Besov-Type Spaces on $\mathbb{R}^d$ and Integrability for the Dunkl Transform*

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Abstract. In this paper, we show the inclusion and the density of the Schwartz space in Besov–Dunkl spaces and we prove an interpolation formula for these spaces by the real method. We give another characterization for these spaces by convolution. Finally, we establish further results concerning integrability of the Dunkl transform of function in a suitable Besov–Dunkl space.

Key words: Dunkl operators; Dunkl transform; Dunkl translations; Dunkl convolution; Besov–Dunkl spaces

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1 Introduction

We consider the differential-difference operators T_i, $1 \leq i \leq d$, on $\mathbb{R}^d$, associated with a positive root system R_+ and a non negative multiplicity function k, introduced by C.F. Dunkl in [9] and called Dunkl operators (see next section). These operators can be regarded as a generalization of partial derivatives and lead to generalizations of various analytic structure, like the exponential function, the Fourier transform, the translation operators and the convolution (see [8, 10, 11, 16, 17, 18, 19, 22]). The Dunkl kernel E_k has been introduced by C.F. Dunkl in [10]. This kernel is used to define the Dunkl transform $\mathcal{F}_k$. K. Trimèche has introduced in [23] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, on the space of infinitely differentiable functions on $\mathbb{R}^d$. At the moment an explicit formula for the Dunkl translation operator of function $\tau_x(f)$ is unknown in general. However, such formula is known when f is a radial function and the L^p-boundedness of τ_x for radial functions is established. As a result, we have the Dunkl convolution $*_k$.

There are many ways to define the Besov spaces (see [6, 15, 21]) and the Besov spaces for the Dunkl operators (see [1, 2, 3, 4, 14]). Let $\beta > 0$, $1 \leq p, q \leq +\infty$, the Besov–Dunkl space denoted by $\mathcal{BD}^{\beta,k}_{p,q}$ in this paper, is the subspace of functions $f \in L^p_k(\mathbb{R}^d)$ satisfying

$$\|f\|_{\mathcal{BD}^{\beta,k}_{p,q}} = \left(\sum_{j\in\mathbb{Z}} (2^{j\beta} \|\varphi_j *_k f\|_{p,k})^q \right)^{\frac{1}{q}} < +\infty \qquad \text{if} \quad q < +\infty$$

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$$\|f\|_{\mathcal{BD}^{\beta,k}_{p,\infty}} = \sup_{j\in\mathbb{Z}} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} < +\infty \qquad \text{if} \quad q = +\infty,$$

where $(\varphi_j)_{j\in\mathbb{Z}}$ is a sequence of functions in $\mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ such that

(i) $\operatorname{supp}\mathcal{F}_k(\varphi_j) \subset A_j = \left\{x \in \mathbb{R}^d\,;\, 2^{j-1} \le \|x\| \le 2^{j+1}\right\}$ for $j \in \mathbb{Z}$;

(ii) $\sup_{j\in\mathbb{Z}} \|\varphi_j\|_{1,k} < +\infty$;

(iii) $\sum_{j\in\mathbb{Z}} \mathcal{F}_k(\varphi_j)(x) = 1$, for $x \in \mathbb{R}^d\backslash\{0\}$.

$\mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ being the subspace of functions in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ which are radial.

Put $\mathcal{A} = \left\{\phi \in \mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}} : \operatorname{supp}\mathcal{F}_k(\phi) \subset \{x \in \mathbb{R}^d;\, 1 \le \|x\| \le 2\}\right\}$. Given $\phi \in \mathcal{A}$, we denote by $\mathcal{C}^{\phi,\beta,k}_{p,q}$ the subspace of functions $f \in L^p_k(\mathbb{R}^d)$ satisfying

$$\left(\int_0^{+\infty} \left(\frac{\|f *_k \phi_t\|_{p,k}}{t^\beta}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < +\infty \qquad \text{if} \quad q < +\infty$$

and

$$\sup_{t\in(0,+\infty)} \frac{\|f *_k \phi_t\|_{p,k}}{t^\beta} < +\infty \qquad \text{if} \quad q = +\infty,$$

where $\phi_t(x) = \frac{1}{t^{2(\gamma+\frac{d}{2})}}\phi(\frac{x}{t})$, for all $t \in (0,+\infty)$ and $x \in \mathbb{R}^d$.

In this paper we show for $\beta > 0$ the inclusion of the Schwartz space in $\mathcal{BD}^{\beta,k}_{p,q}$ for $1 \le p, q \le +\infty$ and the density when $1 \le p, q < +\infty$. We prove an interpolation formula for the Besov–Dunkl spaces by the real method. We compare these spaces with $\mathcal{C}^{\phi,\beta,k}_{p,q}$ which extend to the Dunkl operators on $\mathbb{R}^d$ some results obtained in [4, 5, 21]. Finally we establish further results of integrability of $\mathcal{F}_k(f)$ when f is in a suitable Besov–Dunkl space $\mathcal{BD}^{\beta,k}_{p,q}$ for $1 \le p \le 2$ and $1 \le q \le +\infty$. Using the characterization of the Besov spaces by differences analogous results of integrability have been obtained in the case $q = 1$ by Giang and Móricz in [13] for a classical Fourier transform on $\mathbb{R}$ and for $q = 1, +\infty$ by Betancor and Rodríguez-Mesa in [7] for the Hankel transform on $(0, +\infty)$ in Lipschitz–Hankel spaces. Later Abdelkefi and Sifi in [1, 2] have established similar results of integrability for the Dunkl transform on $\mathbb{R}$ and in radial case on $\mathbb{R}^d$. The argument used in [1, 2] to establish such integrability is the L^p-boundedness of the Dunkl translation operators, making it difficult to extend the results on $\mathbb{R}^d$. We take a different approach based on the the characterization of the Besov spaces by convolution to establish our results on higher dimension.

The contents of this paper are as follows. In Section 2 we collect some basic definitions and results about harmonic analysis associated with Dunkl operators. In Section 3 we show the inclusion and the density of the Schwartz space in $\mathcal{BD}^{\beta,k}_{p,q}$, we prove an interpolation formula for the Besov–Dunkl spaces by the real method and we compare these spaces with $\mathcal{C}^{\phi,\beta,k}_{p,q}$. In Section 4 we establish our results concerning integrability of the Dunkl transform of function in the Besov–Dunkl spaces.

Along this paper we denote by $\langle\cdot,\cdot\rangle$ the usual Euclidean inner product in $\mathbb{R}^d$ as well as its extension to $\mathbb{C}^d \times \mathbb{C}^d$, we write for $x \in \mathbb{R}^d$, $\|x\| = \sqrt{\langle x, x\rangle}$ and we represent by c a suitable positive constant which is not necessarily the same in each occurrence. Furthermore we denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$;
- $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of functions in $\mathcal{E}(\mathbb{R}^d)$ which are rapidly decreasing as well as their derivatives;
- $\mathcal{D}(\mathbb{R}^d)$ the subspace of $\mathcal{E}(\mathbb{R}^d)$ of compactly supported functions.

2 Preliminaries

Let W be a finite reflection group on $\mathbb{R}^d$, associated with a root system R and R_+ the positive subsystem of R (see [8, 10, 11, 12, 18, 19]). We denote by k a nonnegative multiplicity function defined on R with the property that k is W-invariant. We associate with k the index

$$\gamma = \gamma(R) = \sum_{\xi \in R_+} k(\xi) \geq 0,$$

and the weight function w_k defined by

$$w_k(x) = \prod_{\xi \in R_+} |\langle \xi, x\rangle|^{2k(\xi)}, \qquad x \in \mathbb{R}^d.$$

Further we introduce the Mehta-type constant c_k by

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} w_k(x) dx \right)^{-1}.$$

For every $1 \leq p \leq +\infty$ we denote by $L^p_k(\mathbb{R}^d)$ the space $L^p(\mathbb{R}^d, w_k(x)dx)$, $L^p_k(\mathbb{R}^d)^{\mathrm{rad}}$ the subspace of those $f \in L^p_k(\mathbb{R}^d)$ that are radial and we use $\|\cdot\|_{p,k}$ as a shorthand for $\|\cdot\|_{L^p_k(\mathbb{R}^d)}$.

By using the homogeneity of w_k it is shown in [18] that for $f \in L^1_k(\mathbb{R}^d)^{\mathrm{rad}}$ there exists a function F on $[0, +\infty)$ such that $f(x) = F(\|x\|)$, for all $x \in \mathbb{R}^d$. The function F is integrable with respect to the measure $r^{2\gamma+d-1}dr$ on $[0, +\infty)$ and we have

$$\int_{S^{d-1}} w_k(x) d\sigma(x) = \frac{c_k^{-1}}{2^{\gamma + \frac{d}{2} - 1}\Gamma(\gamma + \frac{d}{2})},$$

where S^{d-1} is the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$ and

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) w_k(x) dx &= \int_0^{+\infty} \Big(\int_{S^{d-1}} w_k(ry) d\sigma(y) \Big) F(r) r^{d-1} dr \\ &= \frac{c_k^{-1}}{2^{\gamma + \frac{d}{2} - 1}\Gamma(\gamma + \frac{d}{2})} \int_0^{+\infty} F(r) r^{2\gamma+d-1} dr. \end{aligned} \tag{1}$$

Introduced by C.F. Dunkl in [9] the Dunkl operators T_j, $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k are the first-order differential- difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \qquad f \in \mathcal{E}(\mathbb{R}^d), \qquad x \in \mathbb{R}^d,$$

where $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, e_2, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

The Dunkl kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F. Dunkl in [10]. For $y \in \mathbb{R}^d$ the function $x \mapsto E_k(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{aligned} &T_j u(x, y) = y_j\, u(x, y), \qquad 1 \leq j \leq d, \\ &u(0, y) = 1. \end{aligned}$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. M. Rösler has proved in [17] the following integral representation for the Dunkl kernel

$$E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\mu_x^k(y), \qquad x \in \mathbb{R}^d, \qquad z \in \mathbb{C}^d,$$

where μ_x^k is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$.

We have for all $\lambda \in \mathbb{C}$ and $z, z' \in \mathbb{C}^d$ $E_k(z, z') = E_k(z', z)$, $E_k(\lambda z, z') = E_k(z, \lambda z')$ and for $x, y \in \mathbb{R}^d$ $|E_k(x, iy)| \leq 1$ (see [10, 17, 18, 19, 22]).

The Dunkl transform $\mathcal{F}_k$ which was introduced by C.F. Dunkl in [11] (see also [8]) is defined for $f \in \mathcal{D}(\mathbb{R}^d)$ by

$$\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^d} f(y) E_k(-ix, y) w_k(y) dy, \qquad x \in \mathbb{R}^d.$$

According to [8, 11, 18] we have the following results:

i) The Dunkl transform of a function $f \in L_k^1(\mathbb{R}^d)$ has the following basic property

$$\|\mathcal{F}_k(f)\|_{\infty,k} \leq \|f\|_{1,k}. \tag{2}$$

ii) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under the Dunkl transform $\mathcal{F}_k$.

iii) When both f and $\mathcal{F}_k(f)$ are in $L_k^1(\mathbb{R}^d)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) E_k(ix, y) w_k(y) dy, \qquad x \in \mathbb{R}^d.$$

iv) (Plancherel's theorem) The Dunkl transform on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$.

By (2), Plancherel's theorem and the Marcinkiewicz interpolation theorem (see [20]) we get for $f \in L_k^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|\mathcal{F}_k(f)\|_{p',k} \leq c\|f\|_{p,k}. \tag{3}$$

The Dunkl transform of a function in $L_k^1(\mathbb{R}^d)^{\mathrm{rad}}$ is also radial and could be expressed via the Hankel transform (see [18, Proposition 2.4]).

K. Trimèche has introduced in [23] the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, on $\mathcal{E}(\mathbb{R}^d)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$ we have

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y)\mathcal{F}_k(f)(y)$$

and

$$\tau_x(f)(y) = c_k \int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi) E_k(ix, \xi) E_k(iy, \xi) w_k(\xi) d\xi. \tag{4}$$

Notice that for all $x, y \in \mathbb{R}^d$ $\tau_x(f)(y) = \tau_y(f)(x)$, and for fixed $x \in \mathbb{R}^d$

$$\tau_x \text{ is a continuous linear mapping from } \mathcal{E}(\mathbb{R}^d) \text{ into } \mathcal{E}(\mathbb{R}^d). \tag{5}$$

As an operator on $L_k^2(\mathbb{R}^d)$, τ_x is bounded. A priori it is not at all clear whether the translation operator can be defined for L^p-functions with p different from 2. However, according to [19, Theorem 3.7] the operator τ_x can be extended to $L_k^p(\mathbb{R}^d)^{\mathrm{rad}}$, $1 \leq p \leq 2$ and for $f \in L_k^p(\mathbb{R}^d)^{\mathrm{rad}}$ we have

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}.$$

The Dunkl convolution product $*_k$ of two functions f and g in $L^2_k(\mathbb{R}^d)$ (see [19, 23]) is given by

$$(f \ *_k g)(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y)g(y)w_k(y)dy, \qquad x \in \mathbb{R}^d.$$

The Dunkl convolution product is commutative and for $f, g \in \mathcal{D}(\mathbb{R}^d)$ we have

$$\mathcal{F}_k(f \ *_k \ g) = \mathcal{F}_k(f)\mathcal{F}_k(g). \tag{6}$$

It was shown in [19, Theorem 4.1] that when g is a bounded function in $L^1_k(\mathbb{R}^d)^{\mathrm{rad}}$, then

$$(f *_k g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x(g)(-y)w_k(y)dy, \qquad x \in \mathbb{R}^d,$$

initially defined on the intersection of $L^1_k(\mathbb{R}^d)$ and $L^2_k(\mathbb{R}^d)$ extends to all $L^p_k(\mathbb{R}^d)$, $1 \le p \le +\infty$ as a bounded operator. In particular,

$$\|f *_k g\|_{p,k} \le \|f\|_{p,k}\|g\|_{1,k}. \tag{7}$$

The Dunkl Laplacian Δ_k is defined by $\Delta_k := \sum_{i=1}^{d} T_i^2$. From [16] we have for each $\lambda > 0$ $\lambda I - \Delta_k$ maps $\mathcal{S}(\mathbb{R}^d)$ onto itself and

$$\mathcal{F}_k((\lambda I - \Delta_k)f)(x) = \big(\lambda + \|x\|^2\big)\mathcal{F}_k(f)(x), \qquad \text{for} \quad x \in \mathbb{R}^d. \tag{8}$$

3 Interpolation and characterization for the Besov–Dunkl spaces

In this section we establish the inclusion and the density of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{BD}^{\beta,k}_{p,q}$ and we prove an interpolation formula for the Besov–Dunkl spaces by the real method. Finally we compare the spaces $\mathcal{BD}^{\beta,k}_{p,q}$ with $\mathcal{C}^{\phi,\beta,k}_{p,q}$. Before, we start with some useful results.

We shall denote by $\mathbf{\Phi}$ the set of all sequences of functions $(\varphi_j)_{j\in\mathbb{Z}}$ in $\mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ satisfying

(i) $\operatorname{supp} \mathcal{F}_k(\varphi_j) \subset A_j = \{x \in \mathbb{R}^d;\ 2^{j-1} \le \|x\| \le 2^{j+1}\}$ for $j \in \mathbb{Z}$;

(ii) $\sup_{j\in\mathbb{Z}} \|\varphi_j\|_{1,k} < +\infty$;

(iii) $\sum_{j\in\mathbb{Z}} \mathcal{F}_k(\varphi_j)(x) = 1$, for $x \in \mathbb{R}^d\backslash\{0\}$.

Proposition 1. *Let $\beta > 0$ and $1 \le p, q \le +\infty$, then $\mathcal{BD}^{\beta,k}_{p,q}$ is independent of the choice of the sequence in $\mathbf{\Phi}$.*

Proof. Fix $(\phi_j)_{j\in\mathbb{Z}}$, $(\psi_j)_{j\in\mathbb{Z}}$ in $\mathbf{\Phi}$ and $f \in \mathcal{BD}^{\beta,k}_{p,q}$ for $q < +\infty$. Using the properties (i) for $(\phi_j)_{j\in\mathbb{Z}}$ and (i) and (iii) for $(\psi_j)_{j\in\mathbb{Z}}$, we have for $j \in \mathbb{Z}$ $\phi_j = \phi_j *_k (\psi_{j-1} + \psi_j + \psi_{j+1})$. Then by the property (ii) for $(\phi_j)_{j\in\mathbb{Z}}$, (7) and Hölder's inequality for $j \in \mathbb{Z}$ we obtain

$$\|f *_k \phi_j\|^q_{p,k} \le c\, 3^{q-1} \sum_{s=j-1}^{j+1} \|\psi_s *_k f\|^q_{p,k}.$$

Thus summing over j with weights $2^{j\beta q}$ we get

$$\sum_{j\in\mathbb{Z}} \big(2^{j\beta}\|\phi_j *_k f\|_{p,k}\big)^q \le c \sum_{j\in\mathbb{Z}} \big(2^{j\beta}\|\psi_j *_k f\|_{p,k}\big)^q.$$

Hence by symmetry we get the result of our proposition. When $q = +\infty$ we make the usual modification. ■

Remark 1. Let $\beta > 0$, $1 \le p, q \le +\infty$, we denote by $\ddot{\mathcal{B}}\mathcal{D}^{\beta,k}_{p,q}$ the subspace of functions $f \in L^p_k(\mathbb{R}^d)$ satisfying

$$\left(\sum_{j\in\mathbb{N}}(2^{j\beta}\|\varphi_j *_k f\|_{p,k})^q\right)^{\frac{1}{q}} < +\infty \qquad \text{if} \quad q < +\infty$$

and

$$\sup_{j\in\mathbb{N}} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} < +\infty \qquad \text{if} \quad q = +\infty,$$

where $(\varphi_j)_{j\in\mathbb{N}}$ is a sequence of functions in $\mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ such that

i) $\operatorname{supp}\mathcal{F}_k(\varphi_0) \subset \{x \in \mathbb{R}^d;\ \|x\| \le 2\}$ and $\operatorname{supp}\mathcal{F}_k(\varphi_j) \subset A_j = \{x \in \mathbb{R}^d;\ 2^{j-1} \le \|x\| \le 2^{j+1}\}$ for $j \in \mathbb{N}\backslash\{0\}$;

ii) $\sup\limits_{j\in\mathbb{N}} \|\varphi_j\|_{1,k} < +\infty$;

iii) $\sum\limits_{j\in\mathbb{N}} \mathcal{F}_k(\varphi_j)(x) = 1$, for $x \in \mathbb{R}^d\backslash\{0\}$.

As the Besov–Dunkl spaces these spaces are also independent of the choice of the sequence $(\varphi_j)_{j\in\mathbb{N}}$ satisfying the previous properties.

Proposition 2. *For $\beta > 0$ and $1 \le p, q \le +\infty$ we have*

$$\ddot{\mathcal{B}}\mathcal{D}^{\beta,k}_{p,q} = \mathcal{B}\mathcal{D}^{\beta,k}_{p,q}.$$

Proof. Since both spaces $\ddot{\mathcal{B}}\mathcal{D}^{\beta,k}_{p,q}$ and $\mathcal{B}\mathcal{D}^{\beta,k}_{p,q}$ are in $L^p_k(\mathbb{R}^d)$ and are independent of the specific selection of sequence of functions, then according to [5, Lemma 6.1.7, Theorem 6.3.2] we can take a function $\phi \in \mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ such that

- $\operatorname{supp}\mathcal{F}_k(\phi) \subset \{x \in \mathbb{R}^d;\ \frac{1}{2} \le \|x\| \le 2\}$;
- $\mathcal{F}_k(\phi)(x) > 0$ for $\frac{1}{2} < \|x\| < 2$;
- $\sum\limits_{j\in\mathbb{Z}} \mathcal{F}_k(\phi_{2^{-j}})(x) = 1$, $x \in \mathbb{R}^d\backslash\{0\}$.

If we consider the sequences $(\psi_j)_{j\in\mathbb{Z}}$ and $(\varphi_j)_{j\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ defined respectively for $\mathcal{B}\mathcal{D}^{\beta,k}_{p,q}$ and $\ddot{\mathcal{B}}\mathcal{D}^{\beta,k}_{p,q}$ by $\psi_j = \phi_{2^{-j}}$ $\forall\, j \in \mathbb{Z}$ and $\varphi_0 = \sum\limits_{j\in\mathbb{Z}_-} \phi_{2^{-j}}$, $\varphi_j = \phi_{2^{-j}}$ $\forall\, j \in \mathbb{N}^*$, we can assert that $\ddot{\mathcal{B}}\mathcal{D}^{\beta,k}_{p,q} = \mathcal{B}\mathcal{D}^{\beta,k}_{p,q}$. ∎

Remark 2. By Proposition 2 and [21, Proposition 2] we have the following embeddings.

1. Let $1 \le q_1 \le q_2 \le +\infty$ and $\beta > 0$. Then
$$\mathcal{B}\mathcal{D}^{\beta,k}_{p,q_1} \subset \mathcal{B}\mathcal{D}^{\beta,k}_{p,q_2} \qquad \text{if} \quad 1 \le p \le +\infty.$$
2. Let $1 \le q_1, q_2 \le +\infty$, $\beta > 0$ and $\varepsilon > 0$. Then
$$\mathcal{B}\mathcal{D}^{\beta+\varepsilon,k}_{p,q_1} \subset \mathcal{B}\mathcal{D}^{\beta,k}_{p,q_2} \qquad \text{if} \quad 1 \le p \le +\infty.$$

Proposition 3. *For $\beta > 0$ and $1 \le p, q \le +\infty$ we have*

$$\mathcal{S}(\mathbb{R}^d) \subset \mathcal{B}\mathcal{D}^{\beta,k}_{p,q}.$$

If $1 \le p, q < +\infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{B}\mathcal{D}^{\beta,k}_{p,q}$.

Proof. In order to prove the inclusion, we may restrict ourself to $q = +\infty$. This follows from the fact that $\mathcal{BD}^{\beta,k}_{p,\infty} \subset \mathcal{BD}^{\beta',k}_{p,q}$ for $\beta > \beta' > 0$ and $1 \le p, q \le +\infty$ (see Remark 2, 2). Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $(\varphi_j)_{j\in\mathbb{N}}$ a sequence of functions in $\mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$ satisfying the properties of Remark 1, there exists a sufficiently large natural number L such that

$$\sup_{j\in\mathbb{N}} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} \le \sup_{j\in\mathbb{N}} 2^{j\beta}\|(1+\|x\|^2)^L(\varphi_j *_k f)\|_{\infty,k}.$$

Since $\varphi_j \in \mathcal{S}(\mathbb{R}^d)^{\mathrm{rad}}$, then for $x \in \mathbb{R}^d$ $\mathcal{F}_k^{-1}(\varphi_j)(x) = \mathcal{F}_k(\varphi_j)(-x) = \mathcal{F}_k(\varphi_j)(x)$, so using (8) and the property i) of $(\varphi_j)_{j\in\mathbb{N}}$ (see Remark 1) we obtain

$$\begin{aligned}\sup_{j\in\mathbb{N}^*} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} &\le \sup_{j\in\mathbb{N}^*} 2^{j\beta}\|\mathcal{F}_k[(I-\Delta_k)^L(\mathcal{F}_k(\varphi_j)\mathcal{F}_k^{-1}(f))]\|_{\infty,k}\\ &\le \sup_{j\in\mathbb{N}^*} 2^{j\beta}\|(I-\Delta_k)^L(\mathcal{F}_k(\varphi_j)\mathcal{F}_k^{-1}(f))\|_{1,k}\\ &\le \sup_{j\in\mathbb{N}^*} c_j\, 2^{j\beta} \sup_{x\in A_j} |(I-\Delta_k)^L(\mathcal{F}_k(\varphi_j)\mathcal{F}_k^{-1}(f))(x)|,\end{aligned}$$

where $c_j = \int_{A_j} w_k(x)dx$. Hence there exists a sufficiently large natural number M such that $\frac{c_j 2^{j\beta}}{(1+2^{2(j-1)})^M} \le 1$, $\forall\, j \in \mathbb{N}^*$ and we get

$$\sup_{j\in\mathbb{N}^*} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} \le \sup_{j\in\mathbb{N}^*}\sup_{x\in A_j} |\big(1+\|x\|^2\big)^M (I-\Delta_k)^L(\mathcal{F}_k(\varphi_j)\mathcal{F}_k^{-1}(f))(x)|.$$

Since $(I-\Delta_k)^L$ is linear and continuous from $\mathcal{S}(\mathbb{R}^d)$ into itself, we deduce that

$$\sup_{j\in\mathbb{N}^*} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} \le c \sup_{j\in\mathbb{N}^*}\sup_{x\in\mathbb{R}^d} |\mathcal{F}_k(\varphi_j)(x)| \sup_{x\in\mathbb{R}^d} |(1+\|x\|^2)^M\mathcal{F}_k^{-1}(f)(x)|,$$

which gives by the property ii) of $(\varphi_j)_{j\in\mathbb{N}}$

$$\sup_{j\in\mathbb{N}^*} 2^{j\beta}\|\varphi_j *_k f\|_{p,k} \le c \sup_{j\in\mathbb{N}^*} \|\varphi_j\|_{1,k} \sup_{x\in\mathbb{R}^d} |(1+\|x\|^2)^M\mathcal{F}_k^{-1}(f)(x)| < +\infty.$$

By Proposition 2 we conclude that $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{BD}^{\beta,k}_{p,q}$.

Let us now prove the density of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{BD}^{\beta,k}_{p,q}$ for $p, q < +\infty$. Assume $f \in \mathcal{BD}^{\beta,k}_{p,q}$ and $(\varphi_j)_{j\in\mathbb{Z}} \in \mathbf{\Phi}$, then we put for $N \in \mathbb{N}\backslash\{0\}$, $f_N = \sum_{s=-N}^{N} \varphi_s *_k f$. It's clear that $f_N \in \mathcal{BD}^{\beta,k}_{p,q}$. We have

$$\sum_{j\in\mathbb{Z}} 2^{j\beta q}\|\varphi_j *_k (f_N - f)\|^q_{p,k} = \sum_{j\in\mathbb{Z}} 2^{j\beta q}\left\|\left(\sum_{s=-N}^{N}\varphi_s\right) *_k \varphi_j *_k f - \varphi_j *_k f\right\|^q_{p,k}.$$

Using the properties (i) and (iii) for $(\varphi_j)_{j\in\mathbb{Z}}$ we get

$$\|f - f_N\|^q_{\mathcal{BD}^{\beta,k}_{p,q}} \le c \sum_{|j|\ge N} 2^{j\beta q}\|\varphi_j *_k f\|^q_{p,k}.$$

Since $f \in \mathcal{BD}^{\beta,k}_{p,q}$, then we deduce that

$$\lim_{N\to+\infty}\|f - f_N\|_{\mathcal{BD}^{\beta,k}_{p,q}} = 0. \tag{9}$$

Next we take a function $\theta \in \mathcal{D}(\mathbb{R}^d)$ such that $\theta(0) = 1$. For $n \in \mathbb{N}\backslash\{0\}$ we put $\theta_n(x) = \theta(n^{-1}x)$, $x \in \mathbb{R}^d$. From (5) we have for $N \in \mathbb{N}\backslash\{0\}$ $f_N \in \mathcal{E}(\mathbb{R}^d)$, then $f_N\theta_n \in \mathcal{S}(\mathbb{R}^d)$. Again from the properties (i) and (iii) for $(\varphi_j)_{j\in\mathbb{Z}}$ we can assert that $f_N = \sum_{j=-N}^{N} \varphi_j *_k f_{N+1}$ which gives $f_N\theta_n = \sum_{j=-N}^{N} \varphi_j *_k f_{N+1}\theta_n$. Using the properties (i), (ii), (iii) for $(\varphi_j)_{j\in\mathbb{Z}}$ and (7) we obtain

$$\|f_N - f_N\theta_n\|^q_{\mathcal{BD}^{\beta,k}_{p,q}} \leq c \sum_{j=-N-1}^{N+1} 2^{j\beta q}\|f_{N+1} - f_{N+1}\theta_n\|^q_{p,k}.$$

The dominated convergence theorem implies that

$$\|f_{N+1} - f_{N+1}\theta_n\|_{p,k} \to 0 \qquad \text{as} \quad n \to +\infty.$$

Hence we deduce that

$$\|f_N - f_N\theta_n\|_{\mathcal{BD}^{\beta,k}_{p,q}} \to 0 \qquad \text{as} \quad n \to +\infty, \tag{10}$$

Combining (9) and (10) we conclude that $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{BD}^{\beta,k}_{p,q}$. This completes the proof of Proposition 3. ∎

For $0 < \theta < 1$, $\beta_0, \beta_1 > 0$, $1 \leq p, q_0, q_1 \leq +\infty$ and $1 \leq q \leq +\infty$, the real interpolation Besov–Dunkl space denoted by $(\mathcal{BD}^{\beta_0,k}_{p,q_0}, \mathcal{BD}^{\beta_1,k}_{p,q_1})_{\theta,q}$ is the subspace of functions $f \in \mathcal{BD}^{\beta_0,k}_{p,q_0} + \mathcal{BD}^{\beta_1,k}_{p,q_1}$ satisfying

$$\left(\int_0^{+\infty} \Big(t^{-\theta}\mathcal{K}_{p,k}(t, f; \beta_0, q_0; \beta_1, q_1)\Big)^q \frac{dt}{t}\right)^{\frac{1}{q}} < +\infty \qquad \text{if} \quad q < +\infty,$$

and

$$\sup_{t\in(0,+\infty)} t^{-\theta}\mathcal{K}_{p,k}(t, f; \beta_0, q_0; \beta_1, q_1) < +\infty \qquad \text{if} \quad q = +\infty,$$

with $\mathcal{K}_{p,k}$ is the Peetre $\mathcal{K}$-functional given by

$$\mathcal{K}_{p,k}(t, f; \beta_0, q_0; \beta_1, q_1) = \inf\Big\{\|f_0\|_{\mathcal{BD}^{\beta_0,k}_{p,q_0}} + t\|f_1\|_{\mathcal{BD}^{\beta_1,k}_{p,q_1}}\Big\},$$

where the infinimum is taken over all representations of f of the form

$$f = f_0 + f_1, \qquad f_0 \in \mathcal{BD}^{\beta_0,k}_{p,q_0}, \qquad f_1 \in \mathcal{BD}^{\beta_1,k}_{p,q_1}.$$

Theorem 1. *Let $0 < \theta < 1$ and $1 \leq p, q, q_0, q_1 \leq +\infty$. For $\beta_0, \beta_1 > 0$, $\beta_0 \neq \beta_1$ and $\beta = (1-\theta)\beta_0 + \theta\beta_1$ we have*

$$\big(\mathcal{BD}^{\beta_0,k}_{p,q_0}, \mathcal{BD}^{\beta_1,k}_{p,q_1}\big)_{\theta,q} = \mathcal{BD}^{\beta,k}_{p,q}.$$

Proof. We start with the proof of the inclusion $(\mathcal{BD}^{\beta_0,k}_{p,\infty}, \mathcal{BD}^{\beta_1,k}_{p,\infty})_{\theta,q} \subset \mathcal{BD}^{\beta,k}_{p,q}$. We may assume that $\beta_0 > \beta_1$. Let $q < +\infty$, for $f = f_0 + f_1$ with $f_0 \in \mathcal{BD}^{\beta_0,k}_{p,\infty}$ and $f_1 \in \mathcal{BD}^{\beta_1,k}_{p,\infty}$ we get by Proposition 2

$$\begin{aligned}\sum_{l=0}^{+\infty} 2^{ql\beta}\|\varphi_l *_k f\|^q_{p,k} &\leq c\sum_{l=0}^{+\infty} 2^{-\theta ql(\beta_0-\beta_1)}\Big(2^{l\beta_0}\|\varphi_l *_k f_0\|_{p,k} + 2^{l(\beta_0-\beta_1)}2^{l\beta_1}\|\varphi_l *_k f_1\|_{p,k}\Big)^q \\ &\leq c\sum_{l=0}^{+\infty} 2^{-\theta ql(\beta_0-\beta_1)}\Big(\|f_0\|_{\mathcal{BD}^{\beta_0,k}_{p,\infty}} + 2^{l(\beta_0-\beta_1)}\|f_1\|_{\mathcal{BD}^{\beta_1,k}_{p,\infty}}\Big)^q.\end{aligned}$$

Then we deduce that

$$\sum_{l=0}^{+\infty} 2^{ql\beta}\|\varphi_l *_k f\|_{p,k}^q \le c\sum_{l=0}^{+\infty} 2^{-\theta ql(\beta_0-\beta_1)}\Big(\mathcal{K}_{p,k}(2^{l(\beta_0-\beta_1)}, f;\beta_0,\infty;\beta_1,\infty)\Big)^q$$
$$\le c\int_0^{+\infty}\Big(t^{-\theta}\mathcal{K}_{p,k}(t,f;\beta_0,\infty;\beta_1,\infty)\Big)^q\frac{dt}{t} < +\infty,$$

which proves the result. When $q=+\infty$, we make the usual modification.

For $1\le s\le q_0, q_1$ Remark 2 gives

$$(\mathcal{BD}^{\beta_0,k}_{p,s}, \mathcal{BD}^{\beta_1,k}_{p,s})_{\theta,q} \subset (\mathcal{BD}^{\beta_0,k}_{p,q_0}, \mathcal{BD}^{\beta_1,k}_{p,q_1})_{\theta,q} \subset (\mathcal{BD}^{\beta_0,k}_{p,\infty}, \mathcal{BD}^{\beta_1,k}_{p,\infty})_{\theta,q} \subset \mathcal{BD}^{\beta,k}_{p,q}.$$

Then in order to complete the proof of the theorem we have to show only that

$$\mathcal{BD}^{\beta,k}_{p,q} \subset (\mathcal{BD}^{\beta_0,k}_{p,s}, \mathcal{BD}^{\beta_1,k}_{p,s})_{\theta,q} \quad \text{for} \quad 1\le s\le q.$$

Suppose that $\beta_0>\beta_1$ again. Let $q<+\infty$, we have

$$\int_0^{+\infty}\Big(t^{-\theta}\mathcal{K}_{p,k}(t,f;\beta_0,s;\beta_1,s)\Big)^q\frac{dt}{t} = \int_0^1 + \int_1^{+\infty} = I_1 + I_2.$$

Since $\beta>\beta_1$, by Remark 2 we get

$$\mathcal{K}_{p,k}(t,f;\beta_0,s;\beta_1,s) \le ct\|f\|_{\mathcal{BD}^{\beta_1,k}_{p,s}} \le ct\|f\|_{\mathcal{BD}^{\beta,k}_{p,q}},$$

hence we deduce

$$I_1 \le c\|f\|^q_{\mathcal{BD}^{\beta,k}_{p,q}}.$$

To estimate I_2 take $f_0=\sum\limits_{j=0}^{l}\varphi_j *_k f$ and $f_1=\sum\limits_{j=l+1}^{+\infty}\varphi_j *_k f$. Using the properties of the sequence $(\varphi_j)_{j\in\mathbb{N}}$ we obtain

$$\|f_0\|^s_{\mathcal{BD}^{\beta_0,k}_{p,s}} \le c\sum_{j=0}^{l+1} 2^{j\beta_0 s}\|\varphi_j *_k f\|^s_{p,k} \qquad \text{and} \qquad \|f_1\|^s_{\mathcal{BD}^{\beta_1,k}_{p,s}} \le c\sum_{j=l}^{+\infty} 2^{j\beta_1 s}\|\varphi_j *_k f\|^s_{p,k}.$$

Hence we can write

$$I_2 \le c\sum_{l=0}^{+\infty} 2^{-\theta ql(\beta_0-\beta_1)}\Big(\mathcal{K}_{p,k}(2^{l(\beta_0-\beta_1)}, f;\beta_0,s;\beta_1,s)\Big)^q$$
$$\le c\sum_{l=0}^{+\infty} 2^{-\theta ql(\beta_0-\beta_1)}\left[\left(\sum_{j=0}^{l+1} 2^{j\beta_0 s}\|\varphi_j *_k f\|^s_{p,k}\right)^{1/s} + 2^{l(\beta_0-\beta_1)}\left(\sum_{j=l}^{+\infty} 2^{j\beta_1 s}\|\varphi_j *_k f\|^s_{p,k}\right)^{1/s}\right]^q$$
$$\le c\sum_{l=0}^{+\infty} 2^{ql\beta}\left[\sum_{j=0}^{l+1} 2^{(j-l)\beta_0 s}\|\varphi_j *_k f\|^s_{p,k} + \sum_{j=l}^{+\infty} 2^{(j-l)\beta_1 s}\|\varphi_j *_k f\|^s_{p,k}\right]^{q/s}.$$

For $s=q$ it is easy to see that $I_2\le c\|f\|^q_{\mathcal{BD}^{\beta,k}_{p,q}}$.

For $s<q$ we take $u>s$ such that $\frac{s}{q}+\frac{s}{u}=1$ and $\beta_1<\alpha_1<\beta<\alpha_0<\beta_0$, then by Hölder's inequality we have

$$I_2 \le c\sum_{l=0}^{+\infty} 2^{ql(\beta-\beta_0)}\left(\sum_{j=0}^{l+1} 2^{(\beta_0-\alpha_0)ju}\right)^{q/u}\left(\sum_{j=0}^{l+1} 2^{\alpha_0 jq}\|\varphi_j *_k f\|^q_{p,k}\right)$$

$$+ c\sum_{l=0}^{+\infty} 2^{ql(\beta-\beta_1)}\left(\sum_{j=l}^{+\infty} 2^{(\beta_1-\alpha_1)ju}\right)^{q/u}\left(\sum_{j=l}^{+\infty} 2^{\alpha_1 jq}\|\varphi_j *_k f\|_{p,k}^q\right)$$
$$\leq c\sum_{l=0}^{+\infty} 2^{ql(\beta-\alpha_0)}\sum_{j=0}^{l+1} 2^{\alpha_0 jq}\|\varphi_j *_k f\|_{p,k}^q + c\sum_{l=0}^{+\infty} 2^{ql(\beta-\alpha_1)}\sum_{j=l}^{+\infty} 2^{\alpha_1 jq}\|\varphi_j *_k f\|_{p,k}^q$$
$$\leq c\sum_{j=0}^{+\infty} 2^{\alpha_0 jq}\|\varphi_j *_k f\|_{p,k}^q \sum_{l=j-1}^{+\infty} 2^{ql(\beta-\alpha_0)} + c\sum_{j=0}^{+\infty} 2^{\alpha_1 jq}\|\varphi_j *_k f\|_{p,k}^q \sum_{l=0}^{j} 2^{ql(\beta-\alpha_1)}$$
$$\leq c\|f\|^q_{\mathcal{BD}^{\beta,k}_{p,q}}.$$

Finally we deduce

$$\int_0^{+\infty}\left(t^{-\theta}\mathcal{K}_{p,k}(t,f;\beta_0,s;\beta_1,s)\right)^q \frac{dt}{t} \leq c\|f\|^q_{\mathcal{BD}^{\beta,k}_{p,q}}.$$

Here when $q=+\infty$ we make the usual modification. Our theorem is proved. ■

Theorem 2. *Let $\beta>0$ and $1\leq p,q\leq+\infty$. Then for all $\phi\in\mathcal{A}$, we have*

$$\mathcal{BD}^{\beta,k}_{p,q} \subset \mathcal{C}^{\phi,\beta,k}_{p,q}.$$

Proof. For $\phi\in\mathcal{A}$ and $1\leq u\leq 2$, we get $\operatorname{supp}\mathcal{F}_k(\phi_{2^{-j}u})\subset A_j$, $\forall\, j\in\mathbb{Z}$. Then we can write $\mathcal{F}_k(\phi_{2^{-j}u}) = \mathcal{F}_k(\phi_{2^{-j}u})(\mathcal{F}_k(\varphi_{j-1})+\mathcal{F}_k(\varphi_j)+\mathcal{F}_k(\varphi_{j+1}))$, which gives $\phi_{2^{-j}u} = \phi_{2^{-j}u} *_k (\varphi_{j-1}+\varphi_j+\varphi_{j+1})$, $\forall\, j\in\mathbb{Z}$.

Let $f\in\mathcal{BD}^{\beta,k}_{p,q}$ for $1\leq q<+\infty$, we can assert that

$$\int_0^{+\infty}\left(\frac{\|f *_k \phi_t\|_{p,k}}{t^\beta}\right)^q\frac{dt}{t} \leq \sum_{j\in\mathbb{Z}}\int_1^2\left(\frac{\|f *_k \phi_{2^{-j}u}\|_{p,k}}{(2^{-j}u)^\beta}\right)^q\frac{du}{u}.$$

Using Hölder's inequality for $j\in\mathbb{Z}$ we get

$$\|f *_k \phi_{2^{-j}u}\|^q_{p,k} \leq \|\phi\|^q_{1,k}3^{q-1}\sum_{s=j-1}^{j+1}\|\varphi_s *_k f\|^q_{p,k},$$

hence we obtain

$$\int_0^{+\infty}\left(\frac{\|f *_k \phi_t\|_{p,k}}{t^\beta}\right)^q\frac{dt}{t} \leq c\|\phi\|^q_{1,k}\sum_{s\in\mathbb{Z}}\int_1^2\left(\frac{\|\varphi_s *_k f\|_{p,k}}{(2^{-s}u)^\beta}\right)^q\frac{du}{u}$$
$$\leq c\sum_{s\in\mathbb{Z}}(2^{s\beta}\|\varphi_s *_k f\|_{p,k})^q < +\infty.$$

Here when $q=+\infty$, we make the usual modification. This completes the proof. ■

Theorem 3. *Let $\beta>0$ and $1\leq p,q\leq+\infty$, then for $\phi\in\mathcal{A}$ such that $\sum_{j\in\mathbb{Z}}\mathcal{F}_k(\phi_{2^{-j}u})(x)=1$, for all $1\leq u\leq 2$ and $x\in\mathbb{R}^d$ we have*

$$\mathcal{C}^{\phi,\beta,k}_{p,q} = \mathcal{BD}^{\beta,k}_{p,q}.$$

Proof. By Theorem 2 we have only to show that $\mathcal{C}^{\phi,\beta,k}_{p,q} \subset \mathcal{BD}^{\beta,k}_{p,q}$. Let $\phi \in \mathcal{A}$ such that $\sum_{j\in\mathbb{Z}} \mathcal{F}_k(\phi_{2^{-j}u})(x) = 1$, for $x \in \mathbb{R}^d$ and $1 \le u \le 2$. Then we can assert that

$$\mathcal{F}_k(\varphi_j) = \mathcal{F}_k(\varphi_j)(\mathcal{F}_k(\phi_{2^{-j-1}u}) + \mathcal{F}_k(\phi_{2^{-j}u}) + \mathcal{F}_k(\phi_{2^{-j+1}u})),$$

this implies that $\varphi_j = \varphi_j *_k (\phi_{2^{-j-1}u} + \phi_{2^{-j}u} + \phi_{2^{-j+1}u})$, $\forall\, j \in \mathbb{Z}$.

Let $f \in \mathcal{C}^{\phi,\beta,k}_{p,q}$ for $1 \le q < +\infty$, using Hölder's inequality for $j \in \mathbb{Z}$ and the property $ii)$ of the sequence of functions $(\varphi_j)_{j\in\mathbb{Z}}$ we get

$$\|\varphi_j *_k f\|^q_{p,k} \le \|\varphi_j\|^q_{1,k} 3^{q-1} \sum_{s=j-1}^{j+1} \|f *_k \phi_{2^{-s}u}\|^q_{p,k} \le c \sum_{s=j-1}^{j+1} \|f *_k \phi_{2^{-s}u}\|^q_{p,k}.$$

Integrating with respect to u over $(1,2)$ we obtain

$$\left(2^{j\beta}\|\varphi_j *_k f\|_{p,k}\right)^q \le c \sum_{s=j-1}^{j+1} \int_1^2 \left(\frac{\|f *_k \phi_{2^{-s}u}\|_{p,k}}{(2^{-s}u)^\beta}\right)^q \frac{du}{u}.$$

Hence

$$\sum_{j\in\mathbb{Z}} \left(2^{j\beta}\|\varphi_j *_k f\|_{p,k}\right)^q \le c \int_0^{+\infty} \left(\frac{\|f *_k \phi_t\|_{p,k}}{t^\beta}\right)^q \frac{dt}{t} < +\infty.$$

When $q = +\infty$ we make the usual modification. Our result is proved. ■

Remark 3. We observe that the spaces $\mathcal{C}^{\phi,\beta,k}_{p,q}$ are independent of the specific selection of $\phi \in \mathcal{A}$ satisfying the assumption of Theorem 3.

Remark 4. In the case $d = 1$, $W = \mathbb{Z}_2$, $\alpha > -\frac{1}{2}$ and

$$T_1(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x}\left[\frac{f(x) - f(-x)}{2}\right], \qquad f \in \mathcal{E}(\mathbb{R}),$$

we can characterize the Besov–Dunkl spaces by differences using the Dunkl translation operators. Observe that

$$\left\{\phi \in \mathcal{A} : \sum_{j\in\mathbb{Z}} \mathcal{F}_k(\phi_{2^{-j}u})(x) = 1,\ \forall\, 1 \le u \le 2,\ \forall\, x \in \mathbb{R}\right\} \subset \mathcal{H},$$

where $\mathcal{H} = \left\{\phi \in \mathcal{S}_*(\mathbb{R}) : \int_0^{+\infty} \phi(x) d\mu_\alpha(x) = 0\right\}$ with $d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx$ and $\mathcal{S}_*(\mathbb{R})$ the space of even Schwartz functions on $\mathbb{R}$. Then we can assert from Theorem 3 and [4, Theorem 3.6] that for $1 < p < +\infty$, $1 \le q \le +\infty$ and $0 < \beta < 1$ we have

$$\mathcal{BD}^{\beta,k}_{p,q} = BD^{p,q}_{\alpha,\beta} \subset \widetilde{B}D^{p,q}_{\alpha,\beta},$$

where $BD^{p,q}_{\alpha,\beta}$ is the subspace of functions $f \in L^p(\mu_\alpha)$ satisfying

$$\left(\int_0^{+\infty} \left(\frac{w_{p,\alpha}(f)(x)}{x^\beta}\right)^q \frac{dx}{x}\right)^{\frac{1}{q}} < +\infty \qquad \text{if} \quad q < +\infty$$

and

$$\sup_{x\in(0,+\infty)} \frac{w_{p,\alpha}(f)(x)}{x^\beta} < +\infty \qquad \text{if} \quad q = +\infty,$$

with $w_{p,\alpha}(f)(x) = \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}$. For the space $\widetilde{B}D^{p,q}_{\alpha,\beta}$ we replace $w_{p,\alpha}(f)(x)$ by $\widetilde{w}_{p,\alpha}(f)(x) = \|\tau_x(f) - f\|_{p,\alpha}$.

Note that when f is an even function in $L^p(\mu_\alpha)$ we have $\tau_x(f)(y) = \tau_{-x}(f)(-y)$ for $x, y \in \mathbb{R}$, then we get

$$f \in BD^{p,q}_{\alpha,\beta} \Longleftrightarrow f \in \widetilde{B}D^{p,q}_{\alpha,\beta}.$$

4 Integrability of the Dunkl transform of function in Besov–Dunkl space

In this section, we establish further results concerning integrability of the Dunkl transform of function f on $\mathbb{R}^d$, when f is in a suitable Besov–Dunkl space.

In the following lemma we prove the Hardy–Littlewood inequality for the Dunkl transform.

Lemma 1. *If $f \in L^p_k(\mathbb{R}^d)$ for some $1 < p \le 2$, then*

$$\int_{\mathbb{R}^d} \|x\|^{2(\gamma+\frac{d}{2})(p-2)} |\mathcal{F}_k(f)(x)|^p w_k(x)dx \le c\|f\|^p_{p,k}. \tag{11}$$

Proof. To see (11) we will make use of the Marcinkiewicz interpolation theorem (see [20]). For $f \in L^p_k(\mathbb{R}^d)$ with $1 \le p \le 2$ consider the operator

$$\mathcal{L}(f)(x) = \|x\|^{2(\gamma+\frac{d}{2})}\mathcal{F}_k(f)(x), \quad x \in \mathbb{R}^d.$$

For every $f \in L^2(\mathbb{R}^d)$ we have from Plancherel's theorem

$$\left(\int_{\mathbb{R}^d} |\mathcal{L}(f)(x)|^2 \frac{w_k(x)}{\|x\|^{4(\gamma+\frac{d}{2})}}dx\right)^{1/2} = \|\mathcal{F}_k(f)\|_{2,k} = c_k^{-1}\|f\|_{2,k}, \tag{12}$$

Moreover, according to (1) and (2) we get for $\lambda \in]0,+\infty)$ and $f \in L^1_k(\mathbb{R}^d)$

$$\begin{aligned}\int_{\{x\in\mathbb{R}^d:\mathcal{L}(f)(x)|>\lambda\}} \frac{w_k(x)}{\|x\|^{4(\gamma+\frac{d}{2})}}dx &\le \int_{\|x\|>(\frac{\lambda}{\|f\|_{1,k}})^{\frac{1}{2(\gamma+\frac{d}{2})}}} \frac{w_k(x)}{\|x\|^{4(\gamma+\frac{d}{2})}}dx \\ &\le c\int_{(\frac{\lambda}{\|f\|_{1,k}})^{\frac{1}{2(\gamma+\frac{d}{2})}}}^{+\infty} \frac{r^{2\gamma+d-1}}{r^{4(\gamma+\frac{d}{2})}}dr \le c\,\frac{\|f\|_{1,k}}{\lambda}.\end{aligned} \tag{13}$$

Hence by (12) and (13) $\mathcal{L}$ is an operator of strong-type $(2,2)$ and weak-type $(1,1)$ between the spaces $(\mathbb{R}^d, w_k(x)dx)$ and $(\mathbb{R}^d, \frac{w_k(x)}{\|x\|^{4(\gamma+\frac{d}{2})}}dx)$.

Using Marcinkiewicz interpolation's theorem we can assert that $\mathcal{L}$ is an operator of strong-type (p,p) for $1 < p \le 2$, between the spaces under consideration. We conclude that

$$\int_{\mathbb{R}^d} |\mathcal{L}(f)(x)|^p \frac{w_k(x)}{\|x\|^{4(\gamma+\frac{d}{2})}}dx = \int_{\mathbb{R}^d} \|x\|^{2(\gamma+\frac{d}{2})(p-2)}|\mathcal{F}_k(f)(x)|^p w_k(x)dx \le c\|f\|^p_{p,k},$$

thus we obtain the result. ■

Now in order to prove the following two theorems we denote by $\widetilde{\mathcal{A}}$ the subset of functions ϕ in $\mathcal{A}$ such that

$$\exists\, c > 0; \quad |\mathcal{F}_k(\phi)(x)| \ge c\|x\|^2 \qquad \text{if} \quad 1 \le \|x\| \le 2. \tag{14}$$

Let $\beta > 0$ and $1 \le p, q \le +\infty$. From Theorem 2 we have obviously for all $\phi \in \widetilde{\mathcal{A}}$,

$$\mathcal{BD}^{\beta,k}_{p,q} \subset \mathcal{C}^{\phi,\beta,k}_{p,q}. \tag{15}$$

For $1 \le p \le 2$ we take p' such that $\frac{1}{p}+\frac{1}{p'} = 1$. We recall that $\mathcal{F}_k(f) \in L^{p'}_k(\mathbb{R}^d)$ for all $f \in L^p_k(\mathbb{R}^d)$.

Theorem 4. *Let $1 < p \le 2$. If $f \in \mathcal{BD}^{\frac{2(\gamma+\frac{d}{2})}{p},k}_{p,1}$, then*

$$\mathcal{F}_k(f) \in L^1_k(\mathbb{R}^d).$$

Proof. Let $f \in \mathcal{BD}_{p,1}^{\frac{2(\gamma+\frac{d}{2})}{p},k}$ with $1 < p \leq 2$. For $\phi \in \widetilde{\mathcal{A}}$ we can write from (6) and for $t \in (0,+\infty)$, $\mathcal{F}_k(f *_k \phi_t)(x) = \mathcal{F}_k(f)(x)\mathcal{F}_k(\phi_t)(x)$, a.e. $x \in \mathbb{R}^d$. From Lemma 1 we obtain

$$\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|^p |\mathcal{F}_k(\phi_t)(x)|^p \|x\|^{2(\gamma+\frac{d}{2})(p-2)} w_k(x)dx \leq c\|f *_k \phi_t\|_{p,k}^p.$$

By (14) we get $|\mathcal{F}_k(\phi_t)(x)| \geq c\|tx\|^2$ if $1 \leq \|tx\| \leq 2$, then we can assert that

$$t^2 \left(\int_{\frac{1}{t} \leq \|x\| \leq \frac{2}{t}} |\mathcal{F}_k(f)(x)|^p \|x\|^{2(\gamma+\frac{d}{2})(p-2)+2p} w_k(x)dx \right)^{1/p} \leq c\|f *_k \phi_t\|_{p,k}. \tag{16}$$

Then by Hölder's inequality, (1) and (16) we have

$$\begin{aligned} \int_{\frac{1}{t} \leq \|x\| \leq \frac{2}{t}} \|x\| |\mathcal{F}_k(f)(x)| w_k(x)dx &\leq c\frac{\|f *_k \phi_t\|_{p,k}}{t^2} \left(\int_{\frac{1}{t}}^{\frac{2}{t}} r^{(\frac{1}{1-p})[2(\gamma+\frac{d}{2})(p-2)+p]} r^{2\gamma+d-1} dr \right)^{\frac{1}{p'}} \\ &\leq c\frac{\|f *_k \phi_t\|_{p,k}}{t^{\frac{2(\gamma+\frac{d}{2})}{p}}} \frac{1}{t}. \end{aligned}$$

Integrating with respect to t over $\mathbb{R}_+$, applying Fubini's theorem and using (15), it yields

$$\int_{\mathbb{R}^d} |\mathcal{F}_k(f)(x)|\, w_k(x)dx \leq c \int_0^{+\infty} \frac{\|f *_k \phi_t\|_{p,k}}{t^{\frac{2(\gamma+\frac{d}{2})}{p}}} \frac{dt}{t} < +\infty\,.$$

This complete the proof of the theorem. ■

Theorem 5. *Let $\beta > 0$ and $1 \leq p \leq 2$. If $f \in \mathcal{BD}_{p,\infty}^{\beta,k}$, then*

i) for $p \neq 1$ and $0 < \beta \leq \frac{2(\gamma+\frac{d}{2})}{p}$ we have

$$\mathcal{F}_k(f) \in L_k^s(\mathbb{R}^d) \qquad \text{provided that} \quad \frac{2(\gamma+\frac{d}{2})p}{\beta p + 2(\gamma+\frac{d}{2})(p-1)} < s \leq p';$$

ii) for $\beta > \frac{2(\gamma+\frac{d}{2})}{p}$ we have $\mathcal{F}_k(f) \in L_k^1(\mathbb{R}^d)$.

Proof. Let $f \in \mathcal{BD}_{p,\infty}^{\beta,k}$ with $1 \leq p \leq 2$ and $\phi \in \widetilde{\mathcal{A}}$.

i) Suppose that $p \neq 1$ and $0 < \beta \leq \frac{2(\gamma+\frac{d}{2})}{p}$. Using (3) and (6) we have for $t \in (0,+\infty)$

$$\|\mathcal{F}_k(f *_k \phi_t)\|_{p',k} = \|\mathcal{F}_k(f)\mathcal{F}_k(\phi_t)\|_{p',k} \leq c\|f *_k \phi_t\|_{p,k}.$$

Then from (14) and (15) we obtain

$$t^2 \left(\int_{\frac{1}{t} \leq \|x\| \leq \frac{2}{t}} |\mathcal{F}_k(f)(x)|^{p'} \|x\|^{2p'} w_k(x)dx \right)^{1/p'} \leq c\|f *_k \phi_t\|_{p,k} \leq ct^\beta. \tag{17}$$

Let $s \in \left]\frac{2(\gamma+\frac{d}{2})p}{\beta p+2(\gamma+\frac{d}{2})(p-1)}, p'\right]$. Since $\mathcal{F}_k(f) \in L_k^{p'}(\mathbb{R}^d)$, we have only to show the case $s \neq p'$. For $t \geq 1$ put G_t the set of x in $\mathbb{R}^d$ such that $\frac{1}{t^{1/s}} \leq \|x\| \leq \frac{2}{t^{1/s}}$. By Hölder's inequality, (1) and (17) we have

$$\int_{G_t} |\mathcal{F}_k(f)(x)|^s \, \|x\|^s w_k(x)dx$$

$$\leq \left(\int_{G_t} |\mathcal{F}_k(f)(x)|^{p'} \|x\|^{2p'} w_k(x)dx\right)^{s/p'} \left(\int_{G_t} \|x\|^{\frac{-p's}{p'-s}} w_k(x)dx\right)^{1-\frac{s}{p'}}$$

$$\leq ct^{\beta-2} \left(\int_{\frac{1}{t^{1/s}}}^{\frac{2}{t^{1/s}}} r^{2\gamma+d-1-\frac{p's}{p'-s}} dr\right)^{1-\frac{s}{p'}} \leq ct^{-1+\beta-2(\gamma+\frac{d}{2})(\frac{1}{s}-\frac{1}{p'})}.$$

Integrating with respect to t over $(0,1)$ and applying Fubini's theorem, it yields

$$\int_{\|x\|\geq 1} |\mathcal{F}_k(f)(x)|^s w_k(x)dx \leq c\int_0^1 t^{-1+\beta-2(\gamma+\frac{d}{2})(\frac{1}{s}-\frac{1}{p'})}dt < +\infty.$$

Since $L_k^{p'}(B(0,1), w_k(x)dx) \subset L_k^s(B(0,1), w_k(x)dx)$ we deduce that $\mathcal{F}_k(f)$ is in $L_k^s(\mathbb{R}^d)$.

$ii)$ Assume now $\beta > \frac{2(\gamma+\frac{d}{2})}{p}$. For $p \neq 1$ by proceeding in the same manner as in the proof of $i)$ with $s = 1$, we obtain the desired result.

For $p = 1$, using (3) and (6), we have for $t \in (0, +\infty)$

$$\|\mathcal{F}_k(f *_k \phi_t)\|_{\infty,k} = \|\mathcal{F}_k(f)\mathcal{F}_k(\phi_t)\|_{\infty,k} \leq c\|f *_k \phi_t\|_{1,k}.$$

Then from (14) and (15) we obtain

$$t^2\|h_t\mathcal{F}_k(f)\|_{\infty,k} \leq c\|f *_k \phi_t\|_{1,k} \leq ct^\beta, \tag{18}$$

where $h_t(x) = \chi_t(x)\|x\|^2$ with χ_t is the characteristic function of the set $\{x \in \mathbb{R}^d : \frac{1}{t} \leq \|x\| \leq \frac{2}{t}\}$.

By Hölder's inequality, (1) and (18) we have

$$\int_{\frac{1}{t}\leq\|x\|\leq\frac{2}{t}} |\mathcal{F}_k(f)(x)|\|x\|w_k(x)dx \leq \|h_t\mathcal{F}_k(f)\|_{\infty,k} \int_{\mathbb{R}^d} |\chi_t(x)|\|x\|^{-1}w_k(x)dx$$

$$\leq ct^{\beta-2}\int_{\frac{1}{t}}^{\frac{2}{t}} r^{2\gamma+d-2}\,dr \leq ct^{\beta-2(\gamma+\frac{d}{2})-1}.$$

Integrating with respect to t over $(0,1)$ and applying Fubini's theorem we obtain

$$\int_{\|x\|\geq 1} |\mathcal{F}_k(f)(x)|w_k(x)dx \leq c\int_0^1 t^{\beta-2(\gamma+\frac{d}{2})-1}dt < +\infty.$$

Since $L_k^\infty(B(0,1), w_k(x)dx) \subset L_k^1(B(0,1), w_k(x)dx)$ we deduce that $\mathcal{F}_k(f)$ is in $L_k^1(\mathbb{R}^d)$. Our theorem is proved. ∎

Remark 5.

1. For $\beta > 0$, $1 \leq p \leq 2$ et $1 \leq q \leq +\infty$, using Remark 2, the results of Theorem 5 are true for $\mathcal{BD}_{p,q}^{\beta,k}$.
2. From Remark 2 we get $\mathcal{BD}_{p,\infty}^{\beta,k} \subset \mathcal{BD}_{p,1}^{\frac{2(\gamma+\frac{d}{2})}{p},k}$ for $\beta > \frac{2(\gamma+\frac{d}{2})}{p}$. Using Theorem 4 we recover the result of Theorem 5, $ii)$ with $1 < p \leq 2$.
3. Let $\beta > 2(\gamma + \frac{d}{2})$, by Theorem 5, $ii)$ we can assert that

   $i)$ $\mathcal{BD}_{1,\infty}^{\beta,k}$ is an example of space where we can apply the inversion formula;

   $ii)$ $\mathcal{BD}_{1,\infty}^{\beta,k}$ is contained in $L_k^1(\mathbb{R}^d) \cap L_k^\infty(\mathbb{R}^d)$ and hence is a subspace of $L_k^2(\mathbb{R}^d)$. By (4) we obtain for $f \in \mathcal{BD}_{1,\infty}^{\beta,k}$

$$\tau_y(f)(x) = c_k\int_{\mathbb{R}^d} \mathcal{F}_k(f)(\xi)E_k(ix,\xi)E_k(-iy,\xi)w_k(\xi)d\xi, \qquad x, y \in \mathbb{R}^d.$$

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