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Sonine Transform Associated to the Dunkl Kernel on the Real Line*

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Abstract. We consider the Dunkl intertwining operator V_α and its dual ${}^tV_\alpha$, we define and study the Dunkl Sonine operator and its dual on $\mathbb{R}$. Next, we introduce complex powers of the Dunkl Laplacian Δ_α and establish inversion formulas for the Dunkl Sonine operator $S_{\alpha,\beta}$ and its dual ${}^tS_{\alpha,\beta}$. Also, we give a Plancherel formula for the operator ${}^tS_{\alpha,\beta}$.

Key words: Dunkl intertwining operator; Dunkl transform; Dunkl Sonine transform; complex powers of the Dunkl Laplacian

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1 Introduction

In this paper, we consider the Dunkl operator Λ_α, $\alpha > -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The operators were in general dimension introduced by Dunkl in [2] in connection with a generalization of the classical theory of spherical harmonics; they play a major role in various fields of mathematics [3, 4, 5] and also in physical applications [6].

The Dunkl analysis with respect to $\alpha \geq -1/2$ concerns the Dunkl operator Λ_α, the Dunkl transform $\mathcal{F}_\alpha$ and the Dunkl convolution $*_\alpha$ on $\mathbb{R}$. In the limit case $(\alpha = -1/2)$; Λ_α, $\mathcal{F}_\alpha$ and $*_\alpha$ agree with the operator d/dx, the Fourier transform and the standard convolution respectively.

First, we study the Dunkl Sonine operator $S_{\alpha,\beta}$, $\beta > \alpha$:

$$S_{\alpha,\beta}(f)(x) := \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)} \int_{-1}^{1} f(xt)(1-t^2)^{\beta-\alpha-1}(1+t)|t|^{2\alpha+1}dt,$$

and its dual ${}^tS_{\alpha,\beta}$ connected with these operators. Next, we establish for them the same results as those given in [8, 14] for the Radon transform and its dual; and in [9] for the spherical mean operator and its dual on $\mathbb{R}$. Especially:

- We define and study the complex powers for the Dunkl Laplacian $\Delta_\alpha = \Lambda_\alpha^2$.
- We give inversion formulas for $S_{\alpha,\beta}$ and ${}^tS_{\alpha,\beta}$ associated with integro-differential and integro-differential-difference operators when applied to some Lizorkin spaces of functions (see [9, 1, 13]).
- We establish a Plancherel formula for the operator ${}^tS_{\alpha,\beta}$.

The content of this work is the following. In Section 2, we recall some results about the Dunkl operators. In particular, we give some properties of the operators $S_{\alpha,\beta}$ and ${}^tS_{\alpha,\beta}$.

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In Section 3, we consider the tempered distribution $|x|^\lambda$ for $\lambda \in \mathbb{C}\backslash\{-(\ell+1),\ \ell \in \mathbb{N}\}$ defined by

$$\langle |x|^\lambda, \varphi\rangle := \int_{\mathbb{R}} |x|^\lambda \varphi(x)dx.$$

Also we study the complex powers of the Dunkl Laplacian $(-\Delta_\alpha)^\lambda$, for some complex number λ. In the classical case when $\alpha = -1/2$, the complex powers of the usual Laplacian are given in [16].

In Section 4, we give the following inversion formulas:

$$g = S_{\alpha,\beta}K_1({}^tS_{\alpha,\beta})(g), \qquad f = ({}^tS_{\alpha,\beta})K_2S_{\alpha,\beta}(f),$$

where

$$K_1(f) = \frac{c_\beta}{c_\alpha}(-\Delta_\alpha)^{\beta-\alpha}f, \qquad K_2(f) = \frac{c_\beta}{c_\alpha}(-\Delta_\beta)^{\beta-\alpha}f \qquad \text{and} \qquad c_\alpha = \frac{1}{[2^{\alpha+1}\Gamma(\alpha+1)]^2}.$$

Next, we give the following Plancherel formula for the operator ${}^tS_{\alpha,\beta}$:

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\beta+1}dx = \int_{\mathbb{R}} |K_3({}^tS_{\alpha,\beta}(f))(y)|^2|x|^{2\alpha+1}dy,$$

where

$$K_3(f) = \sqrt{\frac{c_\beta}{c_\alpha}}(-\Delta_\alpha)^{(\beta-\alpha)/2}f.$$

2 The Dunkl intertwining operator and its dual

We consider the Dunkl operator Λ_α, $\alpha \geq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha f(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right]. \tag{1}$$

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial problem:

$$\Lambda_\alpha f(x) = \lambda f(x), \qquad f(0) = 1,$$

has a unique analytic solution $E_\alpha(\lambda x)$ called Dunkl kernel [3, 5] given by

$$E_\alpha(\lambda x) = \Im_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha+1)}\Im_{\alpha+1}(\lambda x),$$

where

$$\Im_\alpha(\lambda x) := \Gamma(\alpha+1)\sum_{n=0}^\infty \frac{(\lambda x/2)^{2n}}{n!\,\Gamma(n+\alpha+1)},$$

is the modified spherical Bessel function of order α.

Notice that in the case $\alpha = -1/2$, we have

$$\Lambda_{-1/2} = d/dx \qquad \text{and} \qquad E_{-1/2}(\lambda x) = e^{\lambda x}.$$

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the Dunkl kernel E_α has the following Bochner-type representation (see [3, 11]):

$$E_\alpha(\lambda x) = a_\alpha \int_{-1}^1 e^{\lambda xt}\left(1-t^2\right)^{\alpha-1/2}(1+t)dt,$$

where

$$a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)},$$

which can be written as:

$$E_\alpha(\lambda x) = a_\alpha \operatorname{sgn}(x)\,|x|^{-(2\alpha+1)} \int_{-|x|}^{|x|} e^{\lambda y}\big(x^2-y^2\big)^{\alpha-1/2}(x+y)dy, \qquad x \neq 0,$$
$$E_\alpha(0) = 1.$$

We notice that, the Dunkl kernel $E_\alpha(\lambda x)$ can be also expanded in a power series [10] in the form:

$$E_\alpha(\lambda x) = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{b_n(\alpha)}, \tag{2}$$

where

$$b_{2n}(\alpha) = \frac{2^{2n} n!}{\Gamma(\alpha+1)}\Gamma(n+\alpha+1), \qquad b_{2n+1}(\alpha) = 2(\alpha+1)b_{2n}(\alpha+1).$$

Let $\alpha > -1/2$ and we define the Dunkl intertwining operator V_α on $\mathcal{E}(\mathbb{R})$ (the space of C^∞-functions on $\mathbb{R}$), by

$$V_\alpha(f)(x) := a_\alpha \int_{-1}^{1} f(xt)\big(1-t^2\big)^{\alpha-1/2}(1+t)dt,$$

which can be written as:

$$V_\alpha(f)(x) = a_\alpha \operatorname{sgn}(x)\,|x|^{-(2\alpha+1)} \int_{-|x|}^{|x|} f(y)\big(x^2-y^2\big)^{\alpha-1/2}(x+y)dy, \qquad x \neq 0,$$
$$V_\alpha(f)(0) = f(0).$$

Remark 1. For $\alpha > -1/2$, we have

$$E_\alpha(\lambda.) = V_\alpha(e^{\lambda.}), \qquad \lambda \in \mathbb{C}.$$

Proposition 1 (see [18], Theorem 6.3). *The operator V_α is a topological automorphism of $\mathcal{E}(\mathbb{R})$, and satisfies the transmutation relation:*

$$\Lambda_\alpha(V_\alpha(f)) = V_\alpha\left(\frac{d}{dx}f\right), \qquad f \in \mathcal{E}(\mathbb{R}).$$

Let $\alpha > -1/2$ and we define the dual Dunkl intertwining operator ${}^tV_\alpha$ on $\mathcal{S}(\mathbb{R})$ (the Schwartz space on $\mathbb{R}$), by

$${}^tV_\alpha(f)(x) := a_\alpha \int_{|y|\geq|x|} \operatorname{sgn}(y)\big(y^2-x^2\big)^{\alpha-1/2}(x+y)f(y)dy,$$

which can be written as:

$${}^tV_\alpha(f)(x) = a_\alpha \operatorname{sgn}(x)|x|^{2\alpha+1} \int_{|t|\geq 1} \operatorname{sgn}(t)\big(t^2-1\big)^{\alpha-1/2}(1+t)f(xt)dt.$$

Proposition 2 (see [19], Theorems 3.2, 3.3).

(i) The operator ${}^tV_\alpha$ is a topological automorphism of $\mathcal{S}(\mathbb{R})$, and satisfies the transmutation relation:

$$ {}^tV_\alpha(\Lambda_\alpha f) = \frac{d}{dx}({}^tV_\alpha(f)), \qquad f \in \mathcal{S}(\mathbb{R}). $$

(ii) For all $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$, we have

$$ \int_{\mathbb{R}} V_\alpha(f)(x)g(x)|x|^{2\alpha+1}dx = \int_{\mathbb{R}} f(x)\,{}^tV_\alpha(g)(x)dx. $$

Remark 2 (see [15]).

(i) For $\alpha > -1/2$ and $f \in \mathcal{E}(\mathbb{R})$, we can write

$$ V_\alpha(f)(x) = \Re_\alpha(f_e)(|x|) + \frac{1}{x}\Re_\alpha(Mf_o)(|x|), $$

where

$$ f_e(x) = \frac{1}{2}(f(x) + f(-x)), \qquad f_o(x) = \frac{1}{2}(f(x) - f(-x)), \qquad Mf_o(x) = xf_o(x), $$

and $\Re_\alpha$ is the Riemann–Liouville transform (see [17], page 75) given by

$$ \Re_\alpha(f_e)(x) := 2a_\alpha \int_0^1 f_e(xt)\big(1-t^2\big)^{\alpha-1/2}dt, \qquad x \geq 0. $$

Thus, we obtain

$$ V_\alpha^{-1}(f)(x) = \Re_\alpha^{-1}(f_e)(|x|) + \frac{1}{x}\Re_\alpha^{-1}(Mf_o)(|x|). $$

Therefore (see also [20], Proposition 2.2), we get

$$ V_\alpha^{-1}(f_e)(x) = d_\alpha \frac{d}{dx}\left(\frac{d}{xdx}\right)^r \left\{x^{2r+1}\int_0^1 f_e(xt)\big(1-t^2\big)^{r-\alpha-1/2}t^{2\alpha+1}dt\right\}, $$
$$ V_\alpha^{-1}(f_o)(x) = d_\alpha \left(\frac{d}{xdx}\right)^{r+1} \left\{x^{2r+2}\int_0^1 f_o(xt)\big(1-t^2\big)^{r-\alpha-1/2}t^{2\alpha+2}dt\right\}, $$

where $r = [\alpha + 1/2]$ denote the integer part of $\alpha + 1/2$, and $d_\alpha = \frac{2^{-r}\pi}{\Gamma(\alpha+1)\Gamma(r-\alpha+1/2)}$.

(ii) For $\alpha > -1/2$ and $f \in \mathcal{S}(\mathbb{R})$, we can write

$$ {}^tV_\alpha(f)(x) = W_\alpha(f_e)(|x|) + xW_\alpha(M^{-1}f_o)(|x|), $$

where

$$ M^{-1}f_o(x) = \frac{1}{2x}(f(x) - f(-x)), $$

and W_α is the Weyl integral transform (see [17, page 85]) given by

$$ W_\alpha(f_e)(x) := 2a_\alpha x^{2\alpha+1}\int_1^\infty f_e(xt)\big(t^2-1\big)^{\alpha-1/2}tdt, \qquad x \geq 0. $$

Thus, we obtain

$$ ({}^tV_\alpha)^{-1}f(x) = W_\alpha^{-1}(f_e)(|x|) + xW_\alpha^{-1}(M^{-1}f_o)(|x|). $$

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on $\mathbb{R}$, which was introduced by Dunkl in [4], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu in [5].

The Dunkl transform of a function $f \in \mathcal{S}(\mathbb{R})$, is given by

$$\mathcal{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x) f(x)|x|^{2\alpha+1}dx, \qquad \lambda \in \mathbb{R}.$$

We notice that $\mathcal{F}_{-1/2}$ agrees with the Fourier transform $\mathcal{F}$ that is given by:

$$\mathcal{F}(f)(\lambda) := \int_{\mathbb{R}} e^{-i\lambda x} f(x)\, dx, \qquad \lambda \in \mathbb{R}.$$

Proposition 3 (see [5]).

(i) For all $f \in \mathcal{S}(\mathbb{R})$, we have

$$\mathcal{F}_\alpha(\Lambda_\alpha f)(\lambda) = i\lambda\, \mathcal{F}_\alpha(f)(\lambda), \qquad \lambda \in \mathbb{R},$$

where Λ_α is the Dunkl operator given by (1).

(ii) $\mathcal{F}_\alpha$ possesses on $\mathcal{S}(\mathbb{R})$ the following decomposition:

$$\mathcal{F}_\alpha(f) = \mathcal{F} \circ {}^tV_\alpha(f), \qquad f \in \mathcal{S}(\mathbb{R}).$$

(iii) $\mathcal{F}_\alpha$ is a topological automorphism of $\mathcal{S}(\mathbb{R})$, and for $f \in \mathcal{S}(\mathbb{R})$ we have

$$f(x) = c_\alpha \int_{\mathbb{R}} E_\alpha(i\lambda x)\mathcal{F}_\alpha(f)(\lambda)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$c_\alpha = \frac{1}{[2^{\alpha+1}\Gamma(\alpha+1)]^2}.$$

(iv) The normalized Dunkl transform $\sqrt{c_\alpha}\,\mathcal{F}_\alpha$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}, |x|^{2\alpha+1}dx)$ onto itself. In particular,

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = c_\alpha \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

For $T \in \mathcal{S}'(\mathbb{R})$, we define the Dunkl transform $\mathcal{F}_\alpha(T)$ of T, by

$$\langle \mathcal{F}_\alpha(T), \varphi\rangle := \langle T, \mathcal{F}_\alpha(\varphi)\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}). \tag{3}$$

Thus the transform $\mathcal{F}_\alpha$ extends to a topological automorphism on $\mathcal{S}'(\mathbb{R})$.

In [19], the author defines:

- The Dunkl translation operators τ_x, $x \in \mathbb{R}$, on $\mathcal{E}(\mathbb{R})$, by

$$\tau_x f(y) := (V_\alpha)_x \otimes (V_\alpha)_y \big[(V_\alpha)^{-1}(f)(x+y)\big], \qquad y \in \mathbb{R}.$$

These operators satisfy for $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the following properties:

$$E_\alpha(\lambda x)E_\alpha(\lambda y) = \tau_x(E_\alpha(\lambda .))(y), \qquad \text{and}$$
$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_k(i\lambda x)\mathcal{F}_\alpha(f)(\lambda), \qquad f \in \mathcal{S}(\mathbb{R}).$$

Proposition 4 (see [11]). *If $f \in \mathcal{C}(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$) and $x, y \in \mathbb{R}$ such that $(x, y) \neq (0, 0)$, then*

$$\tau_x f(y) = a_\alpha \int_0^\pi \left[f_e((x,y)_\theta) + f_o((x,y)_\theta) \frac{x+y}{(x,y)_\theta} \right] [1 - \operatorname{sgn}(xy)\cos\theta] \sin^{2\alpha}\theta d\theta,$$
$$f_e(z) = \tfrac{1}{2}(f(z) + f(-z)), \qquad f_o(z) = \tfrac{1}{2}(f(z) - f(-z)),$$
$$(x,y)_\theta = \sqrt{x^2 + y^2 - 2|xy|\cos\theta}.$$

- The Dunkl convolution product $*_\alpha$ of two functions f and g in $\mathcal{S}(\mathbb{R})$, by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y) |y|^{2\alpha+1} dy, \qquad x \in \mathbb{R}.$$

This convolution is associative, commutative in $\mathcal{S}(\mathbb{R})$ and satisfies (see [19, Theorem 7.2]):

$$\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g).$$

For $T \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, we define the Dunkl convolution product $T *_\alpha f$, by

$$T *_\alpha f(x) := \langle T(y), \tau_x f(-y) \rangle, \qquad x \in \mathbb{R}. \tag{4}$$

Note that $*_{-1/2}$ agrees with the standard convolution $*$:

$$T * f(x) := \langle T(y), f(x-y) \rangle.$$

3 The Dunkl Sonine transform

In this section we study the Dunkl Sonine transform, which also studied by Y. Xu on polynomials in [20]. For thus we consider the following identity, which is a consequence of Xu's result when we extend the result of Lemma 2.1 on $\mathcal{E}(\mathbb{R})$.

Proposition 5. *Let $\alpha, \beta \in]-1/2, \infty[$, such that $\beta > \alpha$. Then*

$$E_\beta(\lambda x) = a_{\alpha,\beta} \int_{-1}^1 E_\alpha(\lambda x t) \left(1 - t^2\right)^{\beta-\alpha-1} (1+t)|t|^{2\alpha+1} dt, \tag{5}$$

where

$$a_{\alpha,\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}.$$

Proof. From (2), we have

$$\int_{-1}^1 E_\alpha(\lambda x t) \left(1 - t^2\right)^{\beta-\alpha-1} (1+t)|t|^{2\alpha+1} dt = \sum_{n=0}^\infty \frac{(\lambda x)^n}{b_n(\alpha)} I_n(\alpha, \beta),$$

where

$$I_n(\alpha,\beta) = \int_{-1}^1 t^n \left(1 - t^2\right)^{\beta-\alpha-1} (1+t)|t|^{2\alpha+1} dt,$$

or

$$I_{2n}(\alpha,\beta) = 2\int_0^1 \left(1 - t^2\right)^{\beta-\alpha-1} t^{2n+2\alpha+1} dt = \int_0^1 (1-y)^{\beta-\alpha-1} y^{n+\alpha} dy$$

$$= \frac{\Gamma(\beta-\alpha)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)},$$

and

$$I_{2n+1}(\alpha,\beta) = 2\int_0^1 \left(1-t^2\right)^{\beta-\alpha-1} t^{2n+2\alpha+3}dt = I_{2n}(\alpha+1,\beta+1).$$

Thus

$$\int_{-1}^1 E_\alpha(\lambda xt)\left(1-t^2\right)^{\beta-\alpha-1}(1+t)|t|^{2\alpha+1}dt = \frac{\Gamma(\beta-\alpha)\Gamma(\alpha+1)}{\Gamma(\beta+1)}\, E_\beta(\lambda x),$$

which gives the desired result. ■

Remark 3. We can write the formula (5) by the following

$$E_\beta(\lambda x) = a_{\alpha,\beta}\, \mathrm{sgn}(x)\, |x|^{-(2\beta+1)} \int_{-|x|}^{|x|} E_\alpha(\lambda y)\left(x^2-y^2\right)^{\beta-\alpha-1}(x+y)|y|^{2\alpha+1}dy, \qquad x \neq 0.$$

Definition 1. Let $\alpha,\beta \in\,]-1/2,\infty[$, such that $\beta > \alpha$. We define the Dunkl Sonine transform $S_{\alpha,\beta}$ on $\mathcal{E}(\mathbb{R})$, by

$$S_{\alpha,\beta}(f)(x) := a_{\alpha,\beta}\int_{-1}^1 f(xt)\left(1-t^2\right)^{\beta-\alpha-1}(1+t)|t|^{2\alpha+1}dt,$$

which can be written as:

$$S_{\alpha,\beta}(f)(x) = a_{\alpha,\beta}\, \mathrm{sgn}(x)\, |x|^{-(2\beta+1)} \int_{-|x|}^{|x|} f(y)\left(x^2-y^2\right)^{\beta-\alpha-1}(x+y)|y|^{2\alpha+1}dy, \qquad x \neq 0,$$

$$S_{\alpha,\beta}(f)(0) = f(0).$$

Remark 4. For $\alpha,\beta \in\,]-1/2,\infty[$, such that $\beta > \alpha$, we have

$$E_\beta(\lambda .) = S_{\alpha,\beta}(E_\alpha(\lambda .)), \qquad \lambda \in \mathbb{C}. \tag{6}$$

Definition 2. Let $\alpha,\beta \in\,]-1/2,\infty[$, such that $\beta > \alpha$. We define the dual Dunkl Sonine transform ${}^tS_{\alpha,\beta}$ on $\mathcal{S}(\mathbb{R})$, by

$${}^tS_{\alpha,\beta}(f)(x) := a_{\alpha,\beta}\int_{|y|\ge|x|} \mathrm{sgn}(y)\left(y^2-x^2\right)^{\beta-\alpha-1}(x+y)f(y)dy,$$

which can be written as:

$${}^tS_{\alpha,\beta}(f)(x) = a_{\alpha,\beta}\, \mathrm{sgn}(x)|x|^{2(\beta-\alpha)} \int_{|t|\ge 1} \mathrm{sgn}(t)\left(t^2-1\right)^{\beta-\alpha-1}(t+1)f(xt)dt.$$

Proposition 6.

(i) For all $f \in \mathcal{E}(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} S_{\alpha,\beta}(f)(x)g(x)|x|^{2\beta+1}dx = \int_{\mathbb{R}} f(x)\, {}^tS_{\alpha,\beta}(g)(x)|x|^{2\alpha+1}dx.$$

(ii) $\mathcal{F}_\beta$ possesses on $\mathcal{S}(\mathbb{R})$ the following decomposition:

$$\mathcal{F}_\beta(f) = \mathcal{F}_\alpha \circ {}^tS_{\alpha,\beta}(f), \qquad f \in \mathcal{S}(\mathbb{R}).$$

Proof. Part (i) follows from Definition 1 by Fubini's theorem. Then part (ii) follows from (i) and (6) by taking $f = E_\alpha(-i\lambda.)$. ■

In [20, Lemma 2.1] Y. Xu proves the identity $S_{\alpha,\beta} = V_\beta \circ V_\alpha^{-1}$ on polynomials. As the intertwiner is a homeomorphism on $\mathcal{E}(\mathbb{R})$ and polynomials are dense in $\mathcal{E}(\mathbb{R})$, this gives the identity also on $\mathcal{E}(\mathbb{R})$. In the following we give a second method to prove this identity.

Theorem 1.
(i) *The operator* ${}^tS_{\alpha,\beta}$ *is a topological automorphism of* $\mathcal{S}(\mathbb{R})$, *and satisfies the following relations:*

$$\begin{aligned}&{}^tS_{\alpha,\beta}(f) = ({}^tV_\alpha)^{-1} \circ {}^tV_\beta(f), \qquad f \in \mathcal{S}(\mathbb{R}),\\ &{}^tS_{\alpha,\beta}(\Lambda_\beta f) = \Lambda_\alpha({}^tS_{\alpha,\beta}(f)), \qquad f \in \mathcal{S}(\mathbb{R}).\end{aligned}$$

(ii) *The operator* $S_{\alpha,\beta}$ *is a topological automorphism of* $\mathcal{E}(\mathbb{R})$, *and satisfies the following relations:*

$$\begin{aligned}&S_{\alpha,\beta}(f) = V_\beta \circ V_\alpha^{-1}(f), \qquad f \in \mathcal{E}(\mathbb{R}),\\ &\Lambda_\beta(S_{\alpha,\beta}(f)) = S_{\alpha,\beta}(\Lambda_\alpha f), \qquad f \in \mathcal{E}(\mathbb{R}).\end{aligned}$$

Proof. (i) From Proposition 6 (ii), we have

$${}^tS_{\alpha,\beta}(f) = (\mathcal{F}_\alpha)^{-1} \circ \mathcal{F}_\beta(f). \tag{7}$$

Using Proposition 3 (ii), we obtain

$${}^tS_{\alpha,\beta}(f) = ({}^tV_\alpha)^{-1} \circ {}^tV_\beta(f), \qquad f \in \mathcal{S}(\mathbb{R}). \tag{8}$$

Thus from Proposition 2 (i),

$${}^tS_{\alpha,\beta}(\Lambda_\beta f) = ({}^tV_\alpha)^{-1} \circ {}^tV_\beta(\Lambda_\beta f) = ({}^tV_\alpha)^{-1}\left(\frac{d}{dx}\,{}^tV_\beta(f)\right).$$

Using the fact that

$${}^tV_\alpha(\Lambda_\alpha f) = \frac{d}{dx}({}^tV_\alpha(f)) \iff \Lambda_\alpha({}^tV_\alpha)^{-1}(f) = ({}^tV_\alpha)^{-1}\left(\frac{d}{dx}f\right),$$

we obtain

$${}^tS_{\alpha,\beta}(\Lambda_\beta f) = \Lambda_\alpha({}^tV_\alpha)^{-1}({}^tV_\beta(f)) = \Lambda_\alpha({}^tS_{\alpha,\beta}(f)).$$

(ii) From Proposition 2 (ii), we have

$$\int_{\mathbb{R}} f(x)\,{}^tV_\beta(g)(x)dx = \int_{\mathbb{R}} V_\beta(f)(x)g(x)|x|^{2\beta+1}dx.$$

On other hand, from (8), Proposition 2 (ii) and Proposition 6 (i) we have

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,{}^tV_\beta(g)(x)dx &= \int_{\mathbb{R}} f(x)\,{}^tV_\alpha \circ {}^tS_{\alpha,\beta}(g)(x)dx = \int_{\mathbb{R}} V_\alpha(f)(x)\,{}^tS_{\alpha,\beta}(g)(x)|x|^{2\alpha+1}dx\\ &= \int_{\mathbb{R}} S_{\alpha,\beta} \circ V_\alpha(f)(x)g(x)|x|^{2\beta+1}dx.\end{aligned}$$

Then

$$S_{\alpha,\beta} \circ V_\alpha(f) = V_\beta(f).$$

Hence from Proposition 1,

$$\Lambda_\beta(S_{\alpha,\beta}(f)) = \Lambda_\beta V_\beta(V_\alpha^{-1}(f)) = V_\beta\left(\frac{d}{dx}V_\alpha^{-1}(f)\right).$$

Using the fact that

$$\Lambda_\alpha(V_\alpha(f)) = V_\alpha\left(\frac{d}{dx}f\right) \iff V_\alpha^{-1}(\Lambda_\alpha f) = \frac{d}{dx}V_\alpha^{-1}(f),$$

we obtain

$$\Lambda_\beta(S_{\alpha,\beta}(f)) = V_\beta \circ V_\alpha^{-1}(\Lambda_\alpha f) = S_{\alpha,\beta}(\Lambda_\alpha f),$$

which completes the proof of the theorem. ■

4 Complex powers of Δ_α

For $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > -1$, we denote by $|x|^\lambda$ the tempered distribution defined by

$$\langle |x|^\lambda, \varphi\rangle := \int_{\mathbb{R}} |x|^\lambda \varphi(x)dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}). \tag{9}$$

We write

$$\langle |x|^\lambda, \varphi\rangle = \int_0^\infty x^\lambda[\varphi(x) + \varphi(-x)]dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}),$$

then from [1], we obtain the following result.

Lemma 1. *Let $\varphi \in \mathcal{S}(\mathbb{R})$. The mapping $g : \lambda \to \langle |x|^\lambda, \varphi\rangle$ is complex-valued function and has an analytic extension to $\mathbb{C}\backslash\{-(1+2\ell),\ \ell \in \mathbb{N}\}$, with simple poles $-(2\ell+1)$, $\ell \in \mathbb{N}$ and*

$$\mathrm{Res}(g, -1-2\ell) = 2\frac{\varphi^{(2\ell)}(0)}{(2\ell)!}.$$

Proposition 7. *Let $\varphi \in \mathcal{S}(\mathbb{R})$.*

(i) The function $\lambda \to \langle |x|^{\lambda+2\alpha+1}, \varphi\rangle$ is analytic on $\mathbb{C}\backslash\{-(2\alpha+2\ell+2),\ \ell \in \mathbb{N}\}$, with simple poles $-(2\alpha+2\ell+2)$, $\ell \in \mathbb{N}$.

(ii) The function $\lambda \to \frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}\langle |x|^{-(\lambda+1)}, \varphi\rangle$ is analytic on $\mathbb{C}\backslash\{-(2\alpha+2\ell+2),\ \ell \in \mathbb{N}\}$, with simple poles $-(2\alpha+2\ell+2)$, $\ell \in \mathbb{N}$.

(iii) For $\lambda \in \mathbb{C}\backslash\{-(2\alpha+2\ell+2),\ \ell \in \mathbb{N}\}$ we have

$$\mathcal{F}_\alpha\big(|x|^{\lambda+2\alpha+1}\big) = \frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}|x|^{-(\lambda+1)}, \qquad \text{in } \mathcal{S}'\text{-sense.}$$

(iv) For $\lambda \in \mathbb{C}\backslash\{-(2\alpha+2\ell+2),\ \ell \in \mathbb{N}\}$ we have

$$|x|^{\lambda+2\alpha+1} = \frac{2^\lambda\,\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(\alpha+1)\Gamma(-\lambda/2)}\mathcal{F}_\alpha(|x|^{-(\lambda+1)}), \qquad \text{in } \mathcal{S}'\text{-sense.}$$

Proof. (i) Follows directly from Lemma 1.

(ii) From [7, pages 2 and 8] the function $\lambda \to \Gamma(\frac{2\alpha+\lambda+2}{2})$ has an analytic extension to $\mathbb{C}\backslash\{-(2\alpha+2\ell+2),\ \ell\in\mathbb{N}\}$, with simple poles $-(2\alpha+2\ell+2)$, $\ell\in\mathbb{N}$, and the function $\lambda\to\frac{1}{\Gamma(-\lambda/2)}$ has zeros 2ℓ, $\ell\in\mathbb{N}$. Thus from Lemma 1 we see that

$$\lambda \to \frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}\langle |x|^{-(\lambda+1)},\varphi\rangle$$

is analytic on $\mathbb{C}\backslash\{-(2\alpha+2\ell+2),\ \ell\in\mathbb{N}\}$, with simple poles $-(2\alpha+2\ell+2)$, $\ell\in\mathbb{N}$.

(iii) Let determine the value of $\mathcal{F}_\alpha(|x|^{\lambda+2\alpha+1})$ in the $\mathcal{S}'$-sense. We put $\psi_t(x):=e^{-tx^2}$, $t>0$. Then $\psi_t\in\mathcal{S}(\mathbb{R})$, and from [12]:

$$\mathcal{F}_\alpha(\psi_t)(x)=\Gamma(\alpha+1)t^{-(\alpha+1)}e^{-x^2/4t},\qquad x\in\mathbb{R}.$$

Furthermore, for $\varphi\in\mathcal{S}(\mathbb{R})$ we have

$$\int_{\mathbb{R}}\mathcal{F}_\alpha(\varphi)(x)\psi_t(x)|x|^{2\alpha+1}dx=\Gamma(\alpha+1)\int_{\mathbb{R}}\varphi(x)t^{-(\alpha+1)}e^{-x^2/4t}|x|^{2\alpha+1}dx.$$

Multiplying both sides by $t^{-\lambda/2-1}$ and integrating over $(0,\infty)$, we obtain for $\mathrm{Re}(\lambda)\in\,]-(2\alpha+2),0[$:

$$\int_{\mathbb{R}}\mathcal{F}_\alpha(\varphi)(x)|x|^{\lambda+2\alpha+1}dx=\frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}\int_{\mathbb{R}}\varphi(x)|x|^{-(\lambda+1)}dx.$$

This and from (3) we get for $\mathrm{Re}(\lambda)\in\,]-(2\alpha+2),0[$:

$$\mathcal{F}_\alpha(|x|^{\lambda+2\alpha+1})=\frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}|x|^{-(\lambda+1)}.$$

The result follows by analytic continuation.

(iv) From (iii) we have

$$|x|^{\lambda+2\alpha+1}=\frac{2^{2\alpha+\lambda+2}\Gamma(\alpha+1)\Gamma(\frac{2\alpha+\lambda+2}{2})}{\Gamma(-\lambda/2)}\mathcal{F}_\alpha^{-1}\big(|x|^{-(\lambda+1)}\big).$$

Using the fact that

$$\langle\mathcal{F}_\alpha^{-1}\big(|x|^{-(\lambda+1)}\big),\varphi\rangle=\langle|x|^{-(\lambda+1)},\mathcal{F}_\alpha^{-1}(\varphi)\rangle,\qquad\varphi\in\mathcal{S}(\mathbb{R}).$$

By applying (9) and Proposition 3 (iii), we obtain

$$\langle\mathcal{F}_\alpha^{-1}(|x|^{-(\lambda+1)}),\varphi\rangle=c_\alpha\int_{\mathbb{R}}|x|^{-(\lambda+1)}\mathcal{F}_\alpha(\varphi)(-x)dx,\qquad\varphi\in\mathcal{S}(\mathbb{R}).$$

Then

$$\mathcal{F}_\alpha^{-1}\big(|x|^{-(\lambda+1)}\big)=c_\alpha\,\mathcal{F}_\alpha\big(|x|^{-(\lambda+1)}\big),$$

which gives the result. ■

Definition 3. For $\lambda\in\mathbb{C}\backslash\{-(\alpha+\ell+1),\ \ell\in\mathbb{N}\}$, the complex powers of the Dunkl Laplacian Δ_α are defined for $f\in\mathcal{S}(\mathbb{R})$ by

$$(-\Delta_\alpha)^\lambda f(x):=\frac{2^{2\lambda}\Gamma(\alpha+\lambda+1)}{\Gamma(\alpha+1)\Gamma(-\lambda)}|x|^{-(2\lambda+1)}*_\alpha f(x),$$

where $*_\alpha$ is the Dunkl convolution product given by (4).

In the next part of this section we use Definition 3 and Proposition 7 (iv) to establish the following result:

$$\mathcal{F}_\alpha\big((-\Delta_\alpha)^\lambda f\big)(x) = |x|^{2\lambda}\mathcal{F}_\alpha(f)(x).$$

Proposition 8. *For* $\lambda \in \mathbb{C}\backslash\{-(\alpha+\ell+1),\ \ell\in\mathbb{N}\}$ *and* $f \in \mathcal{S}(\mathbb{R})$,

$$(-\Delta_\alpha)^\lambda f(x) = b_\alpha(\lambda)\int_{\mathbb{R}}\left[\int_0^\pi \frac{(1+\operatorname{sgn}(xy)\cos\theta)}{(x,y)_\theta^{2(\lambda+\alpha+1)}}\sin^{2\alpha}\theta d\theta\right] f(y)|y|^{2\alpha+1}dy,$$

where

$$b_\alpha(\lambda) = \frac{2^{2\lambda}\Gamma(\alpha+\lambda+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)\Gamma(-\lambda)}, \qquad (x,y)_\theta = \sqrt{x^2+y^2-2|xy|\cos\theta}.$$

Proof. From Definition 3, (4) and (9), we have

$$\begin{aligned}(-\Delta_\alpha)^\lambda f(x) &= \frac{2^{2\lambda}\Gamma(\alpha+\lambda+1)}{\Gamma(\alpha+1)\Gamma(-\lambda)}\langle |y|^{-(2\lambda+1)}, \tau_x f(-y)\rangle\\ &= \frac{2^{2\lambda}\Gamma(\alpha+\lambda+1)}{\Gamma(\alpha+1)\Gamma(-\lambda)}\int_{\mathbb{R}}|y|^{-2(\lambda+\alpha+1)}\tau_x f(-y)|y|^{2\alpha+1}dy.\end{aligned}$$

So

$$(-\Delta_\alpha)^\lambda f(x) = \int_{\mathbb{R}}\tau_x(|y|^{-2(\lambda+\alpha+1)})(-y)f(y)|y|^{2\alpha+1}dy.$$

Then the result follows from Proposition 4. ■

Note 1. We denote by

- Ψ the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions f, such that

$$f^{(k)}(0) = 0, \qquad \forall\, k\in\mathbb{N}.$$

- Φ_α the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions f, such that

$$\int_{\mathbb{R}} f(y)\,y^k|y|^{2\alpha+1}dy = 0, \qquad \forall\, k\in\mathbb{N}.$$

The spaces Ψ and $\Phi_{-1/2}$ are well-known in the literature as Lizorkin spaces (see [1, 9, 13]).

Lemma 2 (see [1]). *The multiplication operator* $M_\lambda : f \to |x|^\lambda f$, $\lambda\in\mathbb{C}$, *is a topological automorphism of* Ψ. *Its inverse operator is* $(M_\lambda)^{-1} = M_{-\lambda}$.

Theorem 2.

(i) *The Dunkl transform* $\mathcal{F}_\alpha$ *is a topological isomorphism from* Φ_α *onto* Ψ.

(ii) *The operator* ${}^tS_{\alpha,\beta}$ *is a topological isomorphism from* Φ_β *onto* Φ_α.

(iii) *For* $\lambda \in \mathbb{C}\backslash\{-(\alpha+\ell+1),\ \ell\in\mathbb{N}\}$ *and* $f\in\Phi_\alpha$, *the function* $(-\Delta_\alpha)^\lambda f$ *belongs to* $\in \Phi_\alpha$, *and*

$$\mathcal{F}_\alpha((-\Delta_\alpha)^\lambda f)(x) = |x|^{2\lambda}\mathcal{F}_\alpha(f)(x). \tag{10}$$

Proof. (i) Let $f \in \Phi_\alpha$, then

$$(\mathcal{F}_\alpha(f))^{(k)}(0) = (-i)^k \frac{k!}{b_k(\alpha)} \int_{\mathbb{R}} f(x)\, x^k |x|^{2\alpha+1} dy = 0, \qquad \forall\, k \in \mathbb{N}.$$

Hence $\mathcal{F}_\alpha(f) \in \Psi$.

Conversely, let $g \in \Psi$. Since $\mathcal{F}_\alpha$ is a topological automorphism of $\mathcal{S}(\mathbb{R})$. There exists $f \in \mathcal{S}(\mathbb{R})$, such that $\mathcal{F}_\alpha(f) = g$. Thus

$$g^{(k)}(0) = (-i)^k \frac{k!}{b_k(\alpha)} \int_{\mathbb{R}} f(x)\, x^k |x|^{2\alpha+1} dy = 0, \qquad \forall\, k \in \mathbb{N}.$$

So $f \in \Phi_\alpha$ and $\mathcal{F}_\alpha(f) = g$.

(ii) follows directly from (i) and (7).

(iii) Similarly to the standard convolution if $f \in \mathcal{S}(\mathbb{R})$ and $S \in \mathcal{S}'(\mathbb{R})$, then $S *_\alpha f \in \mathcal{E}(\mathbb{R})$ and $T_{|x|^{2\alpha+1}\, S*_\alpha f} \in \mathcal{S}'(\mathbb{R})$. Moreover

$$\mathcal{F}_\alpha(T_{|x|^{2\alpha+1}\, S*_\alpha f}) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(S).$$

Let $f \in \Phi_\alpha$ and $\lambda \in \mathbb{C}\backslash\{-(\alpha+\ell+1),\ \ell \in \mathbb{N}\}$. Consequently, from Definition 3, Proposition 7 (iv) and (9) we have

$$\mathcal{F}_\alpha(T_{|x|^{2\alpha+1}(-\Delta_\alpha)^\lambda f}) = |x|^{2\lambda+2\alpha+1}\mathcal{F}_\alpha(f) = T_{|x|^{2\lambda+2\alpha+1}\mathcal{F}_\alpha(f)}. \tag{11}$$

On the other hand from (3),

$$\mathcal{F}_\alpha(T_{|x|^{2\alpha+1}(-\Delta_\alpha)^\lambda f}) = T_{|x|^{2\alpha+1}\mathcal{F}_\alpha((-\Delta_\alpha)^\lambda f)}. \tag{12}$$

From (11) and (12), we obtain

$$\mathcal{F}_\alpha((-\Delta_\alpha)^\lambda f) = |x|^{2\lambda}\mathcal{F}_\alpha(f).$$

Then by Lemma 2 and (i) we deduce that $(-\Delta_\alpha)^\lambda f \in \Phi_\alpha$. ∎

5 Inversion formulas for $S_{\alpha,\beta}$ and ${}^tS_{\alpha,\beta}$

In this section, we establish inversion formulas for the Dunkl Sonine transform and its dual.

Definition 4. We define the operators K_1, K_2 and K_3, by

$$K_1(f) := \frac{c_\beta}{c_\alpha}\, \mathcal{F}_\alpha^{-1}\big(|\lambda|^{2(\beta-\alpha)}\, \mathcal{F}_\alpha(f)\big) = \frac{c_\beta}{c_\alpha}\, (-\Delta_\alpha)^{\beta-\alpha} f, \qquad f \in \Phi_\alpha,$$
$$K_2(f) := \frac{c_\beta}{c_\alpha}\, \mathcal{F}_\beta^{-1}\big(|\lambda|^{2(\beta-\alpha)}\, \mathcal{F}_\beta(f)\big) = \frac{c_\beta}{c_\alpha}\, (-\Delta_\beta)^{\beta-\alpha} f, \qquad f \in \Phi_\beta,$$
$$K_3(f) := \sqrt{\frac{c_\beta}{c_\alpha}}\, \mathcal{F}_\alpha^{-1}\big(|\lambda|^{\beta-\alpha}\, \mathcal{F}_\alpha(f)\big) = \sqrt{\frac{c_\beta}{c_\alpha}}\, (-\Delta_\alpha)^{(\beta-\alpha)/2} f, \qquad f \in \Phi_\alpha.$$

Lemma 3. *For all $g \in \Phi_\beta$, we have*

$$K_1({}^tS_{\alpha,\beta})(g) = ({}^tS_{\alpha,\beta})K_2(g). \tag{13}$$

Proof. Let $g \in \Phi_\beta$. Using Proposition 6 (ii),

$$K_1({}^tS_{\alpha,\beta})(g) = \frac{c_\beta}{c_\alpha}\, \mathcal{F}_\alpha^{-1}\big(|\lambda|^{2(\beta-\alpha)}\, \mathcal{F}_\beta(g)\big) = ({}^tS_{\alpha,\beta})K_2(g).$$ ∎

Theorem 3.

(*i*) *Inversion formulas: For all* $f \in \Phi_\alpha$ *and* $g \in \Phi_\beta$*, we have the inversions formulas:*

$$(a)\quad g = S_{\alpha,\beta}K_1({}^tS_{\alpha,\beta})(g), \qquad (b)\quad f = ({}^tS_{\alpha,\beta})K_2S_{\alpha,\beta}(f).$$

(*ii*) *Plancherel formula: For all* $f \in \Phi_\beta$ *we have*

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\beta+1}dx = \int_{\mathbb{R}} |K_3({}^tS_{\alpha,\beta}(f))(x)|^2|x|^{2\alpha+1}dx.$$

Proof. (*i*) Let $g \in \Phi_\beta$. From Proposition 3 (*iii*), (6) and Proposition 6 (*ii*), we obtain

$$\begin{aligned} g &= c_\beta \int_{\mathbb{R}} S_{\alpha,\beta}(E_\alpha(i\lambda.))\,\mathcal{F}_\beta(g)(\lambda)|\lambda|^{2\beta+1}d\lambda \\ &= c_\beta\, S_{\alpha,\beta}\left[\int_{\mathbb{R}} E_\alpha(i\lambda.)\,\mathcal{F}_\alpha \circ {}^tS_{\alpha,\beta}(g)(\lambda)|\lambda|^{2\beta+1}d\lambda\right] \\ &= \frac{c_\beta}{c_\alpha}\, S_{\alpha,\beta}\left[\mathcal{F}_\alpha^{-1}(|\lambda|^{2(\beta-\alpha)}\,\mathcal{F}_\alpha \circ {}^tS_{\alpha,\beta}(g))\right]. \end{aligned}$$

Thus

$$g = S_{\alpha,\beta}K_1({}^tS_{\alpha,\beta})(g), \qquad g \in \Phi_\beta.$$

From the previous relation and (13), we deduce the relation:

$$f = ({}^tS_{\alpha,\beta})K_2S_{\alpha,\beta}(f), \qquad f \in \Phi_\alpha.$$

(*ii*) Let $f \in \Phi_\beta$. From Proposition 3 (*iv*) and Proposition 6 (*ii*), we deduce that

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\beta+1}dx = c_\beta \int_{\mathbb{R}} \big||\lambda|^{\beta-\alpha}\mathcal{F}_\alpha({}^tS_{\alpha,\beta}(f))(\lambda)\big|^2|\lambda|^{2\alpha+1}d\lambda.$$

Thus we obtain

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\beta+1}dx = c_\alpha \int_{\mathbb{R}} \big|\mathcal{F}_\alpha\big(K_3({}^tS_{\alpha,\beta}(f))\big)(\lambda)\big|^2|\lambda|^{2\alpha+1}d\lambda.$$

Then the result follows from this identity by applying Proposition 3 (*iv*). ■

Remark 5. Let $f \in \Phi_\alpha$ and $g \in \Phi_\beta$. By writing (*a*) and (*b*) respectively for the functions $S_{\alpha,\beta}(f)$ and ${}^tS_{\alpha,\beta}(g)$, we obtain

$$(c)\quad f = K_1({}^tS_{\alpha,\beta})S_{\alpha,\beta}(f), \qquad (d)\quad g = K_2S_{\alpha,\beta}({}^tS_{\alpha,\beta})(g).$$

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