PREFACE

The whole idea behind the Conference *"Global Analysis and Harmonic Analysis"*, held at the Centre International de Rencontres Mathématiques in Marseilles-Luminy in May 1999, was to create the proper conditions for members of two mathematical sub-communities, harmonic analysts and global differential geometers using or studying spinor fields, to meet and discuss problems of common interest. The intense discussions that took place all along the week, and continued later, proved that the time was ripe for such a friendly confrontation.

In doing so, we were in some sense only coming back to the origins of the two disciplines whose birth and development owe a lot to two towering mathematical figures, namely William Kingdon Clifford and Élie Cartan.

Harmonic Analysis can be traced back to the Erlangen Programme and the new definition of a geometry proposed by Felix Klein in his 1872 inaugural address as a *collection of properties invariant under the action of a transitive group, its automorphism group*. It is less known that, some years before, Hermann von Helmholtz in his 1869 *"Über Tatsachen, welche der Geometrie zur Grunde liegen"* did propose a similar, even more comprehensive programme. W.K. Clifford also had a broad view of the way Geometry can be formulated. All these works induced the systematic study of *homogeneous spaces*, i.e. spaces which are acted upon transitively by a group. This grew into a branch of its own through the first systematic study of continuous groups made by Sophus Lie in the last years of the XIXth century.

At that time, the tool of choice in studying the physical world was the theory of partial differential equations. Lie noticed that almost all properties of differential equations that were useful in their *integration*, or solution, had to do with their behavior under groups of transformations of the underlying space. He was led to the idea that one might be able to do for differential equations what Galois had done for algebraic equations, namely to reduce their solution to group theory. The fact that the theory of Lie groups was developed just in time for modern Physics is no coincidence. S. Lie and his successors on one hand, and the physicists on the other were both struggling with the deeper meaning of partial differential equations.

W.K. Clifford laid the ground for the theory of algebras that bear his name, a motherly niche for spinors. He was interested in defining all kinds of generalisations of quaternions, and hypercomplex numbers. The notion of spinors arose formally for the first time in the work of Élie Cartan when, in 1913, he gave the first complete classification of complex representations of the Lie algebras of orthogonal groups. In doing so he was taking up earlier work by Wilhelm Karl Joseph Killing. Besides the already known irreducible tensor representations of matrix algebras, in each dimension he found another fundamental irreducible representation whose dimension was growing exponentially with the dimension of the group (as well as new infinite families of irreducible representations obtained by decomposing its tensor powers). It is only later, in 1914, when he turned to the study of the real representations, that he considered the problem from the group point of view, to discover that the spinor representations were not genuine representations of the orthogonal groups, but representations of 2-fold covers of these groups. These representations were later named "spin representations" in connection with their use by Paul Adrien Maurice Dirac in his relativistic model of the spinning electron. For that purpose he introduced a new operator, since then called the Dirac operator, a version of the Schrödinger equation invariant under the Lorentz group. The Dirac operator could not be a scalar operator but rather was necessarily acting on vectors in this representation space; these were called "spinors". It is therefore natural that the 2-fold covers of orthogonal groups be called "spin groups".

Élie Cartan also contributed greatly to Harmonic Analysis through the systematic development of the theory of symmetric spaces. They provide one of the most beautiful instances of interaction between pure Geometry (the parallelism of the curvature tensor), Lie group theory (the finite dimensionality of the group generated by geodesic symmetries centered at each point), and Algebra (specific pairs of Lie algebras, the famous "symmetric pairs"). The analysis of invariant differential operators on symmetric spaces can be unfolded systematically, and for that theory too the parallel between Geometry and Analysis is one of the most fruitful Leitfaden. In this context, a beautiful duality appears between the compact and the non-compact cases. It too has a geometric side (curvatures have opposite signs) and an analytic one.

The theories of both spinors and homogeneous spaces developed, each in its own way, much further, throughout the entire XXth century. Today, spinor fields lie at the heart of almost all modern theories of physics as wave functions of fermionic particles, the basic constituents of matter. This went very far with the claim of Roger Penrose that all Physics should be rethought in terms of spinors. This philosophy was finally subsumed in the various attempts to define a "super-geometry", in which non commuting variables are treated on an equal footing with (more conventional) commuting variables. The search for supersymmetry (if one is to keep the spirit of the Erlangen Programme) has been one of the driving forces of the development of theoretical physics for the last twenty years.

Harmonic Analysis has developed with considerable success much beyond the case of symmetric spaces. It now is one of the major branches of Analysis, that impacts many other areas of Mathematics. The theory of pseudo-differential operators, a decisive step in the study of elliptic operators, was greatly influenced in its development by the consideration of algebras of operators connected to solvable and other Lie groups, as exemplified in the work of Elias Stein. It was also significantly influenced by the systematic search for irreducible representations of very specific groups by theoretical physicists. They needed to test them as possible symmetry groups of physical theories, whereas vectors in the irreducible representation spaces would model wave functions of elementary particles of the theory in this context.

The coming together of Geometry and Analysis we are talking about here is indeed one of the main events of XXth century mathematics. Depending on the area one is coming from, one sees it as the recognition by geometers of the exceptional power of analytical tools to solve geometric problems or as the remarkable ability of questions rooted in geometry to point to the critical situations in analytical contexts. As a result, everybody today acknowledges the importance of Global Analysis, a new domain covering global properties to be considered in order to solve analytical problems, as well as analytical aspects of global geometric questions. This cross-fertilisation is examplified by the role played by spectral invariants (e.g., eigenvalues of the Laplace-Beltrami operator on Riemannian manifolds either through geometric estimates of the lowest ones, or through their asymptotic behaviour). Other examples are provided by the very fruitful study of geometric non-linear variational problems that came out of the limiting case of Sobolev inequalities, in particular those connected to conformal classes of metrics on spaces admitting non compact groups of automorphisms such as spheres, a critical situation. Many more problems related to conformal geometry remain unsolved, and this theory is still wide open. One of the reasons for this state of affairs is that the analysis required to deal properly with it is not the traditional theory of second order elliptic operators but the more formidable theory of fourth (and higher) order operators.

There is still another source that had a great impact on many aspects of Geometry and Analysis, namely the theory of integrable systems. It has been used extensively by physicists, in particular in the extremely productive and stimulating atmosphere that, for many years, characterized the Soviet school of mathematical physics. It took some time before this theory was considered seriously enough in Western circles. This slowness to recognize the richness and the fruitfulness of points of view that it brings is likely to be related to its missing (so far will optimists say) to fitting in a general theory, a sin for people still under the influence of the Bourbaki era. A direct link to problems we are interested in is provided by the theory of twistors, also stongly advocated by Roger Penrose, which gives special (but very interesting) solutions to a number of outstanding geometric problems, provided one "twists" it properly. This again leads to a very happy and prolific marriage between Harmonic Analysis and Differential Geometry, even Spin Geometry. Again, to name one specific instance, finding local coordinates on the twistor space of a four-dimensional self-dual manifold is nothing but looking for local solutions to the Killing spinor equations, provided one takes into account the close connection that, in four dimensions, ties together complex structures and lines of spinors.

One tool that is in some sense exemplary of the still mysterious interaction between local and global geometric properties through analytical means goes by the name of "Weitzenböck formulas". These formulas, already noticed by Weitzenböck in the early part of the century, compare two natural second-order differential operators on a Riemannian manifold. They have been exploited in Differential Geometry since the far reaching work of Salomon Bochner in the 30s, showing that metrics with positive Ricci curvature, a local assumption, could only exist on manifolds with vanishing first Betti number, a global consequence. Since then, in a number of geometric instances, it was possible to find natural second-order differential operators and compare them in order to derive global results from local assumptions on the curvature. One of the most striking successes of this method is in the spinorial context, and goes back to a 1963 Compte-Rendus Note to the Académie des Sciences de Paris by André Lichnerowicz. The formula he used was in fact known to Erwin Schrödinger. Thanks to it and to the Atiyah-Singer theorem, which by the way is the prototype of results connecting Geometry and Analysis, A. Lichnerowicz could prove that any compact spin manifold with non-vanishing $\hat{A}$-genus does not admit any metric with positive scalar curvature, a far reaching generalisation of the Bochner theorem we quoted above.

A systematic treatment of these questions in fact requires the use of representation theory, and this brings us back to Harmonic Analysis. Indeed, the occurence of curvature terms of a certain type can be explained on a priori representation theoretic grounds. Considering all possible invariant, or natural operators, is indeed the key to obtaining the optimal formulas, and this approach is very similar to what has been the trend in Harmonic Analysis, i.e. the systematic consideration of algebras of invariant differential operators.

Intertwining operators, and notably intertwining differential operators, have for some time been a central feature of representations obtained by *parabolic induction.* The parabolic groups in question are natural choices for the structure group of a geometry. For example, oriented Riemannian geometry is the study of the special orthogonal structure group (or the spin group if we want access to the spinor bundles). Conformal geometry naturally points to the maximal parabolic subgroup of the conformal group of the sphere as its structure group. Within this group are the special orthogonal group, a group of uniform dilations, and a nilpotent part. Similar statements can be made for CR and other geometries. The nilpotent content of the structure group leads to a nontrivial Jordan content in its associated vector bundles. That is, even before taking section spaces, one has a nontrivial composition series, under the structure group, of the vector bundle fibers. The exploitation of this structure, and calculations with its characteristic bundles on arbitrary manifolds admitting the structure, is still a new subject in Differential Geometry, despite the wealth of knowledge one has in the model (homogeneous space) cases from the theory of the principal series and Knapp-Stein intertwinors on one hand, and from the theory of Verma modules on the other.

Work on such structures was initiated in the 1920's by Tracy Thomas, and largely abandoned after his work. (The language of vector bundles was still not available, so it is no wonder that people had trouble figuring out what Thomas was talking about.) After a long hiatus, the subject was taken up again in the 1980's by a number of researchers. Part of the work of the conference was to pick up the different strands of this project (representation theoretic and differential geometric), and get the different groups of practitioners talking together.

The most recent strictly mathematical achievements involving spinors are connected to the topology and geometry of manifolds. It bears the name of Seiberg-Witten theory. At its heart lies a system of coupled non-linear equations associating a connection (and its curvature form) and a spinor field. It was purely motivated by Physics where there are supposed to account for a certain duality. Moduli spaces of solutions bring a lot of information on the geometry of the space on which they are defined.

All this shows that, presently, Mathematics is thriving, and modifies itself through a very dynamic mixing of subdisciplines. As a result, mathematicians are more and more tempted to go even further along the path of specialisation. This can have short-term advantages, sometimes even amplified by the evaluation policies enforced by some research organisations and some organs of our community. We must nevertheless be careful, and make sure that we create conditions to help in particular our younger colleagues form a more global picture of our discipline. This can be done in many different ways. This volume acknowledges one of them, namely the attempt of bringing together two subdisciplines which have developed very rapidly in the last twenty years. In setting up this event, and in disseminating its fruits through these Procedings, we hope to have modestly contributed to the long term health of our discipline. Many more attempts will be needed, but others will create more opportunities.

We would like to thank the “Centre International de Rencontres Mathématiques” (CIRM) Luminy, France for its financial support and for the excellent working conditions we enjoyed during the meeting. Special thanks are due to the “Institut Élie Cartan”, Université Henri Poincaré, Nancy, and to the “Centre de Mathématiques de l’École polytechnique”, Palaiseau, France for their support. Thanks also go to the “Ministère de l’Éducation Nationale, de la recherche et de la Technologie” for covering expenses of participants from Eastern countries, via the “ACCES Program”, to the “CNRS” and to the “Société Mathématiques de France” for publishing the proceedings of this meeting.

Bures-sur-Yvette, Iowa City, Nancy, August 1, 2000

Jean-Pierre Bourguignon, Thomas Branson, Oussama Hijazi