x)(\partial^\iota_{\mathbb{C}} H)(x)$$

for all $H \in \mathcal{O}(B_{\mathbb{C}})$ and $x \in B_{\mathbb{R}}$. Since $\mathcal{P}_\nu$ is $G_{\mathbb{R}}$-invariant, Proposition 3.4 implies

$$(\mathcal{P}_\nu\, F)(x) \sim \sum_{\mathbf{m}} \frac{1}{[\nu]_{\mathbf{m}}}(\mathcal{P}^{\mathbf{m}}\, F)(x) \tag{3.9}$$

for all $F \in \mathcal{C}^\infty(B_{\mathbb{C}})$ and $x \in B_{\mathbb{R}}$.

Now let $\mathbf{m} \in \mathbb{N}^r_+$ and $\varkappa \in \mathbb{N}^d$ be fixed, with $|\varkappa| \le |m|$. Define a (non-invariant) "holomorphic" differential operator

$$\mathcal{P}^{\mathbf{m}}_\varkappa :\ \mathcal{O}(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$$

by the formula

$$(\mathcal{P}^{\mathbf{m}}_\varkappa\, H)(x) = \sum_{\alpha,\beta} P^{\mathbf{m}}_{\alpha,\beta}\partial^\alpha_{\mathbb{C}}(H \circ \gamma_x)(0)\overline{\gamma^\beta_\varkappa(x)} = \sum_{\alpha,\beta,\iota} P^{\mathbf{m}}_{\alpha,\beta}\gamma^\alpha_\iota(x)\overline{\gamma^\beta_\varkappa(x)}\ (\partial^\iota_{\mathbb{C}} H)(x) \tag{3.10}$$

for all $x \in B_{\mathbb{R}}$ and $H \in \mathcal{O}(B_{\mathbb{C}})$, where the constants $P^{\mathbf{m}}_{\alpha,\beta}$ are defined by (3.6).

**Lemma 3.2.** *Let $G, H \in \mathcal{O}(B_{\mathbb{C}})$. Then*

$$\mathcal{P}^{\mathbf{m}}(\overline{G}H)(x) = \sum_\varkappa (\mathcal{P}^{\mathbf{m}}_\varkappa H)(x)\overline{\partial^\varkappa_{\mathbb{C}} G(x)}.$$

**Proof.** Since $\gamma_x$ preserves holomorphy, (3.10) implies

$$\mathcal{P}^{\mathbf{m}}(\overline{G}H)(x) = \sum_{\alpha,\beta} P^{\mathbf{m}}_{\alpha,\beta}\partial^\alpha_{\mathbb{C}}\overline{\partial}^\beta_{\mathbb{C}}(\overline{G \circ \gamma_x}(H \circ \gamma_x))(0)$$

$$= \sum_{\alpha,\beta} P^{\mathbf{m}}_{\alpha,\beta}\partial^\alpha_{\mathbb{C}}(H \circ \gamma_x)(0)\overline{\partial^\beta_{\mathbb{C}}(G \circ \gamma_x)(0)} = \sum_{\alpha,\beta,\iota,\varkappa} P^{\mathbf{m}}_{\alpha,\beta}\gamma^\alpha_\iota(x)(\partial^\iota_{\mathbb{C}} H)(x)\overline{\gamma^\beta_\varkappa(x)(\partial^\varkappa_{\mathbb{C}} G)(x)}$$

$$= \sum_{\alpha,\beta,\iota,\varkappa} P^{\mathbf{m}}_{\alpha,\beta}\gamma^\alpha_\iota(x)\overline{\gamma^\beta_\varkappa(x)}(\partial^\iota_{\mathbb{C}} H)(x)\overline{(\partial^\varkappa_{\mathbb{C}} G)(x)} = \sum_\varkappa (\mathcal{P}^{\mathbf{m}}_\varkappa H)(x)\overline{\partial^\varkappa_{\mathbb{C}} G(x)}. \qquad \blacksquare$$

**Definition 3.1.** For $\mathbf{m} \in \mathbb{N}^r_+$, the $\mathbf{m}$-th *Moyal component* is the differential operator

$$\rho^{\mathbf{m}} :\ \mathcal{O}(B_{\mathbb{C}}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$$

defined by the formula

$$(\rho^{\mathbf{m}}\, H)(x) = h_{\mathbb{C}}(x,x)^{p/2}\sum_\varkappa (-1)^\varkappa \partial^\varkappa_{\mathbb{R}}(h_{\mathbb{C}}^{-p/2}\mathcal{P}^{\mathbf{m}}_\varkappa H)(x) \tag{3.11}$$

for all $x \in B_{\mathbb{R}}$ and $H \in \mathcal{O}(B_{\mathbb{C}})$. Here $\partial^\varkappa_{\mathbb{R}}$ is the "real" partial derivative operator on $B_{\mathbb{R}} \subset Z_{\mathbb{R}}$, and $(-1)^\varkappa\, \partial^\varkappa_{\mathbb{R}}$ is its (Euclidean) adjoint. We also write just $h_{\mathbb{C}}$ for $h_{\mathbb{C}}(x,x)$.

**Proposition 3.5.** *Let $G, H \in \mathcal{O}(B_{\mathbb{C}})$. Then*

$$\int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{\frac{\nu-p}{2}}\overline{G(x)}(\rho^{\mathbf{m}}\, H)(x) = \int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\mathcal{P}^{\mathbf{m}}(\overline{G/I_\nu}H)(x). \tag{3.12}$$

**Proof.** Since $G$ is holomorphic, we have $\partial_{\mathbb{C}}^{\varkappa} G(x) = \partial_{\mathbb{R}}^{\varkappa} G(x)$ for all $\varkappa \in \mathbb{N}^d$ and $x \in B_{\mathbb{R}}$. Applying Lemma 3.2 to $G/I_\nu$ we obtain

$$\int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{\frac{\nu-p}{2}}\overline{G(x)}(\rho^{\mathbf{m}}H)(x) = \int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\overline{(G/I_\nu)(x)}(\rho^{\mathbf{m}}H)(x)$$
$$= \sum_{\varkappa}\int_{B_{\mathbb{R}}} dx\, \overline{(G/I_\nu)(x)}(-1)^{\varkappa}\, \partial_{\mathbb{R}}^{\varkappa}(h_{\mathbb{C}}^{-p/2}\mathcal{P}_{\varkappa}^{\mathbf{m}}H)(x)$$
$$= \sum_{\varkappa}\int_{B_{\mathbb{R}}} dx\, \overline{\partial_{\mathbb{R}}^{\varkappa}(G/I_\nu)(x)}h_{\mathbb{C}}(x,x)^{-p/2}(\mathcal{P}_{\varkappa}^{\mathbf{m}}H)(x)$$
$$= \sum_{\varkappa}\int_{B_{\mathbb{R}}} dx\, \overline{\partial_{\mathbb{C}}^{\varkappa}(G/I_\nu)(x)}h_{\mathbb{C}}(x,x)^{-p/2}(\mathcal{P}_{\varkappa}^{\mathbf{m}}H)(x) = \int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\mathcal{P}^{\mathbf{m}}(\overline{G/I_\nu}H)(x). \qquad \blacksquare$$

As a consequence of Proposition 3.5 we obtain

**Corollary 3.4.** *The differential operators $\rho^{\mathbf{m}}$ are $G_{\mathbb{R}}$-invariant, i.e.,*

$$\rho^{\mathbf{m}}(H \circ g)(x) = (\rho^{\mathbf{m}}H)(g(x))$$

*for all $H \in \mathcal{O}(B_{\mathbb{C}})$, $g \in G_{\mathbb{R}}$ and $x \in B_{\mathbb{R}}$.*

**Proof.** Replacing $G/I_\nu = \Phi$, (3.12) can be written as

$$\int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\overline{\Phi(x)}(\rho^{\mathbf{m}}H)(x) = \int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\mathcal{P}^{\mathbf{m}}(\overline{\Phi}H)(x)$$

for $\Phi, H \in \mathcal{O}(B_{\mathbb{C}})$. Since $\mathcal{P}^{\mathbf{m}}$ is $G_{\mathbb{R}}$-invariant by construction (cf. (3.8)), the assertion follows. $\blacksquare$

The main result of this section yields the desired asymptotic expansion of the Moyal type restriction operator $\rho_\nu$ in terms of the invariant differential operators $\rho^{\mathbf{m}}$:

**Theorem 3.1.** *For $H \in \mathcal{O}(B_{\mathbb{C}})$ we have an asymptotic expansion*

$$\rho_\nu H \sim \sum_{\mathbf{m}} \frac{1}{[\nu]_{\mathbf{m}}}\rho^{\mathbf{m}}H,$$

*where $\rho^{\mathbf{m}} : \mathcal{O}(B_{\mathbb{C}}) \to \mathcal{C}^\infty(D_{\mathbb{R}})$ are $G_{\mathbb{R}}$-invariant holomorphic differential operators independent of $\nu$ and the constants $[\nu]_{\mathbf{m}}$ are determined by (3.7).*

**Proof.** Let $G, H \in \mathcal{O}(B_{\mathbb{C}})$. Applying Proposition 3.2 and 3.3, we obtain with (3.9) and (3.12)

$$\int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{\frac{\nu-p}{2}}\overline{G(x)}(\rho_\nu H)(x) = \int_{B_{\mathbb{C}}} dz\, h_{\mathbb{C}}(z,z)^{-p}\mathcal{B}_\nu(z)\overline{(G/I_\nu)(z)}H(z)$$
$$= \int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\mathcal{P}_\nu(\overline{G/I_\nu}H)(x) = \sum_{\mathbf{m}}\frac{1}{[\nu]_{\mathbf{m}}}\int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{-p/2}\mathcal{P}^{\mathbf{m}}\,(\overline{G/I_\nu}H)(x)$$
$$= \sum_{\mathbf{m}}\frac{1}{[\nu]_{\mathbf{m}}}\int_{B_{\mathbb{R}}} dx\, h_{\mathbb{C}}(x,x)^{\frac{\nu-p}{2}}\overline{G(x)}(\rho^{\mathbf{m}}H)(x).$$

Since $G \in \mathcal{O}(B_{\mathbb{C}})$ is arbitrary, the assertion follows. $\blacksquare$

**Remark 3.1.** Most – probably all – of the above extends also to the case of general $G_{\mathbb{R}}$-invariant smooth retractions $\pi : B_{\mathbb{C}} \to B_{\mathbb{R}}$, i.e. when $\pi^{-1}(0)$ is not necessarily an intersection of $B_{\mathbb{C}}$ with some real subspace $Y$, or that the parameterization $\Lambda : B_{\mathbb{R}} \to \pi^{-1}(0)$ is not necessarily linear but only smooth. In fact, the application of the stationary phase method in the proof of Proposition 3.4 involves only the germs of $F$ and $\pi^{-1}(0)$ (or, equivalently, $\Lambda$) at the origin. Thus we may replace the variety $\pi^{-1}(0)$ by its tangent space at $0 \in \pi^{-1}(0)$, and $\Lambda : B_{\mathbb{R}} \to \pi^{-1}(0)$ by its differential at the origin. We omit the details.

# 4 Asymptotic expansion: the complex case

In the complex case, where

$$B_{\mathbb{C}} = B \times \overline{B} = \{(z, \overline{w}) : \ z, w \in B\},$$
$$B_{\mathbb{R}} = \{(z, \overline{z}) : \ z \in B\}$$

and $B$ is an irreducible complex Hermitian bounded symmetric domain (of rank $r$), the Moyal type restriction operator

$$\rho_\nu : \ \mathcal{C}^\infty(B_{\mathbb{C}}) = \mathcal{C}^\infty(B) \,\overline{\otimes}\, \mathcal{C}^\infty(\overline{B}) \to \mathcal{C}^\infty(B_{\mathbb{R}})$$

can be identified with the Moyal type (star-) product $\sharp_\nu$ via the formula

$$\rho_\nu(f \otimes g) = f \,\sharp_\nu\, g$$

for all $f, g \in \mathcal{C}^\infty(B)$. In this case an asymptotic expansion has been constructed in [18], and here we show that the general construction described in Section 3 yields precisely the expansion of [18]. This is not completely obvious, since the construction in [18] is based on the complex structure of $B$ whereas the general construction of Section 3 uses the "real" structure of $B_{\mathbb{R}}$.

The first step is to identify the Berezin kernel $\mathcal{B}_\nu$ on $B_{\mathbb{C}}$, defined in (3.2), for the complex case. We have

$$h_{\mathbb{C}}((z, \overline{w}), (\zeta, \overline{\omega})) = h(z, \zeta) h(\omega, w)$$

for $z, w, \zeta, \omega \in B$ and the involution is given by

$$(z, \overline{w})^\sharp := (w, \overline{z}).$$

Therefore

$$\mathcal{B}_\nu(z, \overline{w}) = \frac{h_{\mathbb{C}}((z, \overline{w}), (z, \overline{w}))^\nu}{|h_{\mathbb{C}}((z, \overline{w}), (w, \overline{z}))|^\nu} = \frac{h(z, z)^\nu h(w, w)^\nu}{h(z, w)^\nu \ h(w, z)^\nu}$$

coincides with the integral kernel for the $G$-invariant *Berezin transform*

$$\mathcal{B}_\nu : \ \mathcal{C}^\infty(B) \to \mathcal{C}^\infty(B)$$

on $B$. This is clearly invariant under

$$G_{\mathbb{R}} = \{(g, g) : \ g \in G\},$$

where $G = \mathrm{Aut}\,(B)$. The construction in [18] starts with the asymptotic expansion

$$(\mathcal{B}_\nu f)(0) = \int_B dz\, h(z, z)^{\nu - p} f(z) = \sum_{\mathbf{m}} \frac{1}{(\nu)_{\mathbf{m}}} (E^{\mathbf{m}}_{\mathbb{R}} f)(0)$$

of the $\nu$-Berezin transform $\mathcal{B}_\nu$ associated with the usual Toeplitz–Berezin quantization of $B$. Here, for any partition $\mathbf{m} \in \mathbb{N}^r_+$, the Pochhammer symbol

$$(\nu)_{\mathbf{m}} := \frac{\Gamma_\Omega(\nu + \mathbf{m})}{\Gamma_\Omega(\nu)}$$

is defined via the Koecher–Gindikin $\Gamma$-function, and the "sesqui-holomorphic" constant coefficient differential operators $E^{\mathbf{m}}_{\mathbb{R}}$ are defined via the Fock space expansion

$$e^{(z|w)} = \sum_{\mathbf{m}} E^{\mathbf{m}}(z,w)$$

for all $z, w \in Z$. In multi-index notation,

$$E^{\mathbf{m}}(z,w) = \sum_{\alpha,\beta} c^{\mathbf{m}}_{\alpha,\beta} z^\alpha \overline{w}^\beta, \qquad E^{\mathbf{m}}_{\mathbb{R}} = \sum_{\alpha,\beta} c^{\mathbf{m}}_{\alpha\,\beta} \partial^\alpha\, \overline{\partial}^\beta \tag{4.1}$$

for suitable constants $c^{\mathbf{m}}_{\alpha\,\beta}$ and multi-indices $\alpha, \beta \in \mathbb{N}^d$, such that $|\alpha| \le |\mathbf{m}| \ge |\beta|$. Since

$$\overline{E^{\mathbf{m}}(z,w)} = E^{\mathbf{m}}(w,z)$$

it follows that

$$\overline{c^{\mathbf{m}}_{\alpha\,\beta}} = c^{\mathbf{m}}_{\beta\,\alpha}. \tag{4.2}$$

Passing to the complexification $Z_{\mathbb{C}} = Z \times \overline{Z}$, with variables $(z, \overline{w})$ for $z, w \in Z$, we use pairs of multi-indices and write $\partial^{\alpha\overline{\beta}}_{\mathbb{C}}$ and $\overline{\partial}^{\gamma\overline{\delta}}_{\mathbb{C}}$ for the associated Wirtinger derivatives. Thus, for functions on $B_{\mathbb{C}}$ of the form

$$(f \otimes g)(z, \overline{w}) = f(z)g(w), \tag{4.3}$$

we have

$$\partial^{\alpha\overline{\beta}}_{\mathbb{C}} \overline{\partial}^{\gamma\overline{\delta}}_{\mathbb{C}} (f \otimes g) = (\partial^\alpha \overline{\partial}^\gamma f) \otimes (\overline{\partial}^\beta \partial^\delta g). \tag{4.4}$$

Note that the first and second variable are treated differently, since holomorphic functions on $B_{\mathbb{C}}$ correspond to the case where $f$ is holomorphic and $g$ is anti-holomorphic. Let

$$\Lambda : \ B_{\mathbb{R}} \to B_{\mathbb{C}}$$

denote the $\mathbb{R}$-linear mapping

$$\Lambda(z, \overline{z}) = (z, 0)$$

which is clearly $K_{\mathbb{R}}$-invariant. Consider the $G_{\mathbb{R}}$-invariant retraction

$$\pi : \ B_{\mathbb{C}} \to B_{\mathbb{R}}$$

defined by $\pi(z, \overline{w}) := (w, \overline{w})$. Then

$$\pi^{-1}(0) = \{(z, 0) : \ z \in B\} = \Lambda\, B_{\mathbb{R}}.$$

**Lemma 4.1.** *For $F \in \mathcal{C}^\infty(B_{\mathbb{C}})$ we have*

$$E^{\mathbf{m}}_{\mathbb{C}}(F \circ \Lambda)(0) = \sum_{\alpha,\beta} c^{\mathbf{m}}_{\alpha\,\beta} \partial^{\alpha 0}_{\mathbb{C}} \overline{\partial}^{\beta 0}_{\mathbb{C}} F(0).$$

**Proof.** We may assume that $F(z,\overline{w}) = f(z)g(w)$ is of the form (4.3) . Since

$$((f \otimes g) \circ \Lambda)(z, \overline{z}) = (f \otimes g)(z, 0) = f(z)g(0)$$

it follows from (4.4) and (4.1) that

$$\begin{aligned} E_{\mathbb{C}}^{\mathbf{m}}((f \otimes g) \circ \Lambda)(0) = (E^{\mathbf{m}} f)(0)g(0) &= \sum_{\alpha,\beta} c_{\alpha\,\beta}^{\mathbf{m}} (\partial^\alpha \overline{\partial}^\beta f)(0)g(0) \\ &= \sum_{\alpha,\beta} c_{\alpha\,\beta}^{\mathbf{m}} (\partial_{\mathbb{C}}^{\alpha\,0} \overline{\partial}_{\mathbb{C}}^{\beta\,0} (f \otimes g))(0). \end{aligned}$$ ■

Comparing with the coefficients $P^{\mathbf{m}}_{\alpha,\beta}$ introduced by (3.6) in the general case, it follows that

$$P^{\mathbf{m}}_{\alpha 0,\beta 0} = c^{\mathbf{m}}_{\alpha\,\beta} \tag{4.5}$$

for $\alpha, \beta \in \mathbb{N}^d$, whereas all other such coefficients vanish. This reflects the fact that $\Lambda$ is trivial on the second component. For $z \in B$, let as before $\gamma_z \in G$ be the transvection mapping 0 to $z$. Then we have for $\alpha \in \mathbb{N}^d$ and $f \in \mathcal{O}(B)$

$$\partial^\alpha (f \circ \gamma_z)(0) = \sum_{\iota \le \alpha} \gamma_\iota^\alpha(z) (\partial^\iota f)(z),$$

where $\gamma_\iota^\alpha$ are smooth functions on $B$. As in [18] define a $G$-invariant operator

$$\mathcal{E}^{\mathbf{m}} : \ \mathcal{C}^\infty(B) \to \mathcal{C}^\infty(B)$$

by putting

$$(\mathcal{E}^{\mathbf{m}}\, f)(z) := E_{\mathbb{R}}^{\mathbf{m}} (f \circ \gamma_z)(0).$$

Then we have for $f, g \in \mathcal{O}(B)$

$$\begin{aligned} (\mathcal{E}^{\mathbf{m}}(f\overline{g}))(z) &= E_{\mathbb{R}}^{\mathbf{m}}((f\overline{g}) \circ \gamma_z)(0) = E_{\mathbb{R}}^{\mathbf{m}}((f \circ \gamma_z)\overline{g \circ \gamma_z})(0) \\ &= \sum_{\alpha,\beta} c_{\alpha\,\beta}^{\mathbf{m}} \partial^\alpha \overline{\partial}^\beta ((f \circ \gamma_z)\overline{g \circ \gamma_z})(0) = \sum_{\alpha,\beta} c_{\alpha\,\beta}^{\mathbf{m}} \partial^\alpha (f \circ \gamma_z)(0) \overline{\partial^\beta (g \circ \gamma_z)(0)} \\ &= \sum_{\alpha,\beta} \sum_{\varkappa,\iota} c_{\alpha\,\beta}^{\mathbf{m}} \gamma_\varkappa^\alpha(z) (\partial^\varkappa f)(z) \overline{\gamma_\iota^\beta(z)}\, \overline{(\partial^\iota g)(z)}. \end{aligned} \tag{4.6}$$

Following [18, Section 4] one defines (non-invariant) differential operators $\mathcal{R}^{\mathbf{m}}_\varkappa$, for any partition $\mathbf{m} \in \mathbb{N}^r_+$ and any multi-index $\varkappa \in \mathbb{N}^d$ with $|\varkappa| \le |\mathbf{m}|$, via the expansion

$$(\mathcal{E}^{\mathbf{m}}(f\overline{g}))(z) = \sum_\varkappa (\partial^\varkappa f)(z) (\mathcal{R}^{\mathbf{m}}_\varkappa \overline{g})(z),$$

where $f \in \mathcal{O}(B)$, $g \in \mathcal{C}^\infty(B)$. Comparing with (4.6) it follows that

$$(\mathcal{R}^{\mathbf{m}}_\varkappa \overline{g})(z) = \sum_{\alpha,\beta} \sum_\iota c_{\alpha\,\beta}^{\mathbf{m}} \gamma_\varkappa^\alpha(z) \overline{\gamma_\iota^\beta(z)}\, \overline{(\partial^\iota g)(z)}$$

whenever $g$ is holomorphic. On the other hand, putting

$$\gamma_{z,\overline{z}} := (\gamma_z, \gamma_z) \in G_{\mathbb{R}} \subset G \times G$$

we have for the "holomorphic" Wirtinger derivatives

$$\begin{aligned}\partial_{\mathbb{C}}^{\alpha\overline{\beta}}[(f\otimes\overline{g})\circ\gamma_{z,\overline{z}}](0,0) &= \partial_{\mathbb{C}}^{\alpha\overline{\beta}}[(f\circ\gamma_z)\otimes\overline{g\circ\gamma_z}](0,0) = \partial^{\alpha}(f\circ\gamma_z)(0)\overline{\partial^{\beta}(g\circ\gamma_z)(0)} \\ &= \sum_{\varkappa,\iota}\gamma_{\varkappa}^{\alpha}(z)(\partial^{\varkappa}f)(z)\overline{\gamma_{\iota}^{\beta}(z)}\ \overline{\partial^{\iota}g(z)} = \sum_{\varkappa,\iota}\gamma_{\varkappa}^{\alpha}(z)\overline{\gamma_{\iota}^{\beta}(z)}\partial_{\mathbb{C}}^{\varkappa\overline{\iota}}(f\otimes\overline{g})(z,\overline{z}).\end{aligned} \tag{4.7}$$

We will now compute the (non-invariant) operators $\mathcal{P}_{\varkappa}^{\mathbf{m}}$, introduced in (3.10), for the complex case. Combining (4.5) and (4.7) it follows that the non-zero operators correspond to multi-index pairs $(\varkappa,0)$ for $\varkappa\in\mathbb{N}^d$ and, in view of (4.2),

$$\begin{aligned}\mathcal{P}_{\varkappa 0}^{\mathbf{m}}(f\otimes\overline{g})(z,\overline{z}) &= \sum_{\alpha,\beta,\iota}P_{\alpha 0,\,\beta 0}^{\mathbf{m}}\gamma_{\iota}^{\alpha}(z)\overline{\gamma_{\varkappa}^{\beta}(z)}\partial_{\mathbb{C}}^{\iota 0}(f\otimes\overline{g})(z,\overline{z}) \\ &= \sum_{\alpha,\beta,\iota}c_{\alpha\beta}^{\mathbf{m}}\gamma_{\iota}^{\alpha}(z)\overline{\gamma_{\varkappa}^{\beta}(z)}(\partial^{\iota}f)(z)\ \overline{g(z)} = \overline{(\mathcal{R}_{\varkappa}^{\mathbf{m}}\overline{f})(z)g(z)}.\end{aligned}$$

This passing to the complex conjugate (also in the proof of the following Proposition) could be avoided by working with the "anti-holomorphic" second variable instead.

The $G$-invariant bi-differential operators $A_{\mathbf{m}}$ on $B$, introduced in [18, Section 4], satisfy

$$A_{\mathbf{m}}(f,\overline{g})(z) = \sum_{\varkappa}h(z,z)^p(-\partial)^{\varkappa}(h^{-p}\,f(\mathcal{R}_{\varkappa}^{\mathbf{m}}\,\overline{g}))(z)$$

for all $f,g\in\mathcal{O}(B)$, and are uniquely determined by this property since $A_{\mathbf{m}}$ involves only holomorphic derivatives in the first variable and anti-holomorphic derivatives in the second variable. By [18, Proposition 6],

$$\overline{A_{\mathbf{m}}(f,\overline{g})} = A_{\mathbf{m}}(g,\overline{f})$$

for all $f,g\in\mathcal{O}(B)$.

**Proposition 4.1.** *Let $f,g\in\mathcal{O}(B)$. Then*

$$\rho^{\mathbf{m}}(f\otimes\overline{g})(z,\overline{z}) = A_{\mathbf{m}}(f,\overline{g})(z)$$

*for all $z\in B$.*

**Proof.** Since the operators $\rho^{\mathbf{m}}$ are defined by taking suitable adjoints on $B_{\mathbb{R}}$, which requires another identification, we instead verify that both operators satisfy the same integral duality formula. Thus let $f,g,\phi,\psi\in\mathcal{O}(B)$. Then, by (3.11),

$$\begin{aligned}&\int_B dz\,h(z,z)^{-p}\overline{\phi(z)}\ \psi(z)\rho^{\mathbf{m}}(f\otimes\overline{g})(z,\overline{z}) \\ &= \int_{B_{\mathbb{R}}} d(z,\overline{z})\,h_{\mathbb{C}}((z,\overline{z}),(z,\overline{z}))^{-p/2}\overline{(\phi\otimes\overline{\psi})(z,\overline{z})}\rho^{\mathbf{m}}(f\otimes\overline{g})(z,\overline{z}) \\ &= \sum_{\varkappa}\int_{B_{\mathbb{R}}} d(z,\overline{z})\,h_{\mathbb{C}}((z,\overline{z}),(z,\overline{z}))^{-p/2}\overline{\partial_{\mathbb{C}}^{\varkappa 0}(\phi\otimes\overline{\psi})(z,\overline{z})}\mathcal{P}_{\varkappa 0}^{\mathbf{m}}(f\otimes\overline{g})(z,\overline{z}) \\ &= \sum_{\varkappa}\int_B dz\,h(z,z)^{-p}\overline{(\partial^{\varkappa}\phi)(z)}\ \psi(z)\overline{g(z)(\mathcal{R}_{\varkappa}^{\mathbf{m}}\overline{f})(z)} \\ &= \sum_{\varkappa}\overline{\int_B dz\,h(z,z)^{-p}(\partial^{\varkappa}\phi)(z)\overline{\psi(z)}g(z)(\mathcal{R}_{\varkappa}^{\mathbf{m}}\overline{f})(z)} \\ &= \sum_{\varkappa}\overline{\int_B dz\,\phi(z)(-\partial)^{\varkappa}(h^{-p}\overline{\psi}g(\mathcal{R}_{\varkappa}^{\mathbf{m}}\overline{f}))(z)} = \sum_{\varkappa}\overline{\int_B dz\,\phi(z)\overline{\psi(z)}(-\partial)^{\varkappa}(h^{-p}g(\mathcal{R}_{\varkappa}^{\mathbf{m}}\overline{f}))(z)}\end{aligned}$$

$$= \overline{\int_B dz\, h(z,z)^{-p}\phi(z)\overline{\psi(z)}A_{\mathbf{m}}(g,\overline{f})(z)} = \int_B dz\, h(z,z)^{-p}\overline{\phi(z)}\,\psi(z)A_{\mathbf{m}}(f,\overline{g})(z).$$

Since $\phi,\psi\in\mathcal{O}(B)$ are arbitrary, the assertion follows. Note that the formula (3.11) defining $\rho^{\mathbf{m}}$ uses the real derivatives $\partial_{\mathbb{R}}$, whereas in this section we are using rather the Wirtinger derivatives $\partial$ and $\overline{\partial}$ on $B$ (corresponding to viewing $B_{\mathbb{R}}=B$ as a domain in $\mathbb{C}^d$ rather than $\mathbb{R}^{2d}$); this is reflected by the appearance of the Hermitian adjoint $-\overline{\partial}^{\varkappa 0}$ (rather than $-\partial^{\varkappa 0}$) of $\partial^{\varkappa 0}$ on the third line in the computation above. ■

# 5 Examples

We begin with the case of the Euclidean space where everything can be computed explicitly.

**Example 5.1.** Let $B_{\mathbb{R}}=\mathbb{R}$, so that $B_{\mathbb{C}}=\mathbb{C}$, and $\Lambda x:=\varepsilon x$ for some $\varepsilon\in\mathbb{C}\setminus\mathbb{R}$, $|\varepsilon|=1$. The corresponding retraction $\pi$ is just the oblique $\mathbb{R}$-linear projection associated to the direct sum decomposition

$$\mathbb{C}=\mathbb{R}\oplus\varepsilon\mathbb{R};$$

the mapping $\phi$ is just $\Phi(x,y)=x+y$, and $\det\Phi'=1$. The role of the Jordan determinant polynomial $h_{\mathbb{C}}(x,y)$ is played by the function

$$e^{-x\overline{y}}, \qquad x,y\in\mathbb{C}, \tag{5.1}$$

and the "genus" $p=0$ while "rank" $r=1$. The partitions are just nonnegative integers $\mathbf{m}=(m)$, and the polynomials $E^{\mathbf{m}}$ are simply

$$E^{\mathbf{m}}(x)=\frac{x^{2m}}{(2m)!}.$$

Thus $E^{\mathbf{m}}_{\mathbb{C}}=(\partial+\overline{\partial})^{2m}/(2m)!$, and

$$P^{\mathbf{m}}_{\alpha,\beta}=\begin{cases}\dfrac{\varepsilon^{\alpha-\beta}}{\alpha!\beta!} & \text{if } \alpha+\beta=2m,\\ 0 & \text{otherwise.}\end{cases} \tag{5.2}$$

The "transvections" $\gamma_x$ are just the ordinary translations $\gamma_x y=x+y$, which implies that the functions $\gamma^\alpha_\iota$ equal constant one if $\alpha=\iota$, and vanish otherwise. Feeding all this information into (3.10) and (3.11), we get

$$\mathcal{P}^{\mathbf{m}}_{\varkappa}=\frac{\varepsilon^{2m-2\varkappa}}{(2m-\varkappa)!\varkappa!}\,\partial^{2m-\varkappa}$$

and, for $H\in\mathcal{O}(\mathbb{C})$,

$$\rho^{\mathbf{m}}H=\frac{1}{(2m)!}\sum_{\varkappa=0}^{2m}\binom{2m}{\varkappa}\varepsilon^{2m-2\varkappa}(-1)^{\varkappa}\partial^{\varkappa}_{\mathbb{R}}(\partial^{2m-\varkappa}H)=\frac{(\varepsilon-\overline{\varepsilon})^{2m}}{(2m)!}\partial^{2m}H.$$

We next compute the "Pochhammer" symbols $[\nu]_{\mathbf{m}}$, using the formula (3.4). By (5.1),

$$\mathcal{P}_\nu F(0)=\int_{\varepsilon\mathbb{R}}F(y)\frac{e^{-\nu|y|^2}}{|e^{-y^2}|^{\nu}}\,dy=\int_{\mathbb{R}}F(\varepsilon y)e^{-\nu y^2(1-\operatorname{Re}\varepsilon^2)}\,dy.$$

Denoting for brevity $1-\operatorname{Re}\varepsilon^2 = -\frac{1}{2}(\varepsilon-\overline{\varepsilon})^2 =: c > 0$ and making the change of variable $y = x/\sqrt{c\nu}$ yields

$$\mathcal{P}_\nu F(0) = \frac{1}{\sqrt{c\nu}} \int_{\mathbb{R}} F\left(\frac{\varepsilon}{\sqrt{c\nu}} x\right) e^{-x^2}\, dx.$$

We may assume that $F$ is holomorphic; replacing then $F$ by its Taylor expansion, integrating term by term (which is easily justified), and using the fact that $\int_{\mathbb{R}} x^{2j} e^{-x^2}\, dx = \Gamma(j+\frac{1}{2})$, we finally arrive at

$$\mathcal{P}_\nu F(0) = \frac{1}{\sqrt{c\nu}} \sum_{j=0}^{\infty} \left(\frac{\varepsilon}{\sqrt{c\nu}}\right)^{2j} \frac{F^{(2j)}(0)}{(2j)!} \Gamma(j+\tfrac{1}{2}).$$

As $E^{\mathbf{m}}_{\mathbb{C}}(F\circ\Lambda)(0) = \frac{\varepsilon^{2j}}{(2j)!} F^{(2j)}(0)$, we thus get

$$\frac{1}{[\nu]_{\mathbf{m}}} = \frac{\Gamma(m+\frac{1}{2})}{(c\nu)^{m+\frac{1}{2}}} = \frac{(2m)!\Gamma(\frac{1}{2})}{m! 4^m (c\nu)^{m+\frac{1}{2}}} \tag{5.3}$$

where the last equality used the doubling formula for the Gamma function.

This corresponds to the unnormalized Lebesgue measure on $\mathbb{C}$; it is usual to make a normalization so that $\rho_\nu \mathbf{1} = \mathbf{1}$, i.e. $[\nu]_{(0)} = 1$. If this is done then (5.3) gets divided by the same thing with $m=0$, that is, it becomes,

$$\frac{1}{[\nu]_{\mathbf{m}}} = \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})(c\nu)^m} = \frac{(2m)!}{(\varepsilon-\overline{\varepsilon})^{2m}\nu^m m!(-2)^m}.$$

Note that even though both $\rho^{\mathbf{m}}$ and $[\nu]_{\mathbf{m}}$ depend on $\varepsilon$, the sum

$$\rho_\nu = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{[\nu]_{\mathbf{m}}} = \sum_{m=0}^{\infty} \frac{\partial^{2m}}{m!(-2)^m \nu^m} = e^{-\partial^2/2\nu}$$

is independent of it, as it should.

A similar analysis can be done for $B_{\mathbb{R}} = \mathbb{R}^d$, $d>1$; cf. the next example.

**Example 5.2.** $B_{\mathbb{R}} = \mathbb{C}^d \cong \mathbb{R}^{2d}$, so that $B_{\mathbb{C}} = \mathbb{C}^d \times \overline{\mathbb{C}^d}$, where as always we identify $B_{\mathbb{R}}$ with $\{(z,\overline{z}) : z\in\mathbb{C}^d\} \subset B_{\mathbb{C}}$. For $\Lambda$, we let

$$\Lambda z = (z, \overline{az}) \tag{5.4}$$

with some fixed $a\in\mathbb{C}$, $a\neq 1$. The retraction $\pi$ is the oblique real-linear projection associated to the direct sum decomposition

$$\mathbb{C}^d \times \overline{\mathbb{C}^d} = \Lambda\mathbb{C}^d \oplus \Lambda_1 \mathbb{C}^d,$$

where $\Lambda_1$ is as in (5.4) but with $a=1$. Using again the Wirtinger derivatives $\partial, \overline{\partial}$ rather than $\partial_{\mathbb{R}}$ on $\mathbb{C}^d \cong \mathbb{R}^{2d}$, we have for any partition $\mathbf{m} = (m)$

$$E^{\mathbf{m}}_{\mathbb{R}} = \sum_{|\beta|=m} \frac{\partial^\beta \overline{\partial}^\beta}{\beta!}$$

with the usual multi-index notation. Recalling that the numbers $P^{\mathbf{m}}_{\rho,\sigma}$ are, quite generally, defined by

$$\sum_{\rho,\sigma} P^{\mathbf{m}}_{\rho,\sigma} \partial^\rho_{\mathbb{C}} \overline{\partial}^\sigma_{\mathbb{C}} F(0) = E^{\mathbf{m}}_{\mathbb{R}}(F\circ\Lambda)(0), \tag{5.5}$$

it follows that

$$P^{\mathbf{m}}_{\alpha\overline{\beta},\gamma\overline{\delta}} = \begin{cases} \dfrac{\rho!}{\alpha!\beta!\gamma!\delta!} a^{|\delta|}\overline{a}^{|\beta|} & \text{if } \alpha+\delta=\beta+\gamma=\rho,\ |\rho|=m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

(Here we are again using the "double" Wirtinger derivatives $\partial^{\alpha\overline{\beta}}_{\mathbb{C}}$ etc. as in Section 4.) As in the preceding example, the role of the "Jordan determinant" $h_{\mathbb{C}}$ is played by the function

$$h_{\mathbb{C}}((z,\overline{w}),(z_1,\overline{w}_1)) = e^{-(z|z_1)-(w_1|w)} \tag{5.7}$$

and $p=0$. Taking $H \in \mathcal{O}(B_{\mathbb{C}})$ of the form $H(z,\overline{w}) = f(z)\overline{g(w)}$ with $f,g \in \mathcal{O}(\mathbb{C}^d)$, we get as in the preceding example

$$\begin{aligned}
\rho^{\mathbf{m}} H &= \sum_{\varkappa,\lambda} (-\overline{\partial})^{\varkappa}(-\partial)^{\lambda} \sum_{\alpha,\beta,\gamma,\delta} \sum_{\iota,\eta} P^{\mathbf{m}}_{\alpha\overline{\beta},\gamma\overline{\delta}} \gamma^{\alpha\overline{\beta}}_{\iota\overline{\eta}} \overline{\gamma^{\gamma\overline{\delta}}_{\varkappa\overline{\lambda}}} \partial^{\iota\overline{\eta}} H \\
&= \sum_{\varkappa,\lambda} (-\overline{\partial})^{\varkappa}(-\partial)^{\lambda} \sum_{\alpha,\beta} P^{\mathbf{m}}_{\alpha\overline{\beta},\varkappa\overline{\lambda}} \partial^{\alpha\overline{\beta}} H = \sum_{\alpha,\beta,\varkappa,\lambda} (-1)^{|\varkappa+\lambda|} P^{\mathbf{m}}_{\alpha\overline{\beta},\varkappa\overline{\lambda}} \partial^{\alpha+\lambda} f \overline{\partial^{\beta+\varkappa} g} \\
&= \sum_{|\rho|=m} \sum_{\beta,\lambda\le\rho} (-1)^{|\rho-\beta+\lambda|} \binom{\rho}{\beta}\binom{\rho}{\lambda} \frac{a^{|\lambda|}\overline{a}^{|\beta|}}{\rho!} \partial^{\rho} f \overline{\partial^{\rho} g} \\
&= \sum_{|\rho|=m} \frac{(-1)^{|\rho|}}{\rho!} |1-a|^{2|\rho|} \partial^{\rho} f \overline{\partial^{\rho} g} = \frac{(-1)^m}{m!} |1-a|^{2m} \Big( \sum_{j=1}^{d} \partial_{z_j} \overline{\partial}_{w_j} \Big)^m H.
\end{aligned}$$

(Here the appearance of $(-\overline{\partial})^{\varkappa}(-\partial)^{\lambda}$, rather than $(-\partial)^{\varkappa}(-\overline{\partial})^{\lambda}$, is for the same reason as indicated at the end of the proof of Proposition 4.1.) Thus, symbolically,

$$\rho^{\mathbf{m}} = \frac{(-1)^m}{m!} |1-a|^{2m} (\partial \otimes \overline{\partial})^m.$$

To compute $[\nu]_{\mathbf{m}}$, we again start from (3.4). Observe that for the function $F \in \mathcal{O}(B_{\mathbb{C}})$ given by

$$F(z,\overline{w}) = z^{\alpha}\overline{z}^{\beta} w^{\gamma} \overline{w}^{\delta}, \qquad \text{where} \quad \alpha+\gamma=\beta+\delta=\rho,$$

we have by (5.5)

$$E^{\mathbf{m}}_{\mathbb{R}}(F\circ\Lambda)(0) = \alpha!\beta!\gamma!\delta! P^{\mathbf{m}}_{\alpha\overline{\beta},\gamma\overline{\delta}}.$$

Hence

$$\mathcal{P}_{\nu} F(0) = \sum_{\mathbf{m}} \frac{1}{[\nu]_{\mathbf{m}}} E^{\mathbf{m}}_{\mathbb{C}}(F\circ\Lambda)(0) = \frac{\rho!}{[\nu]_{|\rho|}} a^{|\gamma|}\overline{a}^{|\delta|}.$$

On the other hand, from (3.4) and (5.7),

$$\begin{aligned}
\mathcal{P}_{\nu} F(0) &= \int_{\mathbb{C}^d} F(z,\overline{az}) e^{-\nu|z|^2|1-a|^2} \, dz \\
&= a^{|\gamma|}\overline{a}^{|\delta|} \int_{\mathbb{C}^d} z^{\rho}\overline{z}^{\rho} e^{-\nu|1-a|^2|z|^2} \, dz = \frac{\rho!}{(|1-a|^2\nu)^{|\rho|}} a^{|\gamma|}\overline{a}^{|\delta|},
\end{aligned}$$

provided $dz$ is normalized so that $[\nu]_0 = 1$. Thus (under this normalization)

$$[\nu]_{\mathbf{m}} = \nu^m |1-a|^{2m}.$$

Note that, again,

$$\rho_\nu = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{[\nu]_{\mathbf{m}}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\nu^m} (\partial \otimes \overline{\partial})^m = e^{-\partial\otimes\overline{\partial}/\nu}$$

does not depend on $a$, even though $\rho^{\mathbf{m}}$ and $[\nu]_{\mathbf{m}}$ both do.

**Example 5.3.** As a first "non-flat" situation, consider the unit interval $B_{\mathbb{R}} = (-1, +1)$ with complexification $B_{\mathbb{C}} = \mathbb{D}$, the unit disc in $\mathbb{C}$; and we take the same $\Lambda$ as in Example 5.1, i.e. $\Lambda x = \varepsilon x$, $\varepsilon \in \mathbb{T} \setminus \mathbb{R}$. The constants $P^{\mathbf{m}}_{\alpha,\beta}$ are thus still given by (5.2), and $h_{\mathbb{C}}(x, y) = 1 - x\overline{y}$ while $p = 2$. Thus for $H \in \mathcal{O}(\mathbb{D})$,

$$\rho^{\mathbf{m}} H(x) = \big(1 - x^2\big) \sum_{\varkappa} (-1)^{\varkappa} \Big(\frac{d}{dx}\Big)^{\varkappa} \Bigg(\frac{1}{1 - x^2} \sum_{\substack{\alpha,\beta,\iota \\ \alpha+\beta=2m}} \frac{\varepsilon^{\alpha-\beta}}{\alpha!\beta!} \gamma^{\alpha}_{\iota}(x) \overline{\gamma^{\beta}_{\varkappa}(x)} \partial^{\iota} H(x)\Bigg).$$

This time explicit formulas are hard to come by, since the expressions $\gamma^{\alpha}_{\iota}(x)$ are quite complicated. One has, of course, $\rho^{(0)} H = H$, while

$$\rho^{(1)} H(x) = (\varepsilon - \overline{\varepsilon})^2 \big[\big(1 - x^2\big)^2 H''(x) - 2x\big(1 - x^2\big) H'(x)\big] = (\varepsilon - \overline{\varepsilon})^2 (H \circ \gamma_x)''(0)$$

is the $G_{\mathbb{R}}$-invariant operator uniquely determined by $\rho^{(1)} H(0) = (\varepsilon - \overline{\varepsilon})^2 H''(0)$. Computer-aided calculation similarly gives

$$\rho^{(2)} H(0) = 24(\varepsilon - \overline{\varepsilon})^2 H''(0) + (\varepsilon - \overline{\varepsilon})^4 H^{(4)}(0),$$
$$\rho^{(3)} H(0) = 1080(\varepsilon - \overline{\varepsilon})^2 H''(0) + 120(\varepsilon - \overline{\varepsilon})^4 H^{(4)}(0) + (\varepsilon - \overline{\varepsilon})^6 H^{(6)}(0).$$

The leading coefficient in $\rho^{(m)} H(0)$ is always $m^2(2m - 1)!$.

To compute $[\nu]_{\mathbf{m}}$, noting that $\det \Phi'(0, y) = 1 - y^2$ by Corollary 3.2, we get from (3.4) and (3.7),

$$\int_{-1}^{1} F(\varepsilon x) |1 - \varepsilon^2 x^2| \Big(\frac{1 - x^2}{|1 - \varepsilon^2 x^2|}\Big)^{\nu} \frac{dx}{(1 - x^2)^2} \sim \sum_{\mathbf{m}} \frac{(\varepsilon\partial + \overline{\varepsilon}\overline{\partial})^{2m} F(0)}{(2m)![\nu]_{\mathbf{m}}}.$$

Denoting $F(\varepsilon x) =: f(x)$ yields

$$\int_{-1}^{1} f(x) \Big(\frac{1 - x^2}{|1 - \varepsilon^2 x^2|}\Big)^{\nu-1} \frac{dx}{1 - x^2} \sim \sum_{m} \frac{f^{(2m)}(0)}{(2m)![\nu]_{\mathbf{m}}}.$$

Taking in particular $f(x) = x^{2m}$ we obtain

$$\frac{1}{[\nu]_{\mathbf{m}}} = \int_{-1}^{1} x^{2m} \frac{(1 - x^2)^{\nu-2}}{|1 - \varepsilon^2 x^2|^{\nu-1}}\, dx = \int_0^1 t^{m-\frac{1}{2}} \frac{(1 - t)^{\nu-2}}{|1 - \varepsilon^2 t|^{\nu-1}}\, dt. \tag{5.8}$$

Writing

$$\frac{1}{|1 - \varepsilon^2 t|^{\nu-1}} = (1 - \varepsilon^2 t)^{-(\nu-1)/2} (1 - \overline{\varepsilon}^2 t)^{-(\nu-1)/2} = \sum_{j,k=0}^{\infty} \frac{(\frac{\nu-1}{2})_j (\frac{\nu-1}{2})_k}{j!k!} \varepsilon^{2(j-k)} t^{j+k}$$

we arrive at the double series

$$\frac{1}{[\nu]_{\mathbf{m}}} = \frac{\Gamma(m + \frac{1}{2})\Gamma(\nu - 1)}{\Gamma(m + \nu - \frac{1}{2})} \sum_{j,k\geq 0} \frac{(\frac{\nu-1}{2})_j (\frac{\nu-1}{2})_k}{j!k!} \frac{(m + \frac{1}{2})_{j+k}}{(m + \nu - \frac{1}{2})_{j+k}} \varepsilon^{2(j-k)}.$$

The double sum on the right-hand side is the value at $x=\varepsilon$, $y=\overline{\varepsilon}$ of the Horn hypergeometric function of two variables [23, § 5.7.1]

$$F_1\big(m+\tfrac12,\tfrac{\nu-1}{2},\tfrac{\nu-1}{2},m+\nu-\tfrac12,x,y\big)$$

and in general cannot be evaluated in closed form. For particular values of $\varepsilon$, there may be some simplifications; for instance, for $\varepsilon=i$ the integral (5.8) becomes

$$\begin{aligned}\frac{1}{[\nu]_{\mathbf{m}}}&=\int_0^1 t^{m-\frac12}(1-t)^{\nu-2}(1+t)^{1-\nu}\,dt\\&=\frac{\Gamma(m+\frac12)\Gamma(\nu-1)}{\Gamma(m+\nu-\frac12)}\,{}_2F_1\big(m+\tfrac12,\nu-1;m+\nu-\tfrac12;-1\big),\end{aligned}$$

where ${}_2F_1$ is the ordinary Gauss hypergeometric function.

We remark that expressions involving values of ${}_2F_1$ at $-1$ occur as eigenvalues of the Berezin (or "link") transform corresponding to the Weyl calculus on rank 1 real symmetric spaces, cf. [5, Theorem 4.1]. (Also, Horn's hypergeometric functions of another kind – namely, $\Phi_2$ in the notation of [23] – appear in the formula for the *harmonic* Segal–Bargmann kernel on $\mathbb{C}^d$, see [21]; it is however unclear if there is any deeper relationship.)

**Example 5.4.** In this final example we consider $B_{\mathbb{R}}=\mathbb{D}$, embedded in $B_{\mathbb{C}}=\mathbb{D}\times\overline{\mathbb{D}}$ in the usual way as $\{(z,\overline{z}):z\in\mathbb{D}\}$. For $\Lambda$ we take the same map $\Lambda z=(z,\overline{az})$ as in Example 5.2, with some fixed $a\in\mathbb{C}$, $a\neq 1$. The corresponding retraction $\pi:B_{\mathbb{C}}\to B_{\mathbb{R}}$ assigns to $(z,\overline{w})\in\mathbb{D}\times\overline{\mathbb{D}}$ the (unique) point $x\in\mathbb{D}$ such that $\gamma_x w=a\gamma_x z$. (The existence of such $x$ follows by the following argument. For any $z,w,u,v\in\mathbb{D}$, the existence of $g\in G$ such that $gz=u$, $gw=v$ is equivalent to the equality

$$\rho(z,w)=\rho(u,v)\tag{5.9}$$

of the pseudohyperbolic distances $\rho(u,v):=|\frac{u-v}{1-\overline{u}v}|$. On the other hand, if $u$ runs through the interval $[0,\min\{1,\frac{1}{|a|}\})$ and $v=au$, then $\rho(u,v)$ runs from 0 to 1; thus (5.9) holds for some $u$. With $g$ as above, take $x=-g^{-1}(0)$.)

The constants $P^{\mathbf{m}}_{\alpha\overline{\beta},\gamma\overline{\delta}}$ are then still given by the formula (5.6) from Example 5.2, while the corresponding functions $\gamma^{\alpha\overline{\beta}}_{\iota\overline{\eta}}$ are easily seen to be given by $\gamma^{\alpha\overline{\beta}}_{\iota\overline{\eta}}(z,\overline{w})=\gamma^{\alpha}_{\iota}(z)\overline{\gamma^{\beta}_{\eta}(w)}$, where $\gamma^{\alpha}_{\iota}$ are the one-variable functions for the disc from the preceding example. By (3.11) we thus get for $H(z,\overline{w})=f(z)\overline{g(w)}$, $f,g\in\mathcal{O}(\mathbb{D})$, and $\mathbf{m}=(m)$,

$$\begin{aligned}\rho^{\mathbf{m}}H(z,\overline{z})=(1-z\overline{w})^2\sum_{\varkappa,\lambda}(-\overline{\partial}_w)^{\varkappa}(-\partial_z)^{\lambda}\Big[(1-z\overline{w})^{-2}\sum_{\alpha,\beta,\gamma,\delta,\iota,\eta}P^{\mathbf{m}}_{\alpha\overline{\beta},\gamma\overline{\delta}}\\ \gamma^{\alpha}_{\iota}(z)\overline{\gamma^{\beta}_{\eta}(w)}\,\overline{\gamma^{\gamma}_{\varkappa}(w)}\gamma^{\delta}_{\lambda}(z)\partial^{\iota}f(z)\overline{\partial^{\eta}g(w)}\Big]\Big|_{w=z}.\end{aligned}$$

Here again $(-\overline{\partial}_w)^{\varkappa}(-\partial_z)^{\lambda}$ occurs rather than $(-\overline{\partial}_w)^{\lambda}(-\partial_z)^{\varkappa}$, and likewise $\overline{\gamma^{\gamma}_{\varkappa}(w)}\gamma^{\delta}_{\lambda}(z)$ rather than $\gamma^{\gamma}_{\varkappa}(z)\overline{\gamma^{\delta}_{\lambda}(w)}$, for the same reasons as in Example 5.2 and in the proof of Proposition 4.1.

For low values of $m$, one computes that $\rho^{(0)}(f\overline{g})=f\overline{g}$ (of course), while

$$\rho^{(1)}(f\overline{g})(z)=-|1-a|^2\big(1-|z|^2\big)^2f'(z)\overline{g'(z)}$$

is the $G$-invariant operator from $\mathcal{O}(\mathbb{D}\times\overline{\mathbb{D}})$ into $\mathcal{C}^\infty(\mathbb{D})$ uniquely determined by

$$\rho^{(1)}(f\overline{g})(0)=-|1-a|^2f'(0)\overline{g'(0)}.$$

Computer-aided calculations give

$$\rho^{(2)}(f\overline{g})(0) = -\frac{|1-a|^2}{2}\Big[4(1+|a|^2)f'(0)\overline{g'(0)} - |1-a|^2 f''(0)\overline{g''(0)}\Big]$$

and

$$\rho^{(3)}(f\overline{g})(0) = -\frac{|1-a|^2}{6}\Big[36(1+|a|^2+|a|^4)f'(0)\overline{g'(0)} - 18|1-a|^2(1+|a|^2)f''(0)\overline{g''(0)} \\ + |1-a|^4 f'''(0)\overline{g'''(0)}\Big].$$

To compute $[\nu]_{\mathbf{m}}$, we note as in Example 5.2 that for the function $F \in \mathcal{C}^\infty(\mathbb{D}\times\overline{\mathbb{D}})$ given by

$$F(z,\overline{w}) = z^\alpha \overline{z}^\beta w^\gamma \overline{w}^\delta, \qquad \text{where} \quad \alpha+\gamma = \beta+\delta = \rho,$$

one has by (3.7)

$$\mathcal{P}_\nu F(0) = a^\gamma \overline{a}^\delta \frac{\rho!}{[\nu]_\rho}.$$

On the other hand, since now $\det \Phi'(0,\Lambda y) = |1-a|^2(1-|a|^2|y|^4)$ by Lemma 3.1, we have from (3.4)

$$\mathcal{P}_\nu F(0) = a^\gamma \overline{a}^\delta |1-a|^2 \int_{\mathbb{D}} z^\rho \overline{z}^\rho \big(1-|a|^2|z|^4\big)\frac{(1-|z|^2)^\nu(1-|az|^2)^\nu}{|1-\overline{a}|z|^2|^{2\nu}} \frac{dz}{(1-|z|^2)^2(1-|az|^2)^2}.$$

Passing to polar coordinates, we thus obtain (writing $m$ instead of $\rho$),

$$\frac{m!}{[\nu]_{\mathbf{m}}} = |1-a|^2 \int_0^1 t^m \frac{(1-|a|^2t^2)}{(1-t)^2(1-|a|^2t)^2} \frac{(1-t)^\nu(1-|a|^2t)^\nu}{|1-at|^{2\nu}}\, dt. \tag{5.10}$$

Using series expansions, the integral can again be expressed in terms of Horn-type two-variable hypergeometric functions, and simplifies for some special values of $a$. In particular, for $a=0$ the right-hand side of (5.10) is just

$$\int_0^1 t^m(1-t)^{\nu-2}\, dt = \frac{m!\Gamma(\nu+1)}{\Gamma(\nu+m)},$$

so that

$$\frac{1}{[\nu]_{\mathbf{m}}} = \frac{\Gamma(\nu-1)}{\Gamma(\nu+m)},$$

or, upon renormalizing so that $[\nu]_0 = 1$,

$$[\nu]_{\mathbf{m}} = \frac{\Gamma(\nu+m)}{\Gamma(\nu)} = (\nu)_m,$$

in agreement with the result

$$\rho_\nu(f\overline{g}) = \sum_{\mathbf{m}} \frac{A_{\mathbf{m}}(f,g)}{(\nu)_{\mathbf{m}}}$$

from [18] reviewed in Section 4. Similarly, for $a=-1$, (5.10) becomes

$$\frac{1}{[\nu]_{\mathbf{m}}} = \frac{4\Gamma(2\nu-2)}{\Gamma(m+2\nu-1)}\, {}_2F_1(2\nu-1, m+1; m+2\nu-1; -1),$$

or, upon renormalizing $dz$ so that $[\nu]_0 = 1$,

$$\frac{1}{[\nu]_{\mathbf{m}}} = \frac{2}{(2\nu-1)_m}\, {}_2F_1(2\nu-1, m+1; m+2\nu-1; -1).$$

Crude estimates also show that

$$[\nu]_m \sim |1-a|^{2m}\nu^m \left[1 - \frac{2(1+|a|^2)}{|1-a|^2\nu} + O\left(\frac{1}{\nu^2}\right)\right]$$

as $\nu \to +\infty$, which can be used to check at least for the first few terms that, again,

$$\rho_\nu = \sum_{\mathbf{m}} \frac{\rho^{\mathbf{m}}}{[\nu]_{\mathbf{m}}}$$

is indeed independent of $a$, although both $\rho^{\mathbf{m}}$ and $[\nu]_{\mathbf{m}}$ are not. Note that the retraction $\pi$ in this case ($a = -1$) is simply

$$\pi(z, \overline{w}) = m_{z,w},$$

the geodesic mid-point between $z$ and $w$.

# A Table of parameters of real bounded symmetric domains

The table on the next page lists the groups $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, the root type $\Sigma$, the rank $r_{\mathbb{R}}$, characteristic multiplicities $a_{\mathbb{R}}$, $b_{\mathbb{R}}$, $c_{\mathbb{R}}$ and the dimension $d$ of real bounded symmetric domains $B_{\mathbb{R}}$, as well as the analogous parameters $r_{\mathbb{C}}$, $a_{\mathbb{C}}$, $b_{\mathbb{C}}$ of the complex domains $B_{\mathbb{C}}$ and the labellings of $B_{\mathbb{C}}$ and $B_{\mathbb{R}}$ following the notation in [30, Chapter 11]. The table was mostly compiled using [37, 26, 24] and [30]. The low-dimensional isomorphisms between the various types, and the resulting restrictions on subscripts needed in order to make the table entries non-redundant, can be found e.g. in the cited chapter in Loos [30]. As a matter of notation, we use $G_n(\mathbb{K})$ and $U_{p,q}(\mathbb{K})$ for the identity component of the general linear (resp. pseudo-unitary) group over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (= quaternions). $Sp_{2r}(\mathbb{K})$ is the $2r \times 2r$-symplectic group over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, whereas $O_n(\mathbb{H})$ is the quaternion analogue of $O_n(\mathbb{C})$ (usually denoted by $SO^*(2n)$).

The genus of $B_{\mathbb{C}}$ is given in terms of the domain parameters by

$$p = (r_{\mathbb{C}} - 1)a_{\mathbb{C}} + b_{\mathbb{C}} + 2,$$

while the dimension $d = \dim_{\mathbb{R}} B_{\mathbb{R}} = \dim_{\mathbb{C}} B_{\mathbb{C}}$ equals

$$d = \frac{r_{\mathbb{R}}(r_{\mathbb{R}}-1)}{2}a_{\mathbb{R}} + r_{\mathbb{R}} = \frac{r_{\mathbb{C}}(r_{\mathbb{C}}-1)}{2}a_{\mathbb{C}} + r_{\mathbb{C}}$$

for type A, and

$$d = r_{\mathbb{R}}(r_{\mathbb{R}}-1)a_{\mathbb{R}} + r_{\mathbb{R}}b_{\mathbb{R}} + r_{\mathbb{R}}c_{\mathbb{R}} + r_{\mathbb{R}} = \frac{r_{\mathbb{C}}(r_{\mathbb{C}}-1)}{2}a_{\mathbb{C}} + r_{\mathbb{C}}b_{\mathbb{C}} + r_{\mathbb{C}}$$

for all other types. Domains of type $D_2$ turn out to have, in some sense, two multiplicities $a$ instead of one.

Note that the unit interval corresponds to $I^{\mathbb{R}}_{1,1}$, the unit ball of $\mathbb{R}^m$, $m > 1$, to $I^{\mathbb{R}}_{1,m}$, the unit ball of $\mathbb{C}^m$ to $I_{1,m}$, the unit ball of the algebra of quaternions $\mathbb{H}$ to $I^{\mathbb{H}}_{2,2m}$, the unit ball of $\mathbb{H}^m$, $m > 1$, to $I^{\mathbb{H}}_{2,2m}$, and the unit ball of the Cayley plane $\mathbb{O}^2$ to $V^{\mathbb{O}}$. In the "complex" cases, the root type does not quite make sense ("BC×BC") and nor do the parameters $a_{\mathbb{C}}$, $b_{\mathbb{C}}$, $r_{\mathbb{C}}$, while $B_{\mathbb{C}}$ is just the product $B \times \overline{B}$; so these columns are left empty.

| $B_{\mathbb{R}}$ | $G_{\mathbb{R}}/K_{\mathbb{R}}$ | $\Sigma$ | $r_{\mathbb{R}}$ | $a_{\mathbb{R}}$ | $b_{\mathbb{R}}$ | $c_{\mathbb{R}}$ | $d$ | $r_{\mathbb{C}}$ | $a_{\mathbb{C}}$ | $b_{\mathbb{C}}$ | $B_{\mathbb{C}}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $I^{\mathbb{R}}_{r,r+b}$ | $U_{r,r+b}(\mathbb{R})/U_r(\mathbb{R}) \times U_{r+b}(\mathbb{R})$ | $D_r/B_r$ | $r$ | 1 | $b$ | 0 | $r(r+b)$ | $r$ | 2 | $b$ | $I_{r,r+b}$ |
| $I_{r,r+b}$ | $U_{r,r+b}(\mathbb{C})/U_r(\mathbb{C}) \times U_{r+b}(\mathbb{C})$ | | $r$ | 2 | $2b$ | 1 | $2r(r+b)$ | | | | (product case) |
| $I^{\mathbb{H}}_{2r,2r+2b}$ | $U_{r,r+b}(\mathbb{H})/U_r(\mathbb{H}) \times U_{r+b}(\mathbb{H})$ | $C_r/BC_r$ | $r$ | 4 | $4b$ | 3 | $4r(r+b)$ | $2r$ | 2 | $2b$ | $I_{2r,2r+2b}$ |
| $V^{\mathbb{O}_0}$ | $U_{2,2}(\mathbb{H})/U_2(\mathbb{H}) \times U_2(\mathbb{H})$ | $B_2$ | 2 | 3 | 4 | 0 | 16 | 2 | 6 | 4 | $V$ |
| $III^{\mathbb{R}}_r$ | $G_r(\mathbb{R})/U_r(\mathbb{R})$ | $A_r$ | $r$ | 1 | – | – | $\frac{1}{2}r(r+1)$ | $r$ | 1 | 0 | $III_r$ |
| $I^{\mathbb{C}}_{r,r}$ | $G_r(\mathbb{C})/U_r(\mathbb{C})$ | $A_r$ | $r$ | 2 | – | – | $r^2$ | $r$ | 2 | 0 | $I_{r,r}$ |
| $II^{\mathbb{H}}_{2r}$ | $G_r(\mathbb{H})/U_r(\mathbb{H})$ | $A_r$ | $r$ | 4 | – | – | $r(2r-1)$ | $r$ | 4 | 0 | $II_{2r}$ |
| $VI^{\mathbb{O}_0}$ | $G_4(\mathbb{H})/U_4(\mathbb{H})$ | $D_3$ | 3 | 4 | 0 | 0 | 27 | 3 | 8 | 0 | $VI$ |
| $III_r$ | $Sp_{2r}(\mathbb{R})/U_r(\mathbb{C})$ | | $r$ | 1 | 0 | 1 | $r(r+1)$ | | | | (product case) |
| $III^{\mathbb{H}}_{2r}$ | $Sp_{2r}(\mathbb{C})/U_r(\mathbb{H})$ | $C_r$ | $r$ | 2 | 0 | 2 | $r(2r+1)$ | $2r$ | 1 | 0 | $III_{2r}$ |
| $II^{\mathbb{R}}_{2r+\varepsilon}$ | $O_{2r+\varepsilon}(\mathbb{C})/U_{2r+\varepsilon}(\mathbb{R})$ | $D_r/B_r$ | $r$ | 2 | $2\varepsilon$ | 0 | $r(2(r+\varepsilon)-1)$ | $r$ | 4 | 2 | $II_{2r+\varepsilon}$ |
| $II_{2r+\varepsilon}$ | $O_{2r+\varepsilon}(\mathbb{H})/U_{2r+\varepsilon}(\mathbb{C})$ | | $r$ | 4 | $4\varepsilon$ | 1 | $2r(2(r+\varepsilon)-1)$ | | | | (product case) |
| $IV^{\mathbb{R},q}_{p+q}$ | $SO_{p,1} \times SO_{1,q}/SO_{p,0} \times SO_{0,q}$ | $D_2/A_2$ | 2 | n/a | 0 | 0 | $p+q$ | 2 | $p+q-2$ | 0 | $IV_{p+q}$ |
| $IV_n$ | $SO_{n,2}/SO_{n,0} \times SO_{0,2}$ | | 2 | $n-2$ | 0 | 1 | $2n$ | | | | (product case) |
| $V$ | $E_{6(-14)}/Spin(10) \times SO(2)$ | | 2 | 6 | 8 | 1 | 32 | | | | (product case) |
| $IV^{\mathbb{R},0}_n$ | $SO_{n,1}/SO_{n,0}$ | $C_1$ | 1 | – | 0 | $n-1$ | $n$ | 2 | $n-2$ | 0 | $IV_n$ |
| $V^{\mathbb{O}}$ | $F_{4(-20)}/SO(9)$ | $BC_1$ | 1 | – | 8 | 7 | 16 | 2 | 6 | 4 | $V$ |
| $VI$ | $E_{7(-25)}/E_6 \times SO(2)$ | | 3 | 8 | 0 | 1 | 54 | | | | (product case) |
| $VI^{\mathbb{O}}$ | $E_{6(-26)} \times O(2)/F_4 \times O(1)$ | $A_3$ | 3 | 8 | – | – | 27 | 3 | 8 | 0 | $VI$ |

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