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CORRESPONDENCES, FERMAT QUOTIENTS, AND UNIFORMIZATION

by

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Abstract. — Ordinary differential equations have an arithmetic analogue in which functions are replaced by integer numbers and the derivative operator is replaced by a Fermat quotient operator. This paper reviews the basics of this theory and explains some of the applications to the invariant theory of correspondences.

***Résumé* (Correspondances, quotients de Fermat et uniformisation).** — Les équations différentielles ordinaires possèdent un analogue arithmétique où les fonctions et leurs dérivées sont remplacées par des nombres entiers et leurs quotients de Fermat. Cet article présente les principes de cette théorie et quelques applications à la théorie des invariants pour les correspondances.

This paper represents a brief overview of some of the author's work on *arithmetic differential algebra* and its applications to the invariant theory of correspondences. Arithmetic differential algebra is an arithmetic analogue of the Ritt-Kolchin differential algebra [**Rit50**], [**Kol73**] in which derivations are replaced by *Fermat quotient operators*. The main foundational results and first applications of arithmetic differential algebra are contained in [**Bui95**], [**Bui96**], [**Bui00**]. A further study of these matters is contained in [**Bar03**], [**Bui03**], [**Bui04**], [**Bui02**]. A program outlining applications to the invariant theory of correspondences was sketched in the last 2 pages of [**Bui02**]. The present paper reports on recent progress along this program. For a detailed exposition of the results announced here we refer to the research monograph [**Bui05**].

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1. Motivation

Let X and $\tilde{X}$ be two complex algebraic curves and $\sigma = (\sigma_1, \sigma_2)$ a pair of dominant maps:

$$X \xleftarrow{\sigma_1} \tilde{X} \xrightarrow{\sigma_2} X. \tag{1}$$

Assume X is irreducible. Denote by $\mathbf{C}(X)$ the field of rational functions on X and by

$$\mathbf{C}(X)^{\sigma} := \{f \in \mathbf{C}(X) \mid f \circ \sigma_1 = f \circ \sigma_2\} \tag{2}$$

the field of *invariants* of the *correspondence* σ. It is a fact that, "most of the times", there are "no non-constant invariants":

$$\mathbf{C}(X)^{\sigma} = \mathbf{C}. \tag{3}$$

There are, of course, exceptions to this: the whole of the classical Galois theory of curves is an exception. Here, when we say *Galois theory*, we mean the case when $\sigma_2 : \tilde{X} := X \times G \to X$ is a finite group action and σ_1 is the first projection; in this case, of course, we have

$$\mathbf{C}(X)^{\sigma} = \mathbf{C}(X)^{G} \neq \mathbf{C}.$$

In this paper we would like to view Galois theory as an exceptional (and "well understood") situation. On the contrary, the fact that the equality (3) holds "most of the times" will be viewed as a basic pathology in algebraic geometry that we would like to address. Indeed equality (3) says in particular that the "categorical quotient" X/σ in the category of algebraic varieties reduces to a point and, hence, the quotient map $X \to X/\sigma$ cannot be viewed, in any reasonable sense, as a Galois cover. Our aim in this paper is to show how one can construct a "larger geometry" (referred to as $\delta-$*geometry*) in which X/σ ceases, in many interesting situations, to reduce to a point; in this new geometry the quotient map $X \to X/\sigma$ will sometimes "looks like" a Galois cover.

Our theory will be $p-$adic (rather than over the complex numbers $\mathbf{C}$). The basic ring of our theory will be $R = \hat{\mathbf{Z}}_p^{ur}$, the completion of the maximum unramified extension of the $p-$adic integers; recall that this is the unique complete discrete valuation ring with maximal ideal generated by p and residue field equal to the algebraic closure $\mathbf{F}_p^a$ of the prime field $\mathbf{F}_p$. The ring R has a unique automorphism ϕ lifting the Frobenius on R/pR. We can therefore consider the *Fermat quotient operator* $\delta : R \to R$,

$$\delta x = \frac{\phi(x) - x^p}{p}. \tag{4}$$

We will view δ as an arithmetic analogue of a derivation; our $\delta-$geometry will then be an arithmetic analogue of the Ritt-Kolchin differential algebraic geometry [**Rit50**], [**Kol73**], [**Bui94**].

2. Toy examples

To explain what we have in mind we begin by looking at an easy example. Assume, in what follows, that $X = \tilde{X} = \mathbf{A}^1$ is the affine line over R. We assume $\sigma_1 = id$ and $\sigma_2(x) = x^2$. Define the map $\psi : R \to R$,

$$\psi(x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{p^{i-1}}{i} \left(\frac{\delta x}{x^p} \right)^i, \tag{5}$$

and consider the (partially defined) map $f : R \to R$,

$$f(x) = \frac{\phi \circ \psi}{\psi}(x) = \psi^{p-1}(x) + p\frac{\delta\psi}{\psi}(x); \tag{6}$$

note that f is not defined precisely at the roots of 1. It is trivial to check that

$$\psi(x^2) = 2 \cdot \psi(x)$$

and, hence,

$$f(x^2) = f(x),$$

so f is an *invariant* for σ. Note that one can write

$$f(x) = \frac{F(x, \delta x, \delta^2 x, x^{-1})}{G(x, \delta x, x^{-1})}, \tag{7}$$

with F, G restricted power series in 4 respectively 3 variables. This example shows that, although no invariants for σ exist in algebraic geometry, invariants as in Equation 7 (which we shall refer to as $\delta-$*invariants*) may very well exist; this suggests to "adjoin" δ to usual algebraic geometry and this is exactly what we shall soon do.

Before proceeding to the general case, let us explore the above example in further detail. Once we discovered the invariant $\eta_0 := \frac{\phi \circ \psi}{\psi}$ it is easy to come up with more invariants namely $\eta_i := \delta^i \circ \eta_0$. Set $\bar{\eta}_i := \eta_i$ mod p. Moreover set $x' = \delta x$, $x'' = \delta^2 x$, e.t.c. One can prove that the field extension

$$\mathbf{F}_p^a(x, \bar{\eta}_0, \bar{\eta}_1, \bar{\eta}_2, ...) \subset \mathbf{F}_p^a(x, x', x'', x''', ...) \tag{8}$$

is Galois with Galois group $\mathbf{Z}_p^\times$. The left hand side of the above extension (8) can be viewed as the compositum of $\mathbf{F}_p^a(x)$ (the "field of rational functions on $X = \mathbf{A}^1$ mod p in the old algebraic geometry") and the field $\mathbf{F}_p^a(\bar{\eta}_0, \bar{\eta}_1, \bar{\eta}_2, ...)$ (which should be viewed as the "field of rational functions mod p on X/σ in the new geometry). The right hand side of the extension (8) can be viewed as the "field of rational functions mod p on X in the new geometry". As we will see the above picture can be generalized.

Let us further postpone our discussion of the general case by looking at yet another example. Assume in what follows that $X = \tilde{X} = \mathbf{A}^1$ over R and $\sigma_1 = id$, $\sigma_2(x) = x^2 - 2$ (the Chebyshev quadratic polynomial). Again one can show that "$\delta-$invariants" exist, more precisely there exist restricted power series F, G in 4 and 3 variables respectively such that

$$f(x) = \frac{F(x, \delta x, \delta^2 x, (x^2-4)^{-1})}{G(x, \delta x, (x^2-4)^{-1})} \tag{9}$$

satisfies

$$f(x^2 - 2) = f(x).$$

Also there is Galois computation similar to that in the previous example.

A natural question is whether the existence of "$\delta-$invariants" in the above 2 examples generalizes to the situation when $X = \tilde{X} = \mathbf{A}^1$, $\sigma_1 = id$, $\sigma_2(x) = x^2 + c$, $c \in \mathbf{Z}$. The answer to this question is NO! (Cf. [**BZ05**] for a precise statement and for related conjectures.)

The next natural question is: what do $x \mapsto x^2$ and $x \mapsto x^2 - 2$ have in common that does not hold for a general quadratic map $x \mapsto x^2 + c$? One possible answer is that the maps corresponding to $c = 0$ and $c = -2$ possess, over the complex numbers, *analytic uniformizations* in the sense that one has commutative diagrams

$$\begin{array}{ccc} \mathbf{C} & \stackrel{2z}{\rightarrow} & \mathbf{C} \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \mathbf{C}^\times & \stackrel{z^2}{\rightarrow} & \mathbf{C}^\times \end{array} , \quad \begin{array}{ccc} \mathbf{C} & \stackrel{2z}{\rightarrow} & \mathbf{C} \\ \pi_2 \downarrow & & \downarrow \pi_2 \\ \mathbf{C} & \stackrel{z^2-2}{\rightarrow} & \mathbf{C}, \end{array}$$

where $\pi_1(z) = e^{2\pi i z}$ and $\pi_2(z) = e^{2\pi i z} + e^{-2\pi i z}$ respectively.

So the next question one is tempted to ask is: are there other correspondences admitting similar "analytic uniformizations"? The answer to this question is: PLENTY! And they can be all classified.

The final question one would then ask would be: Do "$\delta-$invariants" exist for such "uniformizable" correspondences? Again the answer to the above question tends to be YES and the aim of this paper is to explain the theory that provides this answer.

3. Outline of the theory

To explain our main ideas it is convenient to start with an arbitrary category $\mathcal{C}$; what we have in mind is a category of spaces in some geometry. By a *correspondence* we will understand a pair $\mathbf{X} = (X, \sigma)$ where X is an object in $\mathcal{C}$ and σ is a pair of morphisms in $\mathcal{C}$ as in Equation (1). A *categorical quotient* for $\mathbf{X}$ will mean a pair (Y, π) where $\pi : X \to Y$ is a morphism in $\mathcal{C}$ such that $\pi \circ \sigma_1 = \pi \circ \sigma_2$ and with the property that for any other pair (Y', π') with $\pi' : X \to Y'$, $\pi' \circ \sigma_1 = \pi' \circ \sigma_2$ there exists a unique $\epsilon : Y \to Y'$ such that $\epsilon \circ \pi = \pi'$. We write $Y = X/\sigma$. (Categorical quotients are sometimes called co-equalizers.) We will also give, in each concrete example, a class of objects of $\mathcal{C}$ which we call *trivial.* For instance, if $\mathcal{C}$ is the category of algebraic varieties, the trivial objects will be declared to be the points. If $\mathbf{X}$ is a correspondence between curves, possessing an infinite orbit (i.e., a sequence of distinct points $Q_1, Q_2, ... \in \tilde{X}$ such that $\sigma_2(Q_i) = \sigma_1(Q_{i+1})$ for $i \geq 1$), then clearly X/σ is trivial. To remedy this situation we will proceed as follows.

1) For each p we will "adjoin" the Fermat quotient operator $\delta = \delta_p$ to usual algebraic geometry; this will lead us to consider a category $\mathcal{C}_\delta$ that underlies what we shall refer to as "$\delta-$geometry".

2) For any correspondence $\mathbf{X}_\mathcal{O}$ in the category of smooth curves over the ring of $S-$integers $\mathcal{O}$ of a number field we will consider the correspondences $\mathbf{X}_\wp$ and $\mathbf{X}_\mathbf{C}$ deduced by base change via $\mathcal{O} \subset \hat{\mathcal{O}}_\wp = \hat{\mathbf{Z}}_p^{ur}$ and $\mathcal{O} \subset \mathbf{C}$, where $\wp$ runs through the set of unramified places outside S. To each $\mathbf{X}_\wp = (X_\wp, \sigma_\wp)$ we will associate a correspondence $\mathbf{X}_\delta = (X_\delta, \sigma_\delta)$ in $\mathcal{C}_\delta$, where $\delta = \delta_p$.

3) We will formulate a conjecture (and state results along this conjecture) essentially asserting that if $\mathbf{X}_\mathbf{C}$ has an infinite orbit then X_δ/σ_δ is non-trivial in $\mathcal{C}_\delta$ for almost all places $\wp$ if and only if $\mathbf{X}_\mathbf{C}$ admits an analytic uniformization (in a sense to be explained below).

The rest of the paper is devoted to explaining the above 3 steps.

4. Uniformization

We begin by explaining the concept of analytic uniformization for correspondences on complex algebraic curves. Let $\mathbf{X} = (X, \sigma)$ be a correspondence in the category of complex algebraic curves. We assume $X, \tilde{X}$ are non-singular connected and σ_1 and σ_2 are dominant. We say that $\mathbf{X}$ has an *analytic uniformization* if one can find a diagram of Riemann surfaces

$$\begin{array}{ccccc} \mathbf{S} & \stackrel{\tau_1}{\longleftarrow} & \mathbf{S} & \stackrel{\tau_2}{\longrightarrow} & \mathbf{S} \\ \pi \downarrow & & \downarrow \tilde{\pi} & & \downarrow \pi \\ X' & \stackrel{\sigma_1'}{\longleftarrow} & \tilde{X}' & \stackrel{\sigma_2'}{\longrightarrow} & X' \\ u \uparrow & & \uparrow \tilde{u} & & \uparrow u \\ X & \stackrel{\sigma_1}{\longleftarrow} & \tilde{X} & \stackrel{\sigma_2}{\longrightarrow} & X \end{array}$$

with $\mathbf{S}$ a simply connected Riemann surface, τ_1, τ_2 automorphisms of $\mathbf{S}$, $\pi, \tilde{\pi}$ Galois covers of degree $\leq \infty$, and $u, \tilde{u}$ inclusions with $X' \backslash X$ and $\tilde{X}' \backslash \tilde{X}$ finite sets containing the ramification locus of π and $\tilde{\pi}$ respectively. It is easy to "classify" all correspondences which admit an analytic uniformization and possess an infinite orbit. The details of the classification are tedious and will be skipped here; we content ourselves with a few remarks. There are 3 cases: the spherical, flat and hyperbolic case according as $\mathbf{S}$ is $\mathbf{CP}^1$, $\mathbf{C}$, or $\mathbf{H}$ (the upper half plane) respectively. In the spherical case everything boils down to the (well known) classification of finite groups of automorphisms of $\mathbf{CP}^1$. In the flat case the Galois groups of π and $\tilde{\pi}$ are crystallographic (i.e., contain a normal subgroup of finite index consisting of translations); the resulting list of possible $\mathbf{X}$'s is a variation on "Thurston's list" of postcritically finite non-hyperbolic dynamical systems; cf. [**DH93**]. (The baby examples in the previous section are in this category and so are the higher degree Chebyshev dynamical systems and the so-called Lattès dynamical systems arising from elliptic curves [**Mil99**].) In the hyperbolic case a deep result of Margulis implies that the Galois groups of π and $\tilde{\pi}$ are arithmetic lattices arising from quaternion algebras B over a totally real field F. So X and $\tilde{X}$ are essentially modular curves (if $F = \mathbf{Q}$, $B = M_2(\mathbf{Q})$) or Shimura curves (in all remaining cases).

5. $\delta-$ringed sets

Our next step is to introduce the category $\mathcal{C}_\delta$ which underlies "$\delta-$geometry". We fix, in this section, a prime p and the ring $R = \hat{\mathbf{Z}}_p^{ur}$. Recall that R carries a Fermat quotient operator $\delta : R \to R$. A natural way to develop $\delta-$geometry would be to consider a category of "$\delta-$ringed spaces" (i.e., ringed spaces equipped with a "Fermat quotient type operator"); this would lead us, however, into a lot of general non-sense that we would like to avoid. Instead we adopt a rather naive viewpoint (not involving topology and sheaves). This viewpoint captures, nevertheless, all the features that we think are relevant for our applications.

Here is the basic definition. A *$\delta-$ringed set* X_δ is a set X_{set} together with the following data:

1) A family $(X_s)_{s\in S}$ of subsets of X_{set} indexed by a monoid $S = S_X$ such that $X_{st} = X_s \cap X_t$,
2) A family $(\mathcal{O}_s)_{s\in S}$ of subrings

$$\mathcal{O}_s \subset \{\text{maps } X_s \longrightarrow R\}$$

such that if $f \in \mathcal{O}_s$ and $t \in S$ then $\delta \circ f \in \mathcal{O}_s$ and $f_{|X_{st}} \in \mathcal{O}_{st}$.

A *morphism of $\delta-$ringed sets* $\sigma_\delta : X_\delta \to Y_\delta$ is a pair $\sigma_\delta = (\sigma_{set}, \sigma^\sharp)$ where $\sigma_{set} : X_{set} \to Y_{set}$ is a map of sets and $\sigma^\sharp : S_Y \to S_X$ is a morphism of monoids such that $\sigma_{set}^{-1}(Y_s) = X_{\sigma^\sharp(s)}$ and such that if $f \in \mathcal{O}_s$ then $f \circ \sigma_{set} \in \mathcal{O}_{\sigma^\sharp(s)}$. We call $\mathcal{C}_\delta$ the category of $\delta-$ringed sets. An object X_δ in $\mathcal{C}_\delta$ is called *trivial* if $\mathcal{O}_s = R$ for all s. Any correspondence in $\mathcal{C}_\delta$ has a categorical quotient. So non-triviality (rather than existence) will be the main issue as far as categorical quotients in $\mathcal{C}_\delta$ are concerned.

Here are a few more important definitions.

A $\delta-$ringed set X_δ will be called *$\delta-$localized* if the following conditions hold:

1) If $pf \in \mathcal{O}_s$ with $f : X_s \to R$ then $f \in \mathcal{O}_s$;
2) If $P \in X_s$, $f \in \mathcal{O}_s$, $f(P) \notin pR$ then there exists t such that $P \in X_{st}$ and $f_{|X_{st}} \in \mathcal{O}_{st}^\times$.

For a $\delta-$localized X_δ one can define the $\delta-$ring of *rational functions*,

$$R\langle X_\delta\rangle := \lim_{\to} \mathcal{O}_s;$$

it is a discrete valuation ring with maximal ideal generated by p. We denote by $k\langle X_\delta\rangle$ its residue field. We say that X_δ is $\delta-$*rational* if there exists a family $\eta=(\eta_i)$, $\eta_i\in R\langle X_\delta\rangle$, such that the reductions mod p:

$$\overline{\eta},\overline{\delta\eta},\overline{\delta^2\eta},...\in k\langle X_\delta\rangle$$

are algebraically independent over $k:=\mathbf{F}_p^a$ and generate the field extension $k\subset k\langle X_\delta\rangle$. This is of course a natural $\delta-$analogue of the concept of rational variety in algebraic geometry.

6. Attaching $\delta-$ringed sets to schemes

Our next step is to show how to attach to a smooth scheme X over $R=\hat{\mathbf{Z}}_p^{ur}$ a $\delta-$ringed set. We recall a basic definition from [**Bui95**]. A $\delta-$*function of order* $\leq r$ $f:X(R)\to R$ is a function such that for any point in $X(R)$ there exists a Zariski open neighborhood $U\subset X$, a closed immersion $u:U\to\mathbf{A}^d$, and a restricted power series

$$F\in R[T_1,...,T_{(r+1)d}]\hat{}$$

such that

$$f(P)=F(u(P),\delta(u(P)),...,\delta^r(u(P))),\quad P\in U(R).$$

(Here the upper $\hat{}$ means $p-$adic completion.)

The rule that attaches to any Zariski open set $V\subset X$ the ring

$$\mathcal{O}^r(V):=\{\delta-\text{functions } V(R)\longrightarrow R \text{ of order }\leq r\}$$

defines a sheaf $\mathcal{O}^r$ on X for the Zariski topology. By a $\delta-$*line bundle* on X we understand a locally free sheaf L of $\mathcal{O}^r-$modules of rank 1. If $W=\mathbf{Z}[\phi]$ is the non-commutative subring of $End(R)$ generated by ϕ then W acts on $R^\times$ and hence on $\mathcal{O}^r(V)^\times$ for any V. Acting by elements $w\in W$ on the cocycle defining a $\delta-$line bundle L one can define $\delta-$line bundles L^w. Let W_+ be the set of all $\sum a_i\phi^i\in W$ with $a_i\geq 0$. Then one can form a W_+-graded ring

$$R(X,L)=\bigoplus_{w\in W_+}H^0(X,L^w).$$

Using this ring we can define a $\delta-$ringed set X_δ as follows. The set X_{set} is, by definition, the set $X(R)$. The monoid S is defined by

$$S=\{\text{homogeneous elements of } R(X,L)\backslash pR(X,L) \text{ of degree}\neq 0\}.$$

For $s\in S$ we let X_s be the set of all $P\in X(R)$ such that $s(P)\not\equiv 0$ mod p. Finally we let

$$\mathcal{O}_s=\{\frac{f}{s^w}\mid w\in W_+, deg(f)=deg(s^w)\}\subset\{\text{maps } X_s\longrightarrow R\}.$$

Of course X_δ depends on L; but everywhere, in what follows, we shall take $L=K_X^{-1}$, the anticanonical bundle on X. If $\mathbf{X}=(X,\sigma)$ is a correspondence in the category of smooth schemes over R and if σ_1, σ_2 are étale then we obtain a correspondence $\mathbf{X}_\delta = (X_\delta, \sigma_\delta)$ in the category $\mathcal{C}_\delta$ of $\delta-$ringed sets.

7. Main conjectures

Let $\mathcal{O} \subset \mathbf{C}$ be the ring of $S-$integers in a number field where S is a finite set of finite places containing all the places which are ramified over $\mathbf{Q}$. Assume that $\mathbf{X}_{\mathcal{O}} = (X, \sigma)$ is a correspondence in the category $\mathcal{C}_{\mathcal{O}}$ of smooth curves over $\mathcal{O}$. Assume σ_1, σ_2 are étale. For any finite place $\wp \notin S$ let $R_\wp := \hat{\mathcal{O}}_\wp^{ur}$ be the completion of the maximum unramified extension of the local ring of $\mathcal{O}$ at $\wp$, let $k_\wp$ be the residue field of $R_\wp$, and let $p\mathbf{Z} = \wp \cap \mathbf{Z}$. Denote by

$$\mathbf{X}_{\mathbf{C}} = (X_{\mathbf{C}}, \sigma_{\mathbf{C}}), \quad \mathbf{X}_\wp = (X_\wp, \sigma_\wp) \tag{10}$$

the correspondences over $\mathbf{C}$ and $R_\wp$ obtained by base change respectively. For each $\wp$ we view $\mathbf{X}_\wp$ equipped with the anticanonical bundle and we consider the associated correspondence

$$\mathbf{X}_\delta = (X_\delta, \sigma_\delta)$$

in the category of $\delta-$ringed sets $\mathcal{C}_\delta$. Also recall that we defined in Section 5 the notion of *$\delta-$rational* $\delta-$ringed set.

Conjecture 7.1. — *Assume $\mathbf{X}_{\mathbf{C}}$ admits an analytic uniformization and possesses an infinite orbit. Then X_δ/σ_δ is non-trivial and $\delta-$rational for almost all places $\wp$.*

In the converse direction we propose the following:

Conjecture 7.2. — *Assume $\mathbf{X}_{\mathbf{C}}$ possesses an infinite orbit and assume that, for all but finitely many places $\wp$, X_δ/σ_δ is non-trivial. Then $\mathbf{X}_{\mathbf{C}}$ is commensurable with a correspondence that admits an analytic uniformization.*

Commensurability in the above statement is the equivalence relation generated by the obvious relation of "dominance" between correspondences.

Conjecture 7.1 should be complemented as follows. Note that the quotient map $X_\delta \to X_\delta/\sigma_\delta$ induces a field extension

$$k_\wp\langle X_\delta/\sigma_\delta\rangle \subset k_\wp\langle X_\delta\rangle.$$

If in addition X is affine we get an induced extension

$$k_\wp(X) \cdot k_\wp\langle X_\delta/\sigma_\delta\rangle \subset k_\wp\langle X_\delta\rangle.$$

We expect that, under the assumptions of Conjecture 7.1, the latter extension is always algebraic and its Galois theoretic properties can then be investigated.

8. Main results

One can prove quite general results along Conjecture 7.1 in the following cases:

1) spherical case

2) flat case

3) hyperbolic case corresponding to quaternion algebras over $\mathbf{Q}$.

So the case not covered (yet) by our theory is that of hyperbolic uniformizations corresponding to quaternion algebras over totally real fields $\neq \mathbf{Q}$.

Conjecture 7.2 is more mysterious. But there is a local analogue of this conjecture along which we can prove quite general results. In the local analogue of Conjecture 7.2 correspondences are replaced by power series in $R[[T]]$, analytic uniformization is replaced by "uniformization by automorphisms of formal groups", and $\delta-$invariants are replaced by "invariant" series in $R[[T]][T',...,T^{(r)}]\hat{}$, where $T',...,T^{(r)}$ are "new variables" which morally stand for "$\delta T, ..., \delta^r T$". Describing the local analogue of Conjecture 7.2 would lead us too far afield.

In what follows we shall give a sample of our main results on Conjecture 7.1.

8.1. Spherical case. — Let $\Gamma \subset SL_2(\mathbf{Z})$ be a finite subgroup and let $\tau \in SL_2(\mathbf{Z})$. Let $\mathcal{O} = \mathbf{Z}[1/m]$ for some m. View $SL_2(\mathbf{Z})$ as acting on the projective line $\mathbf{P}^1 = Proj\ \mathcal{O}[x_0, x_1]$ over $\mathcal{O}$. Let $F \in \mathcal{O}[x_0, x_1]$ be a homogeneous $\Gamma-$invariant polynomial such that all geometric points of $\mathbf{P}^1$ fixed by some member of Γ belong to the closed scheme $Z(F)$ defined by F. Consider the schemes

$$Y := \mathbf{P}^1 \backslash Z(F), \quad X = Y/\Gamma, \quad \tilde{X} = Y \cap \tau^{-1}(Y),$$

Let $\pi : \mathbf{P}^1 \to \mathbf{P}^1/\Gamma$ be the canonical projection. and consider the correspondence

$$\mathbf{X}_{\mathcal{O}} := (X, \tilde{X}, \pi, \pi \circ \tau)$$

in $\mathcal{C}_{\mathcal{O}}$.

Theorem 8.1. — *For all but finitely many primes p, X_δ/σ_δ is non-trivial. If in addition the group $\langle \Gamma, \tau \rangle = SL_2(\mathbf{Z})$ then, for infinitely many primes p,*

1) X_δ/σ_δ is $\delta-$rational;

2) $k_p\langle X_\delta \rangle$ can be embedded into an algebraic Galois extension of $k_p\langle X_\delta/\sigma_\delta \rangle$ with Galois group $PSL_2(\mathbf{Z}_p)$ or $PGL_2(\mathbf{Z}_p)$.

8.2. Flat case. — Let $\mathcal{O} \subset \mathbf{C}$ be the ring of $S-$integers in a number field. Let G be either the multiplicative group $\mathbf{G}_m$ or an elliptic curve over $\mathcal{O}$, let $N \in \mathbf{Z}$ be invertible in $\mathcal{O}$ with $N \not\in \{1, -1\}$, let $[N] : G \to G$ be the multiplication by N endomorphism, and let $\epsilon \in \{1, -1\}$. Then $[N]$ induces a morphism of schemes $\sigma : G/\langle [\epsilon] \rangle \to G/\langle [\epsilon] \rangle$. Let $X \subset G/\langle [\epsilon] \rangle$ be an affine Zariski open set such that the natural projection $\pi : G \to G/\langle [\epsilon] \rangle$ is étale above X, let $\tilde{X} = X \cap \sigma^{-1}(X)$ and consider the étale irreducible correspondence in $\mathcal{C}_{\mathcal{O}}$:

$$\mathbf{X}_{\mathcal{O}} = (X, \tilde{X}, \iota, \sigma),$$

where ι is the inclusion.

Theorem 8.2. — *For all but finitely many places $\wp$ the following hold:*

1) X_δ/σ_δ *is non-trivial and* $\delta-$*rational;*
2) *The field extension* $k_\wp(X)\cdot k_\wp\langle X_\delta/\sigma_\delta\rangle\subset k_\wp\langle X_\delta\rangle$ *is algebraic. If* $\epsilon=1$ *and* $G=\mathbf{G}_m$ *the above extension is Galois, with Galois group* $\mathbf{Z}_p^\times$.

8.3. Hyperbolic case. — Fix an integer $N\geq 4$ and a prime l not dividing N. Let $\mathcal{O}=\mathbf{Z}[1/Nl]$ and consider the Hecke correspondence in $\mathcal{C}_\mathcal{O}$:

$$\mathbf{X}_\mathcal{O}:=(X:=Y_1(N),\tilde{X}:=Y_1(N,l),\sigma_1,\sigma_2) \tag{11}$$

where $Y_1(N)$ parameterizes elliptic curves with $\Gamma_1(N)-$level structure, $Y_1(N,l)$ parameterizes isogenies of degree l between elliptic curves with $\Gamma_1(N)-$level structure, and σ_1, σ_2 are the natural projection maps.

Theorem 8.3. — *For all but finitely many primes p,* X_δ/σ_δ *is non-trivial. Moreover, for infinitely many primes p,*

1) X_δ/σ_δ *is* $\delta-$*rational;*
2) *The field extension* $k_p(X)\cdot k_p\langle X_\delta/\sigma_\delta\rangle\subset k_p\langle X_\delta\rangle$ *can be embedded into a Galois extension with pro-solvable Galois group.*

9. Strategy of proofs

Here is a very rough description of the strategy behind this theory. First, following [**Bui95**], one attaches to any smooth scheme X over $R=\hat{\mathbf{Z}}_p^{ur}$ a projective system of formal schemes

$$...\longrightarrow J^r(X)\longrightarrow ...\longrightarrow J^2(X)\longrightarrow J^1(X)\longrightarrow J^0(X)=\hat{X}$$

called the *p−jet spaces* of X. They are arithmetic analogues of the usual jet spaces in differential geometry and have the property that

$$\mathcal{O}^r(X)=\mathcal{O}(J^r(X))$$

for all r. The latter equalities reduce the study of $\delta-$geometry of X_δ to the study of usual algebraic geometry of the projective system $(J^r(X))$. To prove our main results we need to:

1) find methods to produce "$\delta-$invariants" i.e., sections $f\in H^0(X,L^w)$ whose pull-backs via σ_1 and σ_2 coincide (or coincide up to a constant in $\mathbf{Z}_p^\times$);

2) prove that all "$\delta-$invariants" arise by the above methods.

To produce $\delta-$invariants is elementary in the spherical case. In the flat case one needs to use the arithmetic analogue of the theory of the Manin map developed in [**Bui95**] plus the compatibility between $p-$jets and étale Galois quotients [**BZ05**]. In the hyperbolic case one uses crystalline cohomology to construct an analogue of modular forms called $\delta-$*modular forms* which are "covariant" with respect to isogenies; cf. [**Bui00**], [**Bui03**], [**Bui04**].

To prove that all "$\delta-$invariants" arise by the above methods one proceeds as in classical invariant theory: one constructs certain (usual, non-arithmetic) differential operators acting on $\delta-$invariants and one sets up an "induction by degree" argument. The differential operators playing a role in this approach can be viewed as arithmetic analogues of operators acting on functions on jet spaces in classical mechanics. Cf. [**Bar03**], [**BZ05**], [**Bui03**], [**Bui04**], [**Bui05**] for details.

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