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Fermion on Curved Spaces, Symmetries, and Quantum Anomalies*

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Abstract. We review the geodesic motion of pseudo-classical spinning particles in curved spaces. Investigating the generalized Killing equations for spinning spaces, we express the constants of motion in terms of Killing–Yano tensors. Passing from the spinning spaces to the Dirac equation in curved backgrounds we point out the role of the Killing–Yano tensors in the construction of the Dirac-type operators. The general results are applied to the case of the four-dimensional Euclidean Taub–Newman–Unti–Tamburino space. The gravitational and axial anomalies are studied for generalized Euclidean Taub-NUT metrics which admit hidden symmetries analogous to the Runge–Lenz vector of the Kepler-type problem. Using the Atiyah–Patodi–Singer index theorem for manifolds with boundaries, it is shown that the these metrics make no contribution to the axial anomaly.

Key words: spinning particles; Dirac type operators; gravitational anomalies; axial anomalies

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1 Introduction

The aim of this paper is to investigate the quantum objects, namely spin one half particles, in curved spaces. Having in mind the lack of a satisfactory quantum theory for gravitational interaction, this study is justified and not at all trivial.

In order to study the geodesic motions and the conserved classical and quantum quantities for fermions on curved spaces, the symmetries of the backgrounds proved to be very important. We mention that the following two generalization of the Killing (K) vector equation have become of interest in physics:

1. A symmetric tensor field $K_{\mu_1\dots\mu_r}$ is called a Stäckel–Killing (S-K) tensor of valence r if and only if
$$K_{(\mu_1\dots\mu_r;\lambda)} = 0. \tag{1}$$
The usual Killing (K) vectors correspond to valence $r = 1$ while the hidden symmetries are encapsulated in S-K tensors of valence $r > 1$.

2. A tensor $f_{\mu_1\dots\mu_r}$ is called a Killing–Yano (K-Y) [1] tensor of valence r if it is totally antisymmetric and it satisfies the equation
$$f_{\mu_1\dots(\mu_r;\lambda)} = 0. \tag{2}$$

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These objects can be characterized in several equivalent ways. For example, K-Y tensors can be defined as differential forms on a manifold whose covariant derivative is totally antisymmetric.

There are many important occurrences of K-Y tensors in physics. For example, investigating the geodesic equations on curved spaces, the K-Y tensors play an important role in the existence of the constants of motion [2]. K-Y tensors do also appear in the study of the Klein–Gordon and Dirac equations. There is a natural and profound connexion between K-Y tensors, Dirac-type operators and supersymmetries [3, 4, 5, 6] having in mind their anticommuting property.

The models of relativistic particles with spin have been proposed for a long time and the literature on the particle with spin grew vast [7, 4, 8]. The models involving only conventional coordinates are called classical models while the models involving anticommuting coordinates are generally called pseudo-classical.

In the beginning of this paper we discuss the relativistic spin one half particle models involving anticommuting vectorial degrees of freedom which are usually called the spinning particles. Spinning particles are in some sense the classical limit of the Dirac particles. The action of spin one half relativistic particle with spinning degrees of freedom described by Grassmannian (odd) variables was first proposed by Berezin and Marinov [7].

The generalized Killing equations for the configuration space of spinning particles (spinning space) are analyzed and the solutions are expressed in terms of K-Y tensors. We mention that the existence of a K-Y tensor is both a necessary and a sufficient condition for the existence of a new supersymmetry for the spinning space [3, 9].

Passing from the pseudo-classical approach to the Dirac equation in curved spaces, we point out the role of the K-Y tensors in the construction of non-standard Dirac-type operators. The Dirac-type operators constructed with the aid of covariantly constant K-Y tensors are equivalent with the standard Dirac operator. The non-covariantly constant K-Y tensors generates non-standard Dirac operators which are not equivalent to the standard Dirac operator and they are associated with the hidden symmetries of the space.

The general results are applied to the case of the four-dimensional Euclidean Taub–Newman–Unti–Tamburino (Taub-NUT) space. The Taub-NUT metrics were found by Taub [10] and extended by Newman–Unti–Tamburino [11]. The Euclidean Taub-NUT metric has lately attracted much attention in physics. Hawking [12] has suggested that the Euclidean Taub-NUT metric might give rise to the gravitational analog of the Yang–Mills instanton. This metric is the space part of the line element of the celebrated Kaluza–Klein monopole of Gross and Perry [13] and Sorkin [14]. On the other hand, in the long distance limit, neglecting radiation, the relative motion of two monopoles is described by the geodesics of this space [15]. The Taub-NUT family of metrics is also involved in many other modern studies in physics like strings, membranes, etc.

The family of Taub-NUT metrics with their plentiful symmetries provides an excellent background to investigate the classical and quantum conserved quantities on curved spaces. In the Taub-NUT geometry there are four K-Y tensors. Three of these are complex structures realizing the quaternion algebra and the Taub-NUT manifold is hyper-Kähler [16]. In addition to these three vector-like K-Y tensors, there is a scalar one which has a non-vanishing field strength and which exists by virtue of the metric being type D. From the symmetry viewpoint, the geodesic motion in Taub-NUT space admits a "hidden" symmetry of the Kepler type.

The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, D_s [17]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank S-K tensors that generate conserved quantities in classical mechanics or conserved operators which commute with D_s. The construction of Carter and McLenaghan depended upon the remarkable fact that the (symmetric) S-K tensor $K_{\mu\nu}$ involved in the constant of motion quadratic in the four-momentum p_μ

$$Z = \frac{1}{2}K^{\mu\nu}p_\mu p_\nu \tag{3}$$

has a certain square root in terms of K-Y tensors $f_{\mu\nu}$:

$$K_{\mu\nu} = f_{\mu\lambda}f^{\lambda}_{\ \nu}\,. \tag{4}$$

These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry especially when the S-K tensor $K_{\mu\nu}$ in equation (3) is proportional with the metric tensor $g_{\mu\nu}$ and the corresponding K-Y tensors in equation (4) are covariantly constant. Then each tensor of this type, f^i, gives rise to a Dirac-type operator, D^i, representing a supercharge of the superalgebra $\{D^i, D^j\} \propto D_s^2\delta_{ij}$.

The necessary condition that a S-K tensor of valence two be written as the square of a K-Y tensor is that it has at the most two distinct eigenvalues [18]. In the case of the generalized Taub-NUT spaces [19, 20, 21] the S-K tensors involved in the Runge-Lenz vector cannot be expressed as a product of K-Y tensors. The non-existence of the K-Y tensors on generalized Taub-NUT metrics leads to gravitational quantum anomalies proportional to a contraction of the S-K tensor with the Ricci tensor [22].

The index of the Dirac operator is a useful tool to investigate the topological properties of the manifold as well as in computing axial quantum anomalies in field theories. In even-dimensional spaces one can define the index of a Dirac operator as the difference between the number of linearly independent zero modes with eigenvalues $+1$ and -1 under γ_5. A remarkable result states the equality of the indices of the standard and non-standard Dirac operators [23].

For the generalized Taub-NUT spaces, in [22] we computed the axial quantum anomaly, interpreted as the index of the Dirac operator of these metrics, on annular domains and on disks, with the non-local Atiyah–Patodi–Singer boundary condition. We also examined the Dirac operator on the complete Euclidean space with respect to these metrics, acting in the Hilbert space of square-integrable spinors. We found that this operator is not Fredholm, hence even the existence of a finite index is not granted.

The structure of the paper is as follows: We start the next section with a description of the pseudo-classical model for the spinning particles. In Section 3 we present the spinning Taub-NUT space. In the next section we consider the Dirac equation on a curved background with symmetries. In Section 5 the general results are applied to the case of the Euclidean Taub-NUT space. In the next two sections we discuss the gravitational and axial anomalies for generalized Euclidean Taub-NUT metrics. Section 8 is the conclusion. In two appendices we review the symmetries and conserved quantities for geodesic motions in the Euclidean Taub-NUT space and its generalizations.

2 Pseudo-classical approach

Spinning space is an extension of an ordinary Riemannian manifold, parametrized by local coordinates $\{x^\mu\}$, to a graded manifold parametrized by local coordinates $\{x^\mu, \psi^\mu\}$, with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anticommuting) [7].

The dynamics of spinning point-particles in a curved space-time is described by the one-dimensional σ-model with the action:

$$S = \int_a^b d\tau \left(\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{D\tau}\right).$$

In what follows we shall investigate the conserved quantities for geodesic motions in the case of spinning manifolds. For this purpose we consider the world-line Hamiltonian given by

$$H = \tfrac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu,$$

where

$$\Pi_\mu = g_{\mu\nu} \dot{x}^\nu$$

is the covariant momentum.

For any constant of motion $\mathcal{J}(x, \Pi, \psi)$, the bracket with H vanishes

$$\{H, \mathcal{J}\} = 0,$$

where the Poisson–Dirac brackets for functions of the covariant phase space variables (x, Π, ψ) is defined by

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} \mathcal{D}_\mu G - \mathcal{R}_{\mu\nu} \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F} \frac{\partial F}{\partial \psi^\mu} \frac{\partial G}{\partial \psi_\mu}.$$

The notations used are

$$\mathcal{D}_\mu F = \partial_\mu F + \Gamma^\lambda_{\mu\nu} \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} - \Gamma^\lambda_{\mu\nu} \psi^\nu \frac{\partial F}{\partial \psi^\lambda}, \qquad \mathcal{R}_{\mu\nu} = \tfrac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\mu\nu}$$

and a_F is the Grassmann parity of F: $a_F = (0, 1)$ for $F = (\text{even,odd})$.

If we expand $\mathcal{J}(x, \Pi, \psi)$ in a power series in the covariant momentum

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)\mu_1 \ldots \mu_n}(x, \psi) \Pi_{\mu_1} \cdots \Pi_{\mu_n}$$

then the bracket $\{H, \mathcal{J}\}$ vanishes for arbitrary Π_μ if and only if the components of $\mathcal{J}$ satisfy the generalized Killing equations [3]:

$$\mathcal{J}^{(n)}_{(\mu_1 \ldots \mu_n ; \mu_{n+1})} + \frac{\partial \mathcal{J}^{(n)}_{(\mu_1 \ldots \mu_n}}{\partial \psi^\sigma} \Gamma^\sigma_{\;\mu_{n+1})\lambda} \psi^\lambda = \tfrac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\nu(\mu_{n+1}} \mathcal{J}^{(n+1)\nu}{}_{\mu_1 \ldots \mu_n)}, \tag{5}$$

where the parentheses denote symmetrization with norm one over the indices enclosed.

Explicit solutions can be constructed starting with the geometrical symmetries of the space. For each K vector R_μ, i.e. $R_{(\mu;\nu)} = 0$ there is a conserved quantity in the spinning case:

$$\mathcal{J} = \tfrac{i}{2} R_{[\mu;\nu]} \psi^\mu \psi^\nu + R_\mu \dot{x}^\mu. \tag{6}$$

A more involved example is given by a S-K tensor $K_{\mu\nu}$ satisfying equation (1). Let us assume that a Stäckel–Killing tensor $K_{\mu\nu}$ can be written as a symmetrized product of two (different) K-Y tensors $(i \neq j)$ of valence 2:

$$K^{\mu\nu}_{ij} = \tfrac{1}{2}(f^{\mu}_{i\;\lambda} f^{\nu\lambda}_j + f^{\nu}_{i\;\lambda} f^{\mu\lambda}_j).$$

The conserved quantity for the spinning space is [3]

$$\mathcal{J}_{ij} = \tfrac{1}{2!} K^{\mu\nu}_{ij} \dot{x}_\mu \dot{x}_\nu + \mathcal{J}^{(1)\mu}_{ij} \dot{x}_\mu + \mathcal{J}^{(0)}_{ij}, \tag{7}$$

where

$$\mathcal{J}^{(0)}_{ij} = -\tfrac{1}{4} \psi^\lambda \psi^\sigma \psi^\rho \psi^\tau \left(R_{\mu\nu\lambda\sigma} f^{\mu}_{i\;\rho} f^{\nu}_{j\;\tau} + \tfrac{1}{2} c_{i\lambda\sigma}{}^{\pi} c_{j\rho\tau\pi} \right),$$

$$\mathcal{J}_{ij}^{(1)\mu} = \frac{i}{2}\psi^\lambda\psi^\sigma \left(f_{i\ \sigma}^{\nu} D_\nu f_{j\ \lambda}^{\mu} + f_{j\ \sigma}^{\nu} D_\nu f_{i\ \lambda}^{\mu} + \frac{1}{2} f_i^{\mu\rho} c_{j\lambda\sigma\rho} + \frac{1}{2} f_j^{\mu\rho} c_{i\lambda\sigma\rho} \right)$$

with $c_{i\mu\nu\lambda} = -2 f_{i[\nu\lambda;\mu]}$.

In what follows we shall return to the equation (5) looking for solutions depending exclusively on the Grassmann variables $\{\psi^\mu\}$. The existence of such kind of solutions of the Killing equation is one of the specific features of the spinning particle models.

The most remarkable class of solutions is represented by:

$$Q_f = f_{\mu_1\dots\mu_r}\Pi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_r} + \frac{i}{r+1}(-1)^{r+1} f_{[\mu_1\dots\mu_r;\mu_{r+1}]}\cdot\psi^{\mu_1}\cdots\psi^{\mu_{r+1}}. \tag{8}$$

This quantity is a superinvariant

$$\{Q_f, Q_0\} = 0,$$

where Q_0 is the supercharge

$$Q_0 = \Pi_\mu\,\psi^\mu. \tag{9}$$

Equations (6) and (7) describing the conserved quantities in the spinning space contains specific spin terms involving even numbers of Grassmann variables.

3 Spinning Taub-NUT space

In the Taub-NUT case, the pseudo-classical approach sets the spin contributions to the angular momentum, “relative electric charge” (20) and Runge–Lenz vector (23) [24, 25, 26, 27].

We start with the observation that the angular momentum and the “relative electric charge” (20) are constructed with the aid of the K vectors (18). The corresponding conserved quantities in the spinning case are the followings:

$$\vec{J} = \vec{B} + \vec{j}, \qquad J_4 = B_4 + q$$

where we introduced the notation: $\vec{J} = (J_1, J_2, J_3)$, $\vec{B} = (B_1, B_2, B_3)$ and the spin corrections are represented by the scalars B_A

$$B_A = \frac{i}{2} R_{A[\mu;\nu]}\psi^\mu\psi^\nu.$$

Using equation (8) we can construct from the K-Y tensors (21) and (22) the supercharges Q_i and Q_Y. The supercharges Q_i together with Q_0 from equation (9) realize the $N = 4$ supersymmetry algebra:

$$\{Q_A, Q_B\} = -2i\delta_{AB}H, \qquad A, B = 0, \dots, 3$$

making manifest the link between the existence of the K-Y tensors (21) and the hyper-Kähler geometry of the Taub-NUT manifold. Moreover, the supercharges Q_i transform as vectors at spatial rotations

$$\{Q_i, J_j\} = \epsilon_{ijk} Q_k, \qquad i, j, k = 1, 2, 3$$

while Q_Y and Q_0 behave as scalars.

To get the spin correction to the Runge–Lenz vector (23) it is necessary to investigate the generalized Killing equations (5) for $n = 1$ with the S-K tensor $\vec{K}_{\mu\nu}$ in the right hand side. For an analytic expression of the solution of this equation we shall use the decomposition (24) of the S-K tensor $\vec{K}_{\mu\nu}$ in terms of K-Y tensors. Starting with this decomposition of the Runge–Lenz vector $\vec{K}$ from the scalar case, it is possible to express the corresponding conserved quantity $\vec{\mathcal{K}}$ in the spinning case [24]:

$$\mathcal{K}_i = 2m\left(-i\{Q_Y, Q_i\} + \frac{1}{8m^2} J_i J_4 \right).$$

4 Dirac equation on a curved background

In what follows we shall consider the Dirac operator on a curved background which has the form

$$D_s = \gamma^\mu \hat{\nabla}_\mu . \tag{10}$$

In this expression the Dirac matrices γ_μ are defined in local coordinates by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$$

and $\hat{\nabla}_\mu$ denotes the canonical covariant derivative for spinors. The essential properties of this covariant derivative are summarized in the following equations

$$\hat{\nabla}_\mu \gamma^\mu = 0, \qquad \hat{\nabla}_{[\rho}\hat{\nabla}_{\mu]} = \tfrac{1}{4} R_{\alpha\beta\rho\mu}\gamma^\alpha\gamma^\beta ,$$

where $R_{\alpha\beta\rho\mu}$ denotes the components of the Riemann curvature tensor.

Carter and McLenaghan showed that in the theory of Dirac fermions for any isometry with K vector R_μ there is an appropriate operator [17]:

$$X_k = -i\left(R^\mu \hat{\nabla}_\mu - \tfrac{1}{4}\gamma^\mu\gamma^\nu R_{\mu;\nu}\right)$$

which commutes with the *standard* Dirac operator (10).

Moreover each K-Y tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form

$$D_f = -i\gamma^\mu\left(f_\mu{}^\nu \hat{\nabla}_\nu - \tfrac{1}{6}\gamma^\nu\gamma^\rho f_{\mu\nu;\rho}\right) \tag{11}$$

which anticommutes with the standard Dirac operator D_s.

5 Dirac equation in the Taub-NUT space

When one uses Cartesian charts in the Taub-NUT geometry it is useful to consider the local frames given by tetrad fields $\hat{e}(x)$, such that $g_{\mu\nu} = \delta_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}_\mu \hat{e}^{\hat{\beta}}_\nu$. The four Dirac matrices $\hat{\gamma}^{\hat{\alpha}}$ that satisfy $\{\hat{\gamma}^{\hat{\alpha}}, \hat{\gamma}^{\hat{\beta}}\} = 2\delta^{\hat{\alpha}\hat{\beta}}$, can be taken as

$$\hat{\gamma}^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \hat{\gamma}^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \qquad \hat{\gamma}^5 = \hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}.$$

In the Taub-NUT space the standard Dirac operator is

$$D_s = \hat{\gamma}^{\hat{\alpha}}\hat{\nabla}_{\hat{\alpha}} = i\sqrt{V}\vec{\hat{\gamma}}\cdot\vec{P} + \frac{i}{\sqrt{V}}\hat{\gamma}^4 P_4 + \frac{i}{2}V\sqrt{V}\hat{\gamma}^4\vec{\Sigma}^*\cdot\vec{B},$$

where $\hat{\nabla}_{\hat{\alpha}}$ are the components of the spin covariant derivatives with local indices

$$\hat{\nabla}_i = i\sqrt{V}P_i + \frac{i}{2}V\sqrt{V}\varepsilon_{ijk}\Sigma^*_j B_k, \qquad \hat{\nabla}_4 = \frac{i}{\sqrt{V}}P_4 - \frac{i}{2}V\sqrt{V}\vec{\Sigma}^*\cdot\vec{B}.$$

These depend on the momentum operators $P_i = -i(\partial_i - A_i\partial_4)$, $P_4 = -i\partial_4$ which obey the commutation rules $[P_i, P_j] = i\varepsilon_{ijk}B_k P_4$ and $[P_i, P_4] = 0$. The spin matrices giving the spin connection are:

$$\Sigma^*_i = S_i + \tfrac{i}{2}\hat{\gamma}^4\hat{\gamma}^i, \qquad S_i = \tfrac{1}{2}\varepsilon_{ijk}S^{jk},$$

where $S^{\hat{\alpha}\hat{\beta}} = -i[\hat{\gamma}^{\hat{\alpha}}, \hat{\gamma}^{\hat{\beta}}]/4$.

In the above representation of the Dirac matrices, the Hamiltonian operator of the *massless* Dirac field reads [28, 29]:

$$H = \hat{\gamma}^5 D_s = \left(\begin{array}{cc} 0 & V\pi^* \dfrac{1}{\sqrt{V}} \\ \sqrt{V}\pi & 0 \end{array} \right).$$

This is expressed in terms of the operators

$$\pi = \sigma_P - \frac{iP_4}{V}, \qquad \pi^* = \sigma_P + \frac{iP_4}{V}, \qquad \sigma_P = \vec{\sigma}\cdot\vec{P}$$

and the Klein–Gordon operator has the form:

$$\Delta = -\nabla_\mu g^{\mu\nu}\nabla_\nu = V\,\pi^*\pi = V\vec{P}^{\,2} + \frac{1}{V}{P_4}^2.$$

The conserved observables can be found among the operators which commute or anticommute with D_s and $\hat{\gamma}^5$ [28, 30]. That is the case of the total angular momentum $\vec{J} = \vec{L} + \vec{S}$, where the orbital angular momentum is

$$\vec{L} \,=\, \vec{x}\times\vec{P} - 4m\frac{\vec{x}}{r}P_4.$$

Dirac-type operators are constructed from the K-Y tensors f_i $(i = 1, 2, 3)$ and f_Y using equation (11). In the quantum Dirac theory these operators replace the supercharges (8) from the pseudo-classical approach. Moreover we can give a *physical* interpretation of the covariantly constant K-Y tensors defining the *spin-like* operators,

$$\Sigma_i = -\tfrac{i}{4} f^i_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} = \left(\begin{array}{cc} \sigma_i & 0 \\ 0 & 0 \end{array} \right), \tag{12}$$

that have similar properties to those of the Pauli matrices. In the pseudo-classical description of a Dirac particle [3, 31], the covariantly constant K-Y tensors correspond to components of the spin which are separately conserved.

Here, since the Pauli matrices commute with the Klein–Gordon operator, the spin-like operators (12) commute with H^2. Remarkable the existence of the K-Y tensors allows one to construct *Dirac-type* operators [17]

$$Q_i = -i f^i_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\hat{\nabla}^{\hat{\beta}} = \{H,\, \Sigma_i\}$$

which anticommute with D_s and γ^5 and commute with H [29]. and obey the $N = 4$ superalgebra, including $Q_0 = iD_s = i\hat{\gamma}^5 H$:

$$\{Q_A,\, Q_B\} = 2\delta_{AB}H^2, \qquad A, B, \ldots = 0, 1, 2, 3$$

linked to the hyper-Kähler geometry of the Taub-NUT space.

Finally, using equation (11), from the fourth K-Y tensor f_Y of the Taub-NUT space we can construct the Dirac-type operator [28, 32]

$$Q_Y = \frac{r}{4m}\left\{H, \left(\begin{array}{cc} \sigma_r & 0 \\ 0 & -\sigma_r V^{-1} \end{array} \right)\right\} = i\frac{r}{4m}\left[Q_0, \left(\begin{array}{cc} \sigma_r & 0 \\ 0 & \sigma_r V^{-1} \end{array} \right)\right].$$

Let us observe that the Dirac-type operators Q_A $(A = 0, 1, 2, 3)$ are characterized by the fact that their quantal anticommutator close on the square of the Hamiltonian of the theory. No such expectation applies to the non-standard, hidden Dirac operators Q^Y which close on a combination of different conserved operators. The Taub-NUT space has a special geometry where the covariantly constant K-Y tensors exist by virtue of the metric being self-dual and the Dirac-type operators generated by them are equivalent with the standard one. The fourth K-Y tensor f^Y which is not covariantly constant exists by virtue of the metric being of type D. The corresponding non-standard or hidden Dirac operator does not close on H. It is associated with the hidden symmetries of the space allowing the construction of the conserved vector-operator analogous to the Runge–Lenz vector of the Kepler problem.

The hidden symmetries of the Taub-NUT geometry are encapsulated in the non-trivial S-K tensors $K_{i\mu\nu}$, $(i = 1, 2, 3)$. For the Dirac theory the construction of the Runge–Lenz operator can be done using products among the Dirac-type operators Q_Y and Q_i.

Let us define the operator [28]:

$$N_i = m\,\{Q_Y, Q_i\} - J_i P_4.$$

The components of the operator $\vec{N}$ commutes with H and satisfy the following commutation relations

$$[N_i, P_4] = 0, \qquad [N_i, J_j] = i\varepsilon_{ijk} N_k, \qquad [N_i, Q_0] = 0,$$
$$[N_i, Q_j] = i\varepsilon_{ijk} Q_k P_4, \qquad [N_i, N_j] = i\varepsilon_{ijk} J_k F^2 + \tfrac{i}{2}\varepsilon_{ijk} Q_i H$$

where $F^2 = {P_4}^2 - H^2$. In order to put the last commutator in a form close to that from the scalar case [16, 38], we can redefine the components of the Runge–Lenz operator, $\vec{\mathcal{K}}$, as follows:

$$\mathcal{K}_i = N_i + \tfrac{1}{2} H^{-1}(F - P_4) Q_i$$

having the desired commutation relation [28]:

$$[\mathcal{K}_i, \mathcal{K}_j] = i\varepsilon_{ijk} J_k F^2.$$

6 Gravitational anomalies

For the classical motions, a S-K tensor $K_{\mu\nu}$ generate a quadratic constant of motion as in equation (3). In the case of the geodesic motion of classical scalar particles, the fact that $K_{\mu\nu}$ is a S-K tensor satisfying (1), assures the conservation of (3).

If we go from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analog of the quadratic function (3) is [2]:

$$\mathcal{K} = D_\mu K^{\mu\nu} D_\nu, \tag{13}$$

where D_μ is the covariant differential operator on the manifold with the metric $g_{\mu\nu}$. Working out the commutator of (13) with the scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu$$

we get from an explicit calculation gives [33]

$$\begin{aligned}[D_\mu D^\mu, \mathcal{K}] &= 2K^{(\mu\nu;\lambda)} D_\mu D_\nu D_\lambda + 3{K^{(\mu\nu;\lambda)}}_{;\lambda} D_\mu D_\nu \\ &\quad + \left\{ -\tfrac{4}{3} {K_\lambda}^{[\mu} R^{\nu]\lambda} + \tfrac{1}{2} g_{\lambda\sigma}\left(K^{(\lambda\sigma;\mu);\nu} - K^{(\lambda\sigma;\nu);\mu}\right) + {K^{(\lambda\mu;\nu)}}_{;\lambda} \right\}_{;\nu} D_\mu .\end{aligned}$$

Note the very last terms are missing in the corresponding equation in [2].

Concerning the hidden symmetry of the quantized system, the above commutator does not vanishes on the strength of (1). If we take K to be a S-K tensor we are left with:

$$[\mathcal{H},\mathcal{K}] = -\tfrac{4}{3}\{K_\lambda{}^{[\mu}R^{\nu]\lambda}\}_{;\nu}D_\mu \tag{14}$$

which means that in general the quantum operator $\mathcal{K}$ does not define a genuine quantum mechanical symmetry [5]. On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S-K tensor $K_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

It is obvious that for a Ricci-flat manifold this quantum anomaly is absent. However, a more interesting situation is represented by the manifolds in which the S-K tensor $K_{\mu\nu}$ can be written as a product of K-Y tensors $f_{\mu\nu}$ [17].

The integrability condition for any solution of (2), written for K-Y tensors of valence $r=2$, is

$$R_{\mu\nu[\sigma}{}^{\tau}f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^{\tau}f_{\nu]\tau} = 0.$$

Now contracting this integrability condition on the Riemann tensor for any solution of (2) we get

$$f^\rho{}_{(\mu}R_{\nu)\rho} = 0. \tag{15}$$

Let us suppose that there exist a *square* of the S-K tensor $K_{\mu\nu}$ of the form of a K-Y tensor $f_{\mu\nu}$ as in equation (4). In case this should happen, the S-K equation (1) is automatically satisfied and the integrability condition (15) becomes

$$K^\rho{}_{[\mu}R_{\nu]\rho} = 0.$$

It is interesting to observe that in this latter equation an antisymmetrization rather than symmetrization is involved this time as compared to (15). But this relation implies the vanishing of the commutator (14) for S-K tensors which admit a decomposition in terms of K-Y tensors.

Using the S-K tensor components of the Runge–Lenz vector (27) we can proceed to the evaluation of the quantum gravitational anomaly for the generalized Taub-NUT metrics [19, 20, 21]. A direct evaluation [22] shows that the commutator (14) does not vanish.

To serve as a model for the evaluation of the commutator (14) involving the components of the S-K tensors corresponding to the Runge–Lenz vector (27), we limit ourselves to give only the components of the third S-K $k_3^{\mu\nu}$ tensor in spherical coordinates. Its non vanishing components are:

$$\begin{aligned}
&k_3^{rr} = -\frac{ar\cos\theta}{2(a+br)}, \qquad k_3^{r\theta} = k_3^{\theta r} = \frac{\sin\theta}{2}, \qquad k_3^{\theta\theta} = \frac{(a+2br)\cos\theta}{2r(a+br)},\\
&k_3^{\varphi\varphi} = \frac{(a+2br)\cot\theta\csc\theta}{2r(a+br)}, \qquad k_3^{\varphi\chi} = k_3^{\chi\varphi} = -\frac{(2a+3br+br\cos(2\theta)\csc^2\theta}{4r(a+br)},\\
&k_3^{\chi\chi} = \frac{(a-adr^2+br(2+cr)+(a+2br))\cot^2\theta)\cos\theta}{2r(a+br)}.
\end{aligned}$$

Again, just to exemplify, we write down from the commutator (14) only the function which multiplies the covariant derivative D_r:

$$\begin{aligned}
&\frac{3r\cos\theta}{4(a+br)^3(1+cr+dr^2)^2}\{-2bd(2ad-bc)r^3 + [3bd(2b-ac)-(ad+bc)(2ad-bc)]r^2\\
&\qquad + 2(ad+bc)(2b-ac)r + a(2ad-bc) + (b+ac)(2b-ac)\}\,.
\end{aligned} \tag{16}$$

Recall that the commutator (14) vanishes for the standard Euclidean Taub-NUT metric. It is easy to see that the above expression (16) vanishes for all r if and only if the constants a, b, c, d are constrained by (26).

In conclusion the operators constructed from symmetric S-K tensors are in general a source of gravitational anomalies for scalar fields. However, when the S-K tensor is of the form (4), then the anomaly disappears owing to the existence of the K-Y tensors.

7 Index formulas and axial anomalies

Atiyah, Patodi and Singer [34] discovered an index formula for first-order differential operators on manifolds with boundary with a non-local boundary condition. Their index formula contains two terms, none of which is necessarily an integer, namely a bulk term (the integral of a density in the interior of the manifold) and a boundary term defined in terms of the spectrum of the boundary Dirac operator. Endless trouble is caused in this theory by the condition that the metric and the operator be of "product type" near the boundary.

In [22] we computed the index of the Dirac operator on annular domains and on disk, with the non-local APS boundary condition. For the generalized Taub-NUT metrics [19, 20, 21], we found that the index is a number-theoretic quantity which depends on the metrics. In particular, our formula shows that the index vanishes on balls of sufficient large radius, but can be non-zero for some values of the parameters c, d and of the radius.

Theorem 1. *If $c > -\frac{\sqrt{15d}}{2}$ then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof. The proof of this statement can be found in [22]. ■

The result is natural since the index of an operator is unchanged under continuous deformations of that operator. In our case this would amount to a continuous change in the metric. The absence of axial anomalies is due to the fact there exists an underlying structure that does not depend on the metric. However for larger deformations of the metric there could appear discontinuities in the boundary conditions and therefore the index could present jumps. Our formula for the index involves a computable number-theoretic quantity depending on the parameters of the metric.

We also examined the Dirac operator on the complete Euclidean space with respect to this metric, acting in the Hilbert space of square-integrable spinors. We found that this operator is not Fredholm, hence even the existence of a finite index is not granted.

We mentioned in [22] some open problems in connection with unbounded domains. The paper [35] brings new results in this direction. First we showed that the Dirac operator on $\mathbb{R}^4$ with respect to the standard Taub-NUT metric does not have L^2 harmonic spinors. This follows rather easily from the Lichnerowicz formula, since the standard Taub-NUT metric has vanishing scalar curvature. In particular, the index vanishes.

Theorem 2. *For the standard Taub-NUT metric on $\mathbb{R}^4$ the Dirac operator does not have L^2 solutions.*

Proof. Recall that the standard Taub-NUT metric is hyper-Kähler, hence its scalar curvature κ vanishes.

By the Lichnerowicz formula,

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4} = \nabla^*\nabla.$$

Let $\phi \in L^2$ be a solution of D in the sense of distributions. Then, again in distributions, $\nabla^*\nabla\phi = 0$. The operator $\nabla^*\nabla$ is essentially self-adjoint with domain $\mathcal{C}_c^\infty(\mathbb{R}^4, \Sigma_4)$, which implies that its kernel equals the kernel of ∇. Hence $\nabla\phi = 0$. Now a parallel spinor has constant pointwise norm, hence it cannot be in L^2 unless it is 0, because the volume of the metric ${ds_K}^2$ is infinite. Therefore $\phi = 0$. ■

8 Concluding remarks

In the pseudo-classical spinning particle models in curved spaces from covariantly constant K-Y tensors $f_{\mu\nu}$ can be constructed conserved quantities of the type $f_{\mu\nu}\psi^\mu\psi^\nu$ depending on the Grassmann variables $\{\psi^\mu\}$ [24]. The Grassmann variables $\{\psi^\mu\}$ transform as a tangent space vector and describe the spin of the particle. The antisymmetric tensor $S^{\mu\nu} = -i\psi^\mu\psi^\nu$ generates the internal part of the local tangent space rotations. For example, in the spinning Euclidean Taub-NUT space such operators correspond to components of the spin which are separately conserved [31].

The construction of the new supersymmetries in the context of pseudo-classical mechanics can be carried over straightforwardly to the case of quantum mechanics by the usual replacement of phase space coordinates by operators and Poisson–Dirac brackets by anticommutators [7]. In terms of these operators the supercharges are replaced by Dirac-type operators [36]. In both cases, the correspondence principle leads to equivalent algebraic structures making obvious the relations between these approaches [31].

In the study of the Dirac equation in curved spaces, it has been proved that the K-Y tensors play an essential role in the construction of new Dirac-type operators. The Dirac-type operators constructed with the aid of covariantly constant K-Y tensors are equivalent with the standard Dirac operator [37]. The non-covariantly constant K-Y tensors generates non-standard Dirac operators which are not equivalent to the standard Dirac operator and they are associated with the hidden symmetries of the space.

The Taub-NUT space has a special geometry where the covariantly constant K-Y tensors exist by virtue of the metric being self-dual and the Dirac-type operators generated by them are equivalent with the standard one. The fourth K-Y tensor f^Y which is not covariantly constant exists by virtue of the metric being of type D. The corresponding non-standard or hidden Dirac operator does not close on H and is not equivalent to the Dirac-type operators. As it was mentioned, it is associated with the hidden symmetries of the space allowing the construction of the conserved vector-operator analogous to the Runge–Lenz vector of the Kepler problem.

There is a relationship between the absence of anomalies and the existence of the K-Y tensors. For scalar fields, the decomposition (4) of S-K tensors in terms of K-Y tensors guarantees the absence of gravitational anomalies. Otherwise operators constructed from symmetric tensors are in general a source of anomalies proportional to the Ricci tensors.

However for the axial anomaly the role of K-Y tensors is not so obvious. The topological aspects are more important and the absence of K-Y tensors does not imply the appearance of anomalies.

A Euclidean Taub-NUT space

Let us consider the Taub-NUT space and the chart with Cartesian coordinates x^μ $(\mu, \nu = 1, 2, 3, 4)$ having the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(r)(d\vec{x})^2 + \frac{g(r)}{16m^2}(dx^4 + A_i dx^i)^2, \tag{17}$$

where $\vec{x}$ denotes the three-vector $\vec{x} = (r, \theta, \varphi)$, $(d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ and $\vec{A}$ is the gauge field of a monopole

$$\mathrm{div}\vec{A} = 0, \qquad \vec{B} = \mathrm{rot}\,\vec{A} = 4m\frac{\vec{x}}{r^3}.$$

The real number m is the parameter of the theory which enter in the form of the functions

$$f(r) = g^{-1}(r) = V^{-1}(r) = \frac{4m + r}{r}$$

and the so called NUT singularity is absent if x^4 is periodic with period $16\pi m$. Sometimes it is convenient to make the coordinate transformation $4m(\chi + \varphi) = -x^4$ with $0 \le \chi < 4\pi$.

In the Taub-NUT geometry there are four K vectors [16, 38]

$$D_A = R^\mu_A\,\partial_\mu, \qquad A = 1, 2, 3, 4, \tag{18}$$

where

$$\begin{aligned}
D_1 &= -\sin\varphi\,\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\,\frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\chi},\\
D_2 &= \cos\varphi\,\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\,\frac{\partial}{\partial\varphi} + \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\chi},\\
D_3 &= \frac{\partial}{\partial\varphi}, \qquad D_4 = \frac{\partial}{\partial\chi}.
\end{aligned}\tag{19}$$

D_4 which generates the $U(1)$ of χ translations, commutes with the other K vectors. In turn the remaining three vectors, corresponding to the invariance of the metric (17) under spatial rotations ($A = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D_1, D_2] = -D_3, \qquad \textit{etc.}$$

In the bosonic case these invariances would correspond to the conservation of angular momentum and the so called "relative electric charge":

$$\vec{j} = \vec{r}\times\vec{p} + q\frac{\vec{r}}{r}, \qquad q = g(r)(\dot{\theta} + \cos\theta\dot{\varphi}), \tag{20}$$

where $\vec{p} = V^{-1}\dot{\vec{r}}$ is the mechanical momentum.

On the other hand, four K-Y tensors of valence 2 are known to exist in the Taub-NUT geometry. The first three are covariantly constant

$$\begin{aligned}
f_i &= 8m(d\chi + \cos\theta d\varphi)\wedge dx_i - \epsilon_{ijk}\left(1 + \frac{4m}{r}\right)dx_j \wedge dx_k,\\
D_\mu f^\nu_{i\lambda} &= 0, \qquad i, j, k = 1, 2, 3.
\end{aligned}\tag{21}$$

The f^i define three anticommuting complex structures of the Taub-NUT manifold, their components realizing the quaternion algebra

$$f^i f^j + f^j f^i = -2\delta_{ij}, \qquad f^i f^j - f^j f^i = -2\varepsilon_{ijk} f^k.$$

The existence of these K-Y tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the $N = 4$ supersymmetric extension of the theory [3, 31].

The fourth K-Y tensor is

$$f_Y = 8m(d\chi + \cos\theta d\varphi) \wedge dr + 4r(r+2m)\left(1 + \frac{r}{4m}\right)\sin\theta d\theta \wedge d\varphi \tag{22}$$

having a non-vanishing covariant derivative

$$f_{Y\,r\theta;\varphi} = 2\left(1 + \frac{r}{4m}\right) r\sin\theta.$$

In Taub-NUT space there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem [16, 39, 40]

$$\vec{K} = \tfrac{1}{2}\vec{K}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \vec{p}\times\vec{j} + \left(\frac{q^2}{4m} - 4mE\right)\frac{\vec{r}}{r}, \tag{23}$$

where the conserved energy is

$$E = \tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu.$$

The components $K_{i\mu\nu}$ involved with the Runge–Lenz vector (23) are Stäckel–Killing tensors satisfying the equations

$$K_{i(\mu\nu;\lambda)} = 0, \qquad K_{i\mu\nu} = K_{i\nu\mu}$$

and they can be expressed as symmetrized products of the K-Y tensors f_i, f_Y and K vectors R_A [24]

$$K_{i\mu\nu} - \frac{1}{8m}(R_{4\mu}R_{i\nu} + R_{4\nu}R_{i\mu}) = m\big(f_{Y\mu\lambda}{f_i}^\lambda{}_\nu + f_{Y\nu\lambda}{f_i}^\lambda{}_\mu\big). \tag{24}$$

B Generalized Taub-NUT spaces

In what follows we restrict ourselves to the *generalized* Taub-NUT manifolds whose metrics are defined on $\mathbb{R}^4 \setminus \{0\}$ by the line element [19, 20, 21]:

$$\begin{aligned} d{s_K}^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\ &= f(r)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2) + g(r)(d\chi + \cos\theta\, d\varphi)^2, \end{aligned} \tag{25}$$

where the angle variables (θ, φ, χ) parametrize the sphere S^3 with $0 \le \theta < \pi$, $0 \le \varphi < 2\pi$, $0 \le \chi < 4\pi$, while the functions

$$f(r) = \frac{a+br}{r}, \qquad g(r) = \frac{ar + br^2}{1 + cr + dr^2}.$$

depend on the arbitrary real constants a, b, c and d.

Here it is worth pointing out that the above metrics are related to the Berger family of metrics on 3-spheres [41]. These are introduced starting with the Hopf fibration π_H: $S^3 \to S^2$ that defines the vertical subbundle $V \subset TS^3$ and its orthogonal complement $H \subset TS^3$ with respect to the standard metric g_{S^3} on S^3. Denoting with g_H and g_V the restriction of g_{S^3} to the horizontal, respectively the vertical bundle, one finds that the corresponding line elements are $d{s_H}^2 = \frac{1}{4}d{s_2}^2$ and $d{s_V}^2 = \frac{1}{4}(d{s_3}^2 - d{s_2}^2)$. For each constant $\lambda > 0$ the Berger metric on S^3 is defined by the formula

$$g_\lambda = g_H + \lambda^2 g_V.$$

This line element can be written in terms of the Berger metrics as

$$ds_K{}^2 = (ar + br^2)\left(\frac{dr^2}{r^2} + 4 ds_{\lambda(r)}{}^2\right),$$

where $ds_{\lambda(r)}{}^2 = (g_{\lambda(r)})_{\mu\nu} dx^\mu dx^\nu$ and

$$\lambda(r) = \frac{1}{\sqrt{1 + cr + dr^2}}.$$

If one takes the constants

$$c = \frac{2b}{a}, \qquad d = \frac{b^2}{a^2} \tag{26}$$

the generalized Taub-NUT metric becomes the original Euclidean Taub-NUT metric up to a constant factor.

By construction, the spaces with the metric (25) have four K vectors (19). The corresponding constants of motion in generalized Taub-NUT backgrounds consist of a conserved quantity for the cyclic variable χ

$$q = g(r)(\dot{\chi} + \cos\theta\dot{\varphi})$$

and the angular momentum vector

$$\vec{J} = \vec{x} \times \vec{p} + q\frac{\vec{x}}{r}, \qquad \vec{p} = f(r)\dot{\vec{x}}.$$

The remarkable result of Iwai and Katayama [19, 20, 21] is that the generalized Taub-NUT space (25) admits a hidden symmetry represented by a conserved vector, quadratic in 4-velocities, analogous to the Runge–Lenz vector of the following form

$$\vec{K} = \vec{p} \times \vec{J} + \kappa\frac{\vec{x}}{r}. \tag{27}$$

The constant κ involved in the Runge–Lenz vector (27) is $\kappa = -aE + \frac{1}{2}cq^2$ where the conserved energy E is

$$E = \frac{\vec{p}^{\,2}}{2f(r)} + \frac{q^2}{2g(r)}.$$

The components $K_i = k_i^{\mu\nu} p_\mu p_\nu$ of the vector $\vec{K}$ (27) involve three S-K tensors $k_i^{\mu\nu}$, $i = 1, 2, 3$ satisfying (1).

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