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Imaginary Powers of the Dunkl Harmonic Oscillator*

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Abstract. In this paper we continue the study of spectral properties of the Dunkl harmonic oscillator in the context of a finite reflection group on $\mathbb{R}^d$ isomorphic to $\mathbb{Z}_2^d$. We prove that imaginary powers of this operator are bounded on L^p, $1 < p < \infty$, and from L^1 into weak L^1.

Key words: Dunkl operators; Dunkl harmonic oscillator; imaginary powers; Calderón–Zygmund operators

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1 Introduction

In [9] the authors defined and investigated a system of Riesz transforms related to the Dunkl harmonic oscillator $\mathcal{L}_k$. The present article continues the study of spectral properties of operators associated with $\mathcal{L}_k$ by considering the imaginary powers $\mathcal{L}_k^{-i\gamma}$, $\gamma \in \mathbb{R}$. Our objective is to study L^p mapping properties of the operators $\mathcal{L}_k^{-i\gamma}$, and the principal tool is the general Calderón–Zygmund operator theory. The main result we get (Theorem 1) partially extends the result obtained recently by Stempak and Torrea [15, Theorem 4.3] and corresponding to the trivial multiplicity function $k \equiv 0$. Imaginary powers of the Euclidean Laplacian were investigated much earlier by Muckenhoupt [6].

Let us briefly describe the framework of the Dunkl theory of differential-difference operators on $\mathbb{R}^d$ related to finite reflection groups. Given such a group $G \subset O(\mathbb{R}^d)$ and a G-invariant nonnegative multiplicity function $k\colon R \to [0, \infty)$ on a root system $R \subset \mathbb{R}^d$ associated with the reflections of G, the Dunkl differential-difference operators T_j^k, $j = 1, \dots, d$, are defined by

$$T_j^k f(x) = \partial_j f(x) + \sum_{\beta \in R_+} k(\beta)\beta_j \frac{f(x) - f(\sigma_\beta x)}{\langle \beta, x\rangle}, \qquad f \in C^1(\mathbb{R}^d);$$

here ∂_j is the jth partial derivative, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$, R_+ is a fixed positive subsystem of R, and σ_β denotes the reflection in the hyperplane orthogonal to β. The Dunkl operators T_j^k, $j = 1, \dots, d$, form a commuting system (this is an important feature, see [3]) of the first order differential-difference operators, and reduce to ∂_j, $j = 1, \dots, d$, when $k \equiv 0$. Moreover, T_j^k are homogeneous of degree -1 on $\mathcal{P}$, the space of all polynomials in $\mathbb{R}^d$. This means that $T_j^k \mathcal{P}_m \subset \mathcal{P}_{m-1}$, where $m \in \mathbb{N} = \{0, 1, \dots\}$ and $\mathcal{P}_m$ denotes the subspace of $\mathcal{P}$ consisting of polynomials of total degree m (by convention, $\mathcal{P}_{-1}$ consists only of the null function).

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In Dunkl's theory the operator, see [2],

$$\Delta_k = \sum_{j=1}^{d} (T_j^k)^2$$

plays the role of the Euclidean Laplacian (notice that Δ comes into play when $k \equiv 0$). It is homogeneous of degree -2 on $\mathcal{P}$ and symmetric in $L^2(\mathbb{R}^d, w_k)$, where

$$w_k(x) = \prod_{\beta \in R_+} |\langle \beta, x\rangle|^{2k(\beta)},$$

if considered initially on $C_c^\infty(\mathbb{R}^d)$. Note that w_k is G-invariant.

The study of the operator

$$L_k = -\Delta_k + \|x\|^2$$

was initiated by Rösler [11, 12]. It occurs that L_k (or rather its self-adjoint extension $\mathcal{L}_k$) has a discrete spectrum and the corresponding eigenfunctions are the generalized Hermite functions defined and investigated by Rösler [11]. Due to the form of L_k, it is reasonable to call it the *Dunkl harmonic oscillator*. In fact L_k becomes the classic harmonic oscillator $-\Delta + \|x\|^2$ when $k \equiv 0$.

The results of the present paper are naturally related to the authors' articles [8, 9]. In what follows we will use the notation introduced there and invoke certain arguments from [8]. For basic facts concerning Dunkl's theory we refer the reader to the excellent survey article by Rösler [13].

Throughout the paper we use a fairly standard notation. Given a multi-index $n \in \mathbb{N}^d$, we write $|n| = n_1 + \cdots + n_d$ and, for $x, y \in \mathbb{R}^d$, $xy = (x_1y_1, \ldots, x_dy_d)$, $x^n = x_1^{n_1} \cdot \cdots \cdot x_d^{n_d}$ (and similarly x^α for $x \in \mathbb{R}_+^d$ and $\alpha \in \mathbb{R}^d$); $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$, and e_j is the jth coordinate vector in $\mathbb{R}^d$. Given $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ is the Euclidean ball in $\mathbb{R}^d$ centered at x and of radius r. For a nonnegative weight function w on $\mathbb{R}^d$, by $L^p(\mathbb{R}^d, w)$, $1 \le p < \infty$, we denote the usual Lebesgue spaces related to the measure $dw(x) = w(x)dx$ (in the sequel we will often abuse slightly the notation and use the same symbol w to denote the measure induced by a density w). Writing $X \lesssim Y$ indicates that $X \le CY$ with a positive constant C independent of significant quantities. We shall write $X \simeq Y$ when $X \lesssim Y$ and $Y \lesssim X$.

2 Preliminaries

In the setting of general Dunkl's theory Rösler [11] constructed systems of naturally associated multivariable generalized Hermite polynomials and Hermite functions. The system of generalized Hermite polynomials $\{H_n^k : n \in \mathbb{N}^d\}$ is orthogonal and complete in $L^2(\mathbb{R}^d, e^{-\|\cdot\|^2} w_k)$, while the system $\{h_n^k : n \in \mathbb{N}^d\}$ of generalized Hermite functions

$$h_n^k(x) = \left(2^{|n|} c_k\right)^{-1/2} \exp(-\|x\|^2/2) H_n^k(x), \qquad x \in \mathbb{R}^d, \quad n \in \mathbb{N}^d,$$

is an orthonormal basis in $L^2(\mathbb{R}^d, w_k)$, cf. [11, Corollary 3.5 (ii)]; here the normalizing constant c_k equals to $\int_{\mathbb{R}^d} \exp(-\|x\|^2) w_k(x)\, dx$. Moreover, h_n^k are eigenfunctions of L_k,

$$L_k h_n^k = (2|n| + 2\tau + d) h_n^k,$$

where $\tau = \sum_{\beta \in R_+} k(\beta)$. For $k \equiv 0$, h_n^0 are the usual multi-dimensional Hermite functions, see for instance [14] or [15].

Let $\langle\cdot,\cdot\rangle_k$ be the canonical inner product in $L^2(\mathbb{R}^d, w_k)$. The operator

$$\mathcal{L}_k f = \sum_{n\in\mathbb{N}^d} (2|n| + 2\tau + d)\langle f, h_n^k\rangle_k \, h_n^k,$$

defined on the domain

$$\mathrm{Dom}(\mathcal{L}_k) = \Big\{ f \in L^2(\mathbb{R}^d, w_k) : \sum_{n\in\mathbb{N}^d} \big|(2|n| + 2\tau + d)\langle f, h_n^k\rangle_k\big|^2 < \infty \Big\},$$

is a self-adjoint extension of L_k considered on $C_c^\infty(\mathbb{R}^d)$ as the natural domain (the inclusion $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(\mathcal{L}_k)$ may be easily verified). The spectrum of $\mathcal{L}_k$ is the discrete set $\{2m+2\tau+d : m \in \mathbb{N}\}$, and the spectral decomposition of $\mathcal{L}_k$ is

$$\mathcal{L}_k f = \sum_{m=0}^{\infty} (2m + 2\tau + d)\mathcal{P}_m^k f, \qquad f \in \mathrm{Dom}(\mathcal{L}_k),$$

where the spectral projections are

$$\mathcal{P}_m^k f = \sum_{|n|=m} \langle f, h_n^k\rangle_k \, h_n^k.$$

By Parseval's identity, for each $\gamma \in \mathbb{R}$ the operator

$$\mathcal{L}_k^{-i\gamma} f = \sum_{m=0}^{\infty} (2m + 2\tau + d)^{-i\gamma} \, \mathcal{P}_m^k f$$

is an isometry on $L^2(\mathbb{R}^d, w_k)$.

Consider the finite reflection group generated by σ_j, $j = 1, \dots, d$,

$$\sigma_j(x_1, \dots, x_j, \dots, x_d) = (x_1, \dots, -x_j, \dots, x_d),$$

and isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$. The reflection σ_j is in the hyperplane orthogonal to e_j. Thus $R = \{\pm\sqrt{2}e_j : j = 1, \dots, d\}$, $R_+ = \{\sqrt{2}e_j : j = 1, \dots, d\}$, and for a nonnegative multiplicity function $k\colon R \to [0,\infty)$ which is $\mathbb{Z}_2^d$-invariant only values of k on R_+ are essential. Hence we may think $k = (\alpha_1 + 1/2, \dots, \alpha_d + 1/2)$, $\alpha_j \geq -1/2$. We write $\alpha_j + 1/2$ in place of seemingly more appropriate α_j since, for the sake of clarity, it is convenient for us to stick to the notation used in [8] and [9].

In what follows the symbols T_j^α, Δ_α, w_α, L_α, $\mathcal{L}_\alpha$, h_n^α, and so on, denote the objects introduced earlier and related to the present $\mathbb{Z}_2^d$ group setting. Thus the Dunkl differential-difference operators are now given by

$$T_j^\alpha f(x) = \partial_j f(x) + (\alpha_j + 1/2)\frac{f(x) - f(\sigma_j x)}{x_j}, \qquad f \in C^1(\mathbb{R}^d),$$

and the explicit formula for the Dunkl Laplacian is

$$\Delta_\alpha f(x) = \sum_{j=1}^{d} \left(\frac{\partial^2 f}{\partial x_j^2}(x) + \frac{2\alpha_j + 1}{x_j}\frac{\partial f}{\partial x_j}(x) - (\alpha_j + 1/2)\frac{f(x) - f(\sigma_j x)}{x_j^2} \right).$$

The corresponding weight w_α has the form

$$w_\alpha(x) = \prod_{j=1}^{d} |x_j|^{2\alpha_j+1} \simeq \prod_{\beta\in R_+} |\langle\beta, x\rangle_\alpha|^{2k(\beta)}, \qquad x \in \mathbb{R}^d.$$

Given $\alpha \in [-1/2, \infty)^d$, the associated generalized Hermite functions are tensor products

$$h_n^\alpha(x) = h_{n_1}^{\alpha_1}(x_1) \cdot \dots \cdot h_{n_d}^{\alpha_d}(x_d), \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad n = (n_1, \dots, n_d) \in \mathbb{N}^d,$$

where $h_{n_i}^{\alpha_i}$ are the one-dimensional functions (see Rosenblum [10])

$$\begin{aligned} h_{2n_i}^{\alpha_i}(x_i) &= d_{2n_i,\alpha_i} e^{-x_i^2/2} L_{n_i}^{\alpha_i}(x_i^2), \\ h_{2n_i+1}^{\alpha_i}(x_i) &= d_{2n_i+1,\alpha_i} e^{-x_i^2/2} x_i L_{n_i}^{\alpha_i+1}(x_i^2); \end{aligned}$$

here $L_{n_i}^{\alpha_i}$ denotes the Laguerre polynomial of degree n_i and order α_i, cf. [5, p. 76], and

$$d_{2n_i,\alpha_i} = (-1)^{n_i} \left(\frac{\Gamma(n_i+1)}{\Gamma(n_i+\alpha_i+1)} \right)^{1/2}, \qquad d_{2n_i+1,\alpha_i} = (-1)^{n_i} \left(\frac{\Gamma(n_i+1)}{\Gamma(n_i+\alpha_i+2)} \right)^{1/2}.$$

For $\alpha = (-1/2, \dots, -1/2)$ we obtain the usual Hermite functions. The system $\{h_n^\alpha : n \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d, w_\alpha)$ and

$$L_\alpha h_n^\alpha = (2|n| + 2|\alpha| + 2d) h_n^\alpha,$$

where by $|\alpha|$ we denote $|\alpha| = \alpha_1 + \dots + \alpha_d$ (thus $|\alpha|$ may be negative).

The semigroup $T_t^\alpha = \exp(-t\mathcal{L}_\alpha)$, $t \ge 0$, generated by $\mathcal{L}_\alpha$ is a strongly continuous semigroup of contractions on $L^2(\mathbb{R}^d, w_\alpha)$. By the spectral theorem,

$$T_t^\alpha f = \sum_{m=0}^\infty e^{-t(2m+2|\alpha|+2d)} \mathcal{P}_m^\alpha f, \qquad f \in L^2(\mathbb{R}^d, w_\alpha).$$

The integral representation of T_t^α on $L^2(\mathbb{R}^d, w_\alpha)$ is

$$T_t^\alpha f(x) = \int_{\mathbb{R}^d} G_t^\alpha(x,y) f(y)\, dw_\alpha(y), \qquad x \in \mathbb{R}^d, \quad t > 0,$$

where the heat kernel $\{G_t^\alpha\}_{t>0}$ is given by

$$G_t^\alpha(x,y) = \sum_{m=0}^\infty e^{-t(2m+2|\alpha|+2d)} \sum_{|n|=m} h_n^\alpha(x) h_n^\alpha(y). \tag{1}$$

In dimension one, for $\alpha \ge -1/2$ it is known (see for instance [11, Theorem 3.12] and [11, p. 523]) that

$$G_t^\alpha(x,y) = \frac{1}{2\sinh 2t} \exp\left(-\frac{1}{2} \coth(2t)(x^2+y^2) \right) \left[\frac{I_\alpha\left(\frac{xy}{\sinh 2t}\right)}{(xy)^\alpha} + xy \frac{I_{\alpha+1}\left(\frac{xy}{\sinh 2t}\right)}{(xy)^{\alpha+1}} \right],$$

with I_ν being the modified Bessel function of the first kind and order ν,

$$I_\nu(z) = \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}.$$

Here we consider the function $z \mapsto z^\nu$, and thus also the Bessel function $I_\nu(z)$, as an analytic function defined on $\mathbb{C} \backslash \{ix : x \le 0\}$ (usually I_ν is considered as a function on $\mathbb{C}$ cut along the half-line $(-\infty, 0]$). Note that I_ν, as a function on $\mathbb{R}_+$, is real, positive and smooth for any $\nu > -1$, see [5, Chapter 5].

Therefore, in d dimensions,

$$G_t^\alpha(x,y) = \sum_{\varepsilon\in\mathbb{Z}_2^d} G_t^{\alpha,\varepsilon}(x,y),$$

where the component kernels are

$$G_t^{\alpha,\varepsilon}(x,y) = \frac{1}{(2\sinh 2t)^d}\exp\left(-\frac{1}{2}\coth(2t)\big(\|x\|^2+\|y\|^2\big)\right)\prod_{i=1}^d (x_iy_i)^{\varepsilon_i}\frac{I_{\alpha_i+\varepsilon_i}\left(\frac{x_iy_i}{\sinh 2t}\right)}{(x_iy_i)^{\alpha_i+\varepsilon_i}}.$$

Note that $G_t^{\alpha,\varepsilon}(x,y)$ is given by the series (1), with the summation in n restricted to the set of multi-indices

$$\mathcal{N}_\varepsilon = \big\{n\in\mathbb{N}^d : n_i \text{ is even if } \varepsilon_i = 0 \text{ or } n_i \text{ is odd if } \varepsilon_i=1,\ i=1,\ldots,d\big\}.$$

To verify this fact it is enough to restrict to the one-dimensional case and then use the Hille–Hardy formula, cf. [5, (4.17.6)].

In the sequel we will make use of the following technical result concerning $G_t^{\alpha,\varepsilon}(x,y)$. The corresponding proof is given at the end of Section 4.

Lemma 1. *Let $\alpha\in[-1/2,\infty)^d$ and let $\varepsilon\in\mathbb{Z}_2^d$. Then, with $x,y\in\mathbb{R}_+^d$ fixed, $x\neq y$, the kernel $G_t^{\alpha,\varepsilon}(x,y)$ decays rapidly when either $t\to 0^+$ or $t\to\infty$. Further, given any disjoint compact sets $E,F\subset\mathbb{R}_+^d$, we have*

$$\int_0^\infty \big|\partial_t G_t^{\alpha,\varepsilon}(x,y)\big|\,dt \lesssim 1, \tag{2}$$

uniformly in $x\in E$ and $y\in F$.

We end this section with pointing out that there is a general background for the facts considered here for an arbitrary reflection group, see [13] for a comprehensive account. In particular, the heat (or Mehler) kernel (1) has always a closed form involving the so-called Dunkl kernel, and is always strictly positive. This implies that the corresponding semigroup is contractive on $L^\infty(\mathbb{R}^d,w_k)$, and as its generator is self-adjoint and positive in $L^2(\mathbb{R}^d,w_k)$, the semigroup is also contractive on the latter space. Hence, by duality and interpolation, it is in fact contractive on all $L^p(\mathbb{R}^d,w_k)$, $1\le p\le\infty$.

3 Main result

From now on we assume $\gamma\in\mathbb{R}$, $\gamma\neq 0$, to be fixed. Recall that the operator $\mathcal{L}_\alpha^{-i\gamma}$ is given on $L^2(\mathbb{R}^d,w_\alpha)$ by the spectral series,

$$\mathcal{L}_\alpha^{-i\gamma} f = \sum_{n\in\mathbb{N}^d} (2|n|+2|\alpha|+2d)^{-i\gamma}\langle f, h_n^\alpha\rangle_\alpha\, h_n^\alpha.$$

Our main result concerns mapping properties of $\mathcal{L}_\alpha^{-i\gamma} f$ on L^p spaces.

Theorem 1. *Assume $\alpha\in[-1/2,\infty)^d$. Then $\mathcal{L}_\alpha^{-i\gamma}$, defined initially on $L^2(\mathbb{R}^d,w_\alpha)$, extends uniquely to a bounded operator on $L^p(\mathbb{R}^d,w_\alpha)$, $1<p<\infty$, and to a bounded operator from $L^1(\mathbb{R}^d,w_\alpha)$ to $L^{1,\infty}(\mathbb{R}^d,w_\alpha)$.*

The proof we give relies on splitting $\mathcal{L}_\alpha^{-i\gamma}$ in $L^2(\mathbb{R}^d, w_\alpha)$ into a finite number of suitable L^2-bounded operators and then treating each of the operators separately. More precisely, we decompose

$$\mathcal{L}_\alpha^{-i\gamma} = \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathcal{L}_{\alpha,\varepsilon}^{-i\gamma},$$

where (with the set $\mathcal{N}_\varepsilon$ introduced in the previous section)

$$\mathcal{L}_{\alpha,\varepsilon}^{-i\gamma} f = \sum_{n \in \mathcal{N}_\varepsilon} (2|n| + 2|\alpha| + 2d)^{-i\gamma} \langle f, h_n^\alpha \rangle_\alpha \, h_n^\alpha, \qquad f \in L^2(\mathbb{R}^d, w_\alpha).$$

Clearly, each $\mathcal{L}_{\alpha,\varepsilon}^{-i\gamma}$ is a contraction in $L^2(\mathbb{R}^d, w_\alpha)$.

It is now convenient to introduce the following terminology: given $\varepsilon \in \mathbb{Z}_2^d$, we say that a function f on $\mathbb{R}^d$ is *ε-symmetric* if for each $i = 1, \ldots, d$, f is either even or odd with respect to the ith coordinate according to whether $\varepsilon_i = 0$ or $\varepsilon_i = 1$, respectively. Thus f is ε-symmetric if and only if $f \circ \sigma_i = (-1)^{\varepsilon_i} f$, $i = 1, \ldots, d$. Any function f on $\mathbb{R}^d$ can be split uniquely into a sum of ε-symmetric functions f_ε,

$$f = \sum_{\varepsilon \in \mathbb{Z}_2^d} f_\varepsilon, \qquad f_\varepsilon(x) = \frac{1}{2^d} \sum_{\eta \in \{-1,1\}^d} \eta^\varepsilon f(\eta x).$$

For $f \in L^2(\mathbb{R}^d, w_\alpha)$ this splitting is orthogonal in $L^2(\mathbb{R}^d, w_\alpha)$. Finally, notice that h_n^α is ε-symmetric if and only if $n \in \mathcal{N}_\varepsilon$. Consequently, $\mathcal{L}_{\alpha,\varepsilon}^{-i\gamma}$ is invariant on the subspace of $L^2(\mathbb{R}^d, w_\alpha)$ of ε-symmetric functions and vanishes on the orthogonal complement of that subspace.

Observe that in order to prove Theorem 1 it is sufficient to show the analogous result for each $\mathcal{L}_{\alpha,\varepsilon}^{-i\gamma}$. Moreover, since

$$\mathcal{L}_\alpha^{-i\gamma} f = \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathcal{L}_{\alpha,\varepsilon}^{-i\gamma} f = \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathcal{L}_{\alpha,\varepsilon}^{-i\gamma} f_\varepsilon$$

and since for a fixed $1 \le p < \infty$ (recall that $w_\alpha(\xi x) = w_\alpha(x)$, $\xi \in \{-1,1\}^d$)

$$\|f\|_{L^p(\mathbb{R}^d, w_\alpha)} \simeq \sum_{\varepsilon \in \mathbb{Z}_2^d} \|f_\varepsilon\|_{L^p(\mathbb{R}^d_+, w_\alpha^+)},$$

it is enough to restrict the situation to the space $(\mathbb{R}^d_+, w_\alpha^+)$, where w_α^+ is the restriction of w_α to $\mathbb{R}^d_+$. Thus we are reduced to considering the operators

$$\mathcal{L}_{\alpha,\varepsilon,+}^{-i\gamma} f = \sum_{n \in \mathcal{N}_\varepsilon} (2|n| + 2|\alpha| + 2d)^{-i\gamma} \langle f, h_n^\alpha \rangle_{L^2(\mathbb{R}^d_+, w_\alpha^+)} \, h_n^\alpha, \qquad f \in L^2(\mathbb{R}^d_+, w_\alpha^+), \tag{3}$$

which are bounded on $L^2(\mathbb{R}^d_+, w_\alpha^+)$ since the system $\{2^{d/2} h_n^\alpha : n \in \mathcal{N}_\varepsilon\}$ is orthonormal in $L^2(\mathbb{R}^d_+, w_\alpha^+)$. Now, Theorem 1 will be justified once we prove the following.

Lemma 2. *Assume that $\alpha \in [-1/2, \infty)^d$ and $\varepsilon \in \mathbb{Z}_2^d$. Then $\mathcal{L}_{\alpha,\varepsilon,+}^{-i\gamma}$, defined initially on $L^2(\mathbb{R}^d_+, w_\alpha^+)$, extends uniquely to a bounded operator on $L^p(\mathbb{R}^d_+, w_\alpha^+)$, $1 < p < \infty$, and to a bounded operator from $L^1(\mathbb{R}^d_+, w_\alpha^+)$ to $L^{1,\infty}(\mathbb{R}^d_+, w_\alpha^+)$.*

The proof of Lemma 2 will be furnished by means of the general Calderón–Zygmund theory. In fact, we shall show that each $\mathcal{L}_{\alpha,\varepsilon,+}^{-i\gamma}$ is a Calderón–Zygmund operator in the sense of the space of homogeneous type $(\mathbb{R}^d_+, w_\alpha^+, \|\cdot\|)$. It is well known that the classical Calderón–Zygmund theory works, with appropriate adjustments, when the underlying space is of homogeneous type. Thus we shall use properly adjusted facts from the classic Calderón–Zygmund theory (presented, for instance, in [4]) in the setting of the space $(\mathbb{R}^d_+, w^+_\alpha, \|\cdot\|)$ without further comments.

A formal computation based on the formula

$$\lambda^{-i\gamma} = \frac{1}{\Gamma(i\gamma)} \int_0^\infty e^{-t\lambda} t^{i\gamma-1}\, dt, \qquad \lambda > 0,$$

suggests that $\mathcal{L}^{-i\gamma}_{\alpha,\varepsilon,+}$ should be associated with the kernel

$$K^{\alpha,\varepsilon}_\gamma(x,y) = \frac{1}{\Gamma(i\gamma)} \int_0^\infty G^{\alpha,\varepsilon}_t(x,y) t^{i\gamma-1}\, dt, \qquad x, y \in \mathbb{R}^d_+ \tag{4}$$

(note that for $x \neq y$ the last integral is absolutely convergent due to the decay of $G^{\alpha,\varepsilon}_t(x,y)$ at $t \to 0^+$ and $t \to \infty$, see Lemma 1). The next result shows that this is indeed the case, at least in the Calderón–Zygmund theory sense.

Proposition 1. *Let $\alpha \in [-1/2, \infty)^d$ and $\varepsilon \in \mathbb{Z}^d_2$. Then for $f, g \in C^\infty_c(\mathbb{R}^d_+)$ with disjoint supports*

$$\langle \mathcal{L}^{-i\gamma}_{\alpha,\varepsilon,+} f, g\rangle_{L^2(\mathbb{R}^d_+, w^+_\alpha)} = \int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} K^{\alpha,\varepsilon}_\gamma(x,y) f(y) \overline{g(x)}\, dw^+_\alpha(y)\, dw^+_\alpha(x). \tag{5}$$

Proof. We follow the lines of the proof of [15, Proposition 4.2], see also [14, Proposition 3.2]. By Parseval's identity and (3),

$$\langle \mathcal{L}^{-i\gamma}_{\alpha,\varepsilon,+} f, g\rangle_{L^2(\mathbb{R}^d_+, w^+_\alpha)} = \sum_{n \in \mathcal{N}_\varepsilon} (2|n| + 2|\alpha| + 2d)^{-i\gamma} \langle f, h^\alpha_n\rangle_{L^2(\mathbb{R}^d_+, w^+_\alpha)} \langle h^\alpha_n, g\rangle_{L^2(\mathbb{R}^d_+, w^+_\alpha)}. \tag{6}$$

To finish the proof it is now sufficient to verify that the right-hand sides of (5) and (6) coincide. This task means justifying the possibility of changing the order of integration, summation and differentiation in the relevant expressions, see the proof of Proposition 4.2 in [15]. The details are rather elementary and thus are omitted. The key estimate

$$\int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \int_0^\infty \left|\partial_t G^{\alpha,\varepsilon}_t(x,y)\right| dt\, |\overline{g(x)} f(y)|\, dy\, dx < \infty$$

is easily verified by means of Lemma 1. Another important ingredient (implicit in the proof of [15, Proposition 4.2]) is a suitable estimate for the growth of the underlying eigenfunctions. In the present setting it is sufficient to know that

$$|h^\alpha_n(x)| \lesssim \prod_{i=1}^d \Phi^{\alpha_i}_{n_i}(x_i), \qquad x \in \mathbb{R}^d_+,$$

where

$$\Phi^{\alpha_i}_{n_i}(x_i) = x_i^{-\alpha_i - 1/2} \begin{cases} 1, & 0 < x_i \le 4(n_i + \alpha_i + 1); \\ \exp(-c x_i), & x_i > 4(n_i + \alpha_i + 1). \end{cases}$$

This follows from Muckenhoupt's generalization [7] of the classical estimates due to Askey and Wainger [1]. ■

The theorem below says that the kernel $K^{\alpha,\varepsilon}_\gamma(x,y)$ satisfies standard estimates in the sense of the homogeneous space $(\mathbb{R}^d_+, w^+_\alpha, \|\cdot\|)$. The corresponding proof is located in Section 4 below. Denote $B^+(x,r) = B(x,r) \cap \mathbb{R}^d_+$.

Theorem 2. *Given $\alpha \in [-1/2,\infty)^d$ and $\varepsilon \in \mathbb{Z}_2^d$, the kernel $K_\gamma^{\alpha,\varepsilon}(x,y)$ satisfies the growth condition*

$$|K_\gamma^{\alpha,\varepsilon}(x,y)| \lesssim \frac{1}{w_\alpha^+(B^+(x,\|y-x\|))}, \qquad x,y\in\mathbb{R}_+^d, \quad x\neq y,$$

and the smoothness condition

$$\|\nabla_{x,y}K_\gamma^{\alpha,\varepsilon}(x,y)\| \lesssim \frac{1}{\|x-y\|}\frac{1}{w_\alpha^+(B^+(x,\|y-x\|))}, \qquad x,y\in\mathbb{R}_+^d, \quad x\neq y.$$

From Theorem 2 and Proposition 1 we conclude that $\mathcal{L}_{\alpha,\varepsilon,+}^{-i\gamma}$ is a Calderón–Zygmund operator. Thus Lemma 2 follows from the general theory, see [4].

Remark 1. The results of this section can be generalized in a straightforward manner by considering weighted L^p spaces. By the general theory, each $\mathcal{L}_{\alpha,\varepsilon,+}^{-i\gamma}$ extends to a bounded operator on $L^p(\mathbb{R}_+^d, W dw_\alpha^+)$, $W\in A_p^\alpha$, $1<p<\infty$, and to a bounded operator from $L^1(\mathbb{R}_+^d, W dw_\alpha^+)$ to $L^{1,\infty}(\mathbb{R}_+^d, W dw_\alpha^+)$, $W\in A_1^\alpha$; here A_p^α stands for the Muckenhoupt class of A_p weights associated with the space $(\mathbb{R}_+^d, w_\alpha^+, \|\cdot\|)$. Consequently, analogous mapping properties hold for $\mathcal{L}_\alpha^{-i\gamma}$, with reflection invariant weights satisfying A_p^α conditions when restricted to $\mathbb{R}_+^d$ (or, equivalently, satisfying A_p conditions related to the whole space $(\mathbb{R}^d, w_\alpha, \|\cdot\|)$).

Remark 2. With the particular $\varepsilon_0=(0,\dots,0)$ the operator $\mathcal{L}_{\alpha,\varepsilon_0,+}^{-i\gamma}$ coincides, up to a constant factor, with the same imaginary power of the Laguerre Laplacian investigated in [8]. Therefore the results of this section deliver also analogous results in the setting of [8].

4 Kernel estimates

This section is mainly devoted to the proof of the standard estimates stated in Theorem 2. The proof follows the pattern of the proof of Proposition 3.1 in [8], see also [9]. We use the formula

$$G_t^{\alpha,\varepsilon}(x,y) = \frac{1}{2^d}\Big(\frac{1-\zeta^2}{2\zeta}\Big)^{d+|\alpha|+|\varepsilon|}(xy)^\varepsilon \int_{[-1,1]^d} \exp\Big(-\frac{1}{4\zeta}q_+(x,y,s) - \frac{\zeta}{4}q_-(x,y,s)\Big)\Pi_{\alpha+\varepsilon}(ds),$$

where

$$q_\pm(x,y,s) = \|x\|^2 + \|y\|^2 \pm 2\sum_{i=1}^d x_i y_i s_i$$

(for the sake of brevity we shall often write shortly q_+ or q_- omitting the arguments) and $t\in(0,\infty)$ and $\zeta\in(0,1)$ are related by $\zeta=\tanh t$, so that

$$t = t(\zeta) = \frac12 \log\frac{1+\zeta}{1-\zeta}; \tag{7}$$

eventually, Π_α denotes the product measure $\bigotimes_{i=1}^d \Pi_{\alpha_i}$, where Π_{α_i} is determined by the density

$$\Pi_{\alpha_i}(ds) = \frac{(1-s^2)^{\alpha_i-1/2}ds}{\sqrt{\pi}2^{\alpha_i}\Gamma(\alpha_i+1/2)}, \qquad s\in(-1,1),$$

when $\alpha_i > -1/2$, and in the limiting case of $\alpha_i=-1/2$,

$$\Pi_{-1/2} = \frac{1}{\sqrt{2\pi}}\big(\eta_{-1}+\eta_1\big)$$

(η_{-1} and η_1 denote point masses at -1 and 1, respectively).

By the change of variable (7) the kernels (4) can be expressed as

$$K_\gamma^{\alpha,\varepsilon}(x,y) = \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \beta_{d,\alpha+\varepsilon}(\zeta)\psi_\zeta^\varepsilon(x,y,s)\, d\zeta, \tag{8}$$

where

$$\psi_\zeta^\varepsilon(x,y,s) = (xy)^\varepsilon \exp\left(-\frac{1}{4\zeta}q_+(x,y,s) - \frac{\zeta}{4}q_-(x,y,s)\right)$$

and

$$\beta_{d,\alpha}(\zeta) = \frac{2^{1-d-i\gamma}}{\Gamma(i\gamma)}\left(\frac{1-\zeta^2}{2\zeta}\right)^{d+|\alpha|}\frac{1}{1-\zeta^2}\left(\log\frac{1+\zeta}{1-\zeta}\right)^{i\gamma-1}.$$

Notice that $|\beta_{d,\alpha}(\zeta)|$ coincides, up to a constant factor, with $\beta^0_{d,\alpha}(\zeta)$ defined in [8, (5.4)].

The application of Fubini's theorem that was necessary to get (8) is also justified since, in fact, the proof (to be given below) of the first estimate in Theorem 2 contains the proof of

$$\int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \left|\beta_{d,\alpha+\varepsilon}(\zeta)\psi_\zeta^\varepsilon(x,y,s)\right| d\zeta < \infty, \qquad x \neq y.$$

For proving Theorem 2 we need a specified version of [8, Corollary 5.2] and a slight extension of [8, Lemma 5.5 (b)] (the proof of the latter result in [8] is given under assumption $k \geq 1$, but in fact it is also valid for any real k provided that the constant factor in the definition of $\beta^k_{d,\alpha}(\zeta)$ is neglected; in particular, $k = 0$ can be admitted).

Lemma 3. *Assume that $\alpha \in [-1/2,\infty)^d$. Let $b \geq 0$ and $c > 0$ be fixed. Then, we have*

(a) $$\big(|x_1 \pm y_1 s_1| + |y_1 \pm x_1 s_1|\big)\exp\left(-c\frac{1}{\zeta}q_\pm(x,y,s)\right) \lesssim \zeta^{\pm 1/2},$$

(b) $$\int_0^1 \big|\beta_{d,\alpha}(\zeta)\big|\zeta^{-b}\exp\left(-c\frac{1}{\zeta}q_+(x,y,s)\right)d\zeta \lesssim \big(q_+(x,y,s)\big)^{-d-|\alpha|-b},$$

uniformly in $x,y \in \mathbb{R}^d_+$, $s \in [-1,1]^d$, and also in $\zeta \in (0,1)$ if (a) is considered.

We also need the following generalization of [8, Proposition 5.9], cf. [9, Lemma 5.3].

Lemma 4. *Assume that $\alpha \in [-1/2,\infty)^d$ and let $\delta,\kappa \in [0,\infty)^d$ be fixed. Then for $x,y \in \mathbb{R}^d_+$, $x \neq y$,*

$$(x+y)^{2\delta}\int_{[-1,1]^d}\Pi_{\alpha+\delta+\kappa}(ds)\,\big(q_+(x,y,s)\big)^{-d-|\alpha|-|\delta|} \lesssim \frac{1}{w^+_\alpha(B^+(x,\|y-x\|))}$$

and

$$(x+y)^{2\delta}\int_{[-1,1]^d}\Pi_{\alpha+\delta+\kappa}(ds)\,\big(q_+(x,y,s)\big)^{-d-|\alpha|-|\delta|-1/2} \lesssim \frac{1}{\|x-y\|}\,\frac{1}{w^+_\alpha(B^+(x,\|y-x\|))}.$$

Proof of Theorem 2. The growth estimate is rather straightforward. Using Lemma 3 (b) with $b = 0$ and observing that $(xy)^\varepsilon \leq (x+y)^{2\varepsilon}$ gives

$$|K_\gamma^{\alpha,\varepsilon}(x,y)| \lesssim (x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon|}.$$

Now Lemma 4, taken with $\delta = \varepsilon$ and $\kappa = (0,\dots,0)$, provides the desired bound.

It remains to prove the smoothness estimate. Notice that by symmetry reasons it is enough to show that

$$|\partial_{x_1} K_\gamma^{\alpha,\varepsilon}(x,y)| + |\partial_{y_1} K_\gamma^{\alpha,\varepsilon}(x,y)| \lesssim \frac{1}{\|x-y\|} \frac{1}{w_\alpha^+(B^+(x,\|y-x\|))}, \qquad x,y \in \mathbb{R}^d_+, \quad x \neq y.$$

Moreover, we can focus on estimating the x_1-derivative only. This is because in the final stroke we shall use Lemma 4, where the left-hand sides are symmetric in x and y. Thus we are reduced to estimating the quantity

$$\mathcal{J} = \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds) \int_0^1 \left|\beta_{d,\alpha+\varepsilon}(\zeta)\partial_{x_1}\psi_\zeta^\varepsilon(x,y,s)\right| d\zeta$$

(passing with ∂_{x_1} under the integral signs is legitimate, the justification being implicitly contained in the estimates below, see the argument in [8, pp. 671–672]).

An elementary computation produces

$$\partial_{x_1}\psi_\zeta^\varepsilon(x,y,s) = \left[(xy)^\varepsilon\left(-\frac{1}{2\zeta}(x_1+y_1s_1) - \frac{\zeta}{2}(x_1-y_1s_1)\right) + \varepsilon_1 y_1 (xy)^{\varepsilon-e_1}\right] \times \exp\left(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\right).$$

Hence, by Lemma 3 (a), we have

$$\begin{aligned}
&\left|\partial_{x_1}\psi_\zeta^\varepsilon(x,y,s)\right| \\
&\quad\lesssim (xy)^\varepsilon(\zeta^{-1/2}+\zeta^{1/2})\exp\left(-\frac{1}{8\zeta}q_+ - \frac{\zeta}{8}q_-\right) + \varepsilon_1 y_1 (xy)^{\varepsilon-e_1}\exp\left(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\right) \\
&\quad\lesssim (x+y)^{2\varepsilon}\zeta^{-1/2}\exp\left(-\frac{1}{8\zeta}q_+\right) + \varepsilon_1 (x+y)^{2(\varepsilon-e_1/2)}\exp\left(-\frac{1}{4\zeta}q_+\right)
\end{aligned}$$

(notice that the second term above vanishes when $\varepsilon_1 = 0$). Consequently,

$$\begin{aligned}
\mathcal{J} &\lesssim (x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1 \left|\beta_{d,\alpha+\varepsilon}(\zeta)\right|\zeta^{-1/2}\exp\left(-\frac{1}{8\zeta}q_+\right)d\zeta \\
&\quad + \varepsilon_1 (x+y)^{2(\varepsilon-e_1/2)}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\int_0^1\left|\beta_{d,\alpha+\varepsilon}(\zeta)\right|\exp\left(-\frac{1}{4\zeta}q_+\right)d\zeta.
\end{aligned}$$

Now, applying Lemma 3 (b) with either $b = 1/2$ or $b = 0$ leads to

$$\begin{aligned}
\mathcal{J} &\lesssim (x+y)^{2\varepsilon}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon|-1/2} \\
&\quad + \varepsilon_1 (x+y)^{2(\varepsilon-e_1/2)}\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)(q_+)^{-d-|\alpha|-|\varepsilon|}.
\end{aligned}$$

Finally, Lemma 4 with either $\delta = \varepsilon$ and $\kappa = (0,\ldots,0)$ or (in case $\varepsilon_1 = 1$) $\delta = \varepsilon - e_1/2$ and $\kappa = e_1/2$ delivers the required smoothness bound for $\mathcal{J}$.

The proof of Theorem 2 is complete. ∎

Proof of Lemma 1. Recall that

$$G_t^{\alpha,\varepsilon}(x,y) = \frac{1}{2^d}\left(\frac{1-\zeta^2}{2\zeta}\right)^{d+|\alpha|+|\varepsilon|}(xy)^\varepsilon\int_{[-1,1]^d}\Pi_{\alpha+\varepsilon}(ds)\,\exp\left(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\right),$$

where t and ζ are related by $\zeta = \tanh t$. Since $\zeta \in (0,1)$ and $\|x-y\|^2 \le q_\pm \le \|x+y\|^2$, we see that

$$G_t^{\alpha,\varepsilon}(x,y) \lesssim \left(\frac{1-\zeta}{\zeta}\right)^{d+|\alpha|+|\varepsilon|} (xy)^\varepsilon \exp\left(-\frac{1}{4\zeta}\|x-y\|^2\right).$$

From this estimate the rapid decay easily follows.

To verify (2) we need first to compute $\partial_t G_t^{\alpha,\varepsilon}(x,y)$. We get

$$\partial_t G_t^{\alpha,\varepsilon}(x,y) = \big(d+|\alpha|+|\varepsilon|\big)\frac{1+\zeta^2}{\zeta} G_t^{\alpha,\varepsilon}(x,y) + \frac{1}{2^d}\left(\frac{1-\zeta^2}{2\zeta}\right)^{d+|\alpha|+|\varepsilon|}(1-\zeta^2)(xy)^\varepsilon \\ \times \int_{[-1,1]^d} \Pi_{\alpha+\varepsilon}(ds)\left(\frac{1}{4\zeta^2}q_+ - \frac{1}{4}q_-\right)\exp\left(-\frac{1}{4\zeta}q_+ - \frac{\zeta}{4}q_-\right)$$

(here passing with ∂_t under the integral can be easily justified). Consequently, taking into account the estimates above,

$$\big|\partial_t G_t^{\alpha,\varepsilon}(x,y)\big| \lesssim \frac{1}{\zeta}\left(\frac{1-\zeta}{\zeta}\right)^{d+|\alpha|+|\varepsilon|}(xy)^\varepsilon \exp\left(-\frac{1}{4\zeta}\|x-y\|^2\right) \\ + (1-\zeta)\left(\frac{1-\zeta}{\zeta}\right)^{d+|\alpha|+|\varepsilon|}(xy)^\varepsilon \frac{\|x+y\|^2}{\zeta^2}\exp\left(-\frac{1}{4\zeta}\|x-y\|^2\right).$$

This implies

$$\int_0^\infty \big|\partial_t G_t^{\alpha,\varepsilon}(x,y)\big|\,dt = \int_0^1 \Big|\partial_t G_t^{\alpha,\varepsilon}(x,y)\big|_{t=\tanh^{-1}\zeta}\Big| \frac{d\zeta}{1-\zeta^2} \\ \lesssim (xy)^\varepsilon \int_0^1 (1-\zeta)^{d+|\alpha|+|\varepsilon|-1}\zeta^{-(d+|\alpha|+|\varepsilon|-1)}\exp\left(-\frac{1}{4\zeta}\|x-y\|^2\right)d\zeta \\ + (xy)^\varepsilon\|x+y\|^2\int_0^1 \zeta^{-(d+|\alpha|+|\varepsilon|-2)}\exp\left(-\frac{1}{4\zeta}\|x-y\|^2\right)d\zeta.$$

Now using the fact that $d+|\alpha|+|\varepsilon|>0$ and $\sup_{u>0} u^a \exp(-Au) < \infty$ for any fixed $A>0$ and $a \ge 0$, leads to the bound

$$\int_0^\infty \big|\partial_t G_t^{\alpha,\varepsilon}(x,y)\big|\,dt \lesssim \frac{(xy)^\varepsilon}{\|x-y\|^{2(d+|\alpha|+|\varepsilon|+1)}} + \frac{(xy)^\varepsilon\|x+y\|^2}{\|x-y\|^{2(d+|\alpha|+|\varepsilon|+2)}}.$$

The conclusion follows. ■

References

[1] Askey R., Wainger S., Mean convergence of expansions in Laguerre and Hermite series, *Amer. J. Math.* **87** (1965), 695–708.

[2] Dunkl C.F., Reflection groups and orthogonal polynomials on the sphere, *Math. Z.* **197** (1988), 33–60.

[3] Dunkl C.F., Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[4] Duoandikoetxea J., Fourier analysis, *Graduate Studies in Mathematics*, Vol. 29, American Mathematical Society, Providence, RI, 2001.

[5] Lebedev N.N., Special functions and their applications, Dover Publications, Inc., New York, 1972.

[6] Muckenhoupt B., On certain singular integrals, *Pacific J. Math.* **10** (1960), 239–261.

[7] Muckenhoupt B., Mean convergence of Hermite and Laguerre series. II, *Trans. Amer. Math. Soc.* **147** (1970), 433–460.

[8] Nowak A., Stempak K., Riesz transforms for multi-dimensional Laguerre function expansions, *Adv. Math.* **215** (2007), 642–678.

[9] Nowak A., Stempak K., Riesz transforms for the Dunkl harmonic oscillator, *Math. Z.*, to appear, arXiv:0802.0474.

[10] Rosenblum M., Generalized Hermite polynomials and the Bose-like oscillator calculus, in Nonselfadjoint Operators and Related Topics (Beer Sheva, 1992), *Oper. Theory Adv. Appl.*, Vol. 73, Birkhäuser, Basel, 1994, 369–396.

[11] Rösler M., Generalized Hermite polynomials and the heat equation for Dunkl operators, *Comm. Math. Phys.* **192** (1998), 519–542, q-alg/9703006.

[12] Rösler M., One-parameter semigroups related to abstract quantum models of Calogero types, in Infinite Dimensional Harmonic Analysis (Kioto, 1999), Gräbner, Altendorf, 2000, 290–305.

[13] Rösler M., Dunkl operators: theory and applications, in Orthogonal Polynomials and Special Functions (Leuven, 2002), *Lecture Notes in Math.*, Vol. 1817, Springer, Berlin, 2003, 93–135, math.CA/0210366.

[14] Stempak K., Torrea J.L., Poisson integrals and Riesz transforms for Hermite function expansions with weights, *J. Funct. Anal.* **202** (2003), 443–472.

[15] Stempak K., Torrea J.L., Higher Riesz transforms and imaginary powers associated to the harmonic oscillator, *Acta Math. Hungar.* **111** (2006), 43–64.