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ANALYTIC CONTINUATION IN REPRESENTATION THEORY AND HARMONIC ANALYSIS

by

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Abstract. **—** In this paper we discuss topics in harmonic analysis and representation theory related to two different real forms G/H and G^c/H of a complex semisimple symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$. We connect representations of G and G^c using the theory of involutive representations of semi-groups and reflection symmetry. We discuss how to generalize the Segal-Bargmann transform to real forms of bounded symmetric domains. This transform maps $L^2(H/H\cap K)$ into the representation space of a highest weight representation of G. We show how this transform is related to reflection symmetry, which shows that it is a natural transform related to representation theory. Finally we discuss the connection of the H-spherical characters of the representations and relate them to spherical functions.

Résumé **(Prolongement analytique en théorie des représentations et analyse harmonique)**
Dans cet article, nous considérons des questions en analyse harmonique et en théorie des représentations concernant deux formes réelles différentes G/H et G^c/H d'un espace symétrique semi-simple complexe $G_{\mathbb{C}}/H_{\mathbb{C}}$. Nous établissons un lien entre les représentations de G et de G^c à l'aide de la théorie des représentations involutives des semi-groupes et la symétrie de réflexion. On examine la question de la généralisation de la transformée de Segal-Bargmann aux formes réelles des domaines symétriques bornés. Cette transformée envoie l'espace $L^c(H/H\cap K)$ dans l'espace de représentations d'une représentation du poids maximum de G. Nous montrons comment cette transformée est liée à la symétrie de réflexion, ce qui montre que c'est une transformée naturelle liée à la théorie des représentations. Finalement, on étudie la relation entre les caractères $H-$sphériques des représentations et les fonctions sphériques.

1. Introduction

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G_{\mathbb{C}}$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. We will for simplicity assume that $G\subset G_{\mathbb{C}}$ even if most of what we say holds also for the universal covering

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group $\tilde{G}$ and other connected groups locally isomorphic to G. Let $\theta : G \to G$ a Cartan involution on G and denote by $K = G^\theta$ the corresponding maximal compact subgroup of G. We will be interested in a special class of symmetric spaces, that are closely related to real forms of *bounded symmetric domains*. We will therefore assume that $D = G/K$ is a bounded symmetric domain. Let $\tau : D \to D$ by a *conjugation*, i.e., an anti-holomorphic involution, fixing the point $\{K\} \in D$. Those involutions were classified by A. Jaffee in [**25, 26**]. We will give the list later. We lift τ to an involution on G, $G_\mathbb{C}$, $\mathfrak{g}$, and $\mathfrak{g}_\mathbb{C}$. We will also denote those involutions by τ. Then τ commutes with θ. Let $H = G^\tau \subset H_\mathbb{C} := G_\mathbb{C}^\tau$. Then $D^\tau = H/H \cap K$.

On the Lie algebra level we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{q} \\ &= \mathfrak{k} \oplus \mathfrak{p} \\ &= (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})\,, \end{aligned}$$

$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\}$, $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}$, $\mathfrak{h} = \{X \in \mathfrak{g} \mid \tau X = X\}$, and $\mathfrak{q} = \{X \in \mathfrak{g} \mid \tau X = -X\}$. Define a new real form of $\mathfrak{g}_\mathbb{C}$ by

$$\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$$

and let G^c be the corresponding real analytic subgroup of $G_\mathbb{C}$. Denote also by τ the restriction of τ to G^c. Let $H^c := G^{c\tau}$. Then $H = H^c = G \cap G^c$.

We have the following diagrams

$$\begin{array}{ccccc} & & M_\mathbb{C} \underset{\text{Complex}}{:=} G_\mathbb{C}/H_\mathbb{C} & & \\ & \nearrow & & \nwarrow & \\ M := G/H & & \text{Real forms} & & M^c := G^c/H \end{array}$$

and

$$D^\tau = H/H \cap K \underset{\text{Real form}}{\hookrightarrow} D = G/K\,.$$

The ideas that we discuss here are *how to analyze representations of* G, G^c, *and* H *via analytic continuation to open domains in* $M_\mathbb{C}$ *or by restriction to a real form.* The main tools are *involutive representations and positive definite kernels.* This can be expressed by the following simple diagram:

$$\begin{tikzcd}
\begin{array}{c}\text{Unitary}\\ \text{highest weight}\\ \text{representations}\\ \text{of } G\end{array} \arrow[rr, leftrightarrow] \arrow[dd, leftrightarrow] & & \begin{array}{c}\text{Generalized}\\ \text{principal series}\\ \text{representations}\\ \text{of } G^c\end{array} \arrow[dd, leftrightarrow] \\
& \begin{array}{c}\text{Reflection positivity}\\ H\text{-spherical characters}\\ \text{and spherical}\\ \text{functions}\end{array} & \\
\begin{array}{c}\text{Restriction of}\\ \text{holomorphic functions}\\ \text{The Segal-Barmann}\\ \text{transform}\end{array} \arrow[dr] & & \begin{array}{c}\text{The subspace of}\\ \text{functions with support}\\ \text{in the open}\\ \text{orbit } HAN\end{array} \arrow[dl] \\
& \begin{array}{c}\text{Representations of}\\ H\text{, Canonical}\\ \text{representations}\end{array} &
\end{tikzcd}$$

Most of the ideas discussed here have been explained before in [**30, 31, 49, 51**]. We would in particular like to point to [**31**] for discussion on reflection positivity and highest weight representations. Several other people have been working on similar projects. We would like to point out here the following papers and preprints [**1, 2, 7, 8, 19, 41, 43, 56, 64, 68**].

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2. Unitary highest weight representations

We use the same notation and assumptions as in the introduction. In particular we assume that $G \subset G_{\mathbb{C}}$ is a Hermitian group, and $G_{\mathbb{C}}$ is simply connected. Thus $D = G/K$ is a bounded symmetric domain. The complex structure on D corresponds to an element $Z^0 \in \mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(Z^0)$ has eigenvalues $0, i, -i$. The eigenspace corresponding to 0 is $\mathfrak{k}_{\mathbb{C}}$, and we denote by $\mathfrak{p}^+$, respectively $\mathfrak{p}^-$ the eigenspace corresponding to i respectively $-i$. Then both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$ and

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^- .$$

Let $K_{\mathbb{C}} := \exp(\mathfrak{k}_{\mathbb{C}})$, and $P^{\pm} := \exp(\mathfrak{p}^{\pm})$. The restriction of the exponential map is an isomorphism of $\mathfrak{p}^{\pm}$ onto $P^{\pm}$. The set $P^+K_{\mathbb{C}}P^-$ is open and dense in $G_{\mathbb{C}}$. Furthermore multiplication induces a diffeomorphism

$$P^+ \times K_{\mathbb{C}} \times P^- \ni (p,k,q) \mapsto pkq \in P^+K_{\mathbb{C}}P^- \subset G_{\mathbb{C}} .$$

We denote the inverse map by $x \mapsto (p(x), k(x), q(x))$. The Harish-Chandra bounded realization of D is given by

$$G/K \ni gK \mapsto \log(p(g)) \in \Omega_{\mathbb{C}} \subset \mathfrak{p}^+ \tag{2.1}$$

and $\Omega_{\mathbb{C}}$ is a bounded symmetric domain in $\mathfrak{p}^+$. Let $(\pi, \mathbf{H})$ be a representation of G in a Hausdorff, locally convex complete topological vector space $\mathbf{H}$, and let L be a closed subgroup of G. A vector $\mathbf{v} \in \mathbf{H}$ is called L-finite if $\pi(L)\mathbf{v}$ spans a finite dimensional subspace of $\mathbf{H}$. We call $\mathbf{v}$ smooth or analytic if for all $X \in \mathfrak{g}$ the map

$$\mathbb{R} \ni t \mapsto \pi(\exp(tX))\mathbf{v} \in \mathbf{H}$$

is smooth, or analytic respectively. We denote by $\mathbf{H}_L$, $\mathbf{H}^\infty$, $\mathbf{H}^\omega$ the space of L-finite, smooth, respectively analytic vectors in $\mathbf{H}$. Define a representation of $\mathfrak{g}$ on $\mathbf{H}^\infty$ by

$$d\pi(X)\mathbf{u} := \lim_{t\to 0} \frac{\pi(\exp tX)\mathbf{u} - \mathbf{u}}{t}, \quad \mathbf{u} \in \mathbf{H}^\infty .$$

We extend $d\pi$ by linearity to a representation of $\mathfrak{g}_{\mathbb{C}}$ and then of $U(\mathfrak{g})$, *the universal enveloping algebra of* $\mathfrak{g}$. The representations of G that we are mainly interested in are the *unitary highest weight representations* of G (see [**5, 6, 9, 10, 15, 16, 22, 27, 42, 61, 65, 67**] for further information.) Let $(\pi, \mathbf{H})$ be an admissible representation of G in a Banach space, and assume that the center of $U(\mathfrak{g})$ acts by scalars. Then $\mathbf{H}_K \subset \mathbf{H}^\omega$ and $\mathbf{H}_K$ is an $(U(\mathfrak{g}), K)$-module in the sense that it is both an $U(\mathfrak{g})$ and a K-module such that

$$X \cdot (k \cdot \mathbf{u}) = (\mathrm{Ad}(k)X) \cdot \mathbf{u}, \quad \forall k \in K, X \in \mathfrak{g}, \mathbf{u} \in \mathbf{H}_K .$$

We say that an $(U(\mathfrak{g}), K)$-module $(\pi, \mathbf{H})$ is *admissible* if the multiplicity of each irreducible representation of K in $\mathbf{H}$ is finite. Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g}$ containing Z^0. Then $\mathfrak{t} \subset \mathfrak{z}_{\mathfrak{g}}(Z^0) \subset \mathfrak{k}$ so $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$.

Definition 2.1. — *Let* $\mathbf{H}$ *be an* $(U(\mathfrak{g}), K)$*-module. Then* π *is called a highest weight representation if there exists a Borel subalgebra* $\mathfrak{p} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{u} \subset \mathfrak{g}_{\mathbb{C}}$, $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, *and* $\mathbf{v} \in \mathbf{H}$ *such that the following holds:*

1. $X \cdot \mathbf{v} = \lambda(X)\mathbf{v}$ *for all* $X \in \mathfrak{t}$*;*
2. $\pi(\mathfrak{u})\mathbf{v} = 0$*;*
3. $U(\mathfrak{g})\mathbf{v} = \mathbf{H}$.

The element $\mathbf{v}$ *is called* primitive element of weight λ.

All the irreducible unitary highest weight representations of G can be constructed in a space of holomorphic functions on D for an appropriate choice of Z^0 or, which is the same, complex structure on D. For that let π be an irreducible representation of K. Let Δ denote the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. A root α is called *compact* if $\mathfrak{g}_{\mathbb{C}\alpha} \subset \mathfrak{k}_{\mathbb{C}}$. Otherwise α is called *non-compact*. Let Δ_c respectively Δ_n be the set of compact respectively non-compact roots. We choose the set Δ^+ of positive roots such that $\Delta_c^+ := \Delta^+ \cap \Delta_c$ is a system of positive roots for Δ_c and $\Delta_n^+ := \Delta^+ \cap \Delta_n = \{\alpha \in \Delta \mid \mathfrak{g}_{\mathbb{C}\alpha} \subset \mathfrak{p}^+\}$. Choose $H_\alpha \in i\mathfrak{t}$, such that $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(H_\alpha) = 2$. Let $W_K = W(\Delta_c) \subset W = W(\Delta)$ be the Weyl group generated by the reflections $s_\alpha(X) = X - \alpha(X)H_\alpha$ for $\alpha \in \Delta_c$ respectively $\alpha \in \Delta$. We denote the corresponding reflection on $i\mathfrak{t}^*$ by the same letter, i.e, $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. Then Δ_n^+ is invariant under W_K. Let $\sigma : G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the conjugation with respect to G. We sometimes write W or $\bar{g}$ for $\sigma(W)$, respectively $\sigma(g)$. We will usually use capital letters for the elements of the Lie algebra $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ except where we are viewing them as complex variables or elements in $\Omega_{\mathbb{C}}$.

Let $(\pi, \mathbf{V})$ be an irreducible representation of K with highest weight $\mu = \mu(\pi) \in i\mathfrak{t}^*$. For $z, v, w \in \mathfrak{p}^+$ and $g \in G_{\mathbb{C}}$ such that $g\exp(z), \exp(-w)\exp(v) \in P^+K_{\mathbb{C}}P^-$, let

$$g \cdot z := \log(p(g \exp z)) \tag{2.2}$$

$$J(g, z) := k(g \exp z)\,, \text{ and} \tag{2.3}$$

$$\kappa(v, w) := k(\exp(-\overline{w})\exp(v)). \tag{2.4}$$

Then the isomorphism in (2.1) intertwines the natural G-action on D with the action $(g, z) \mapsto g \cdot z$ of G on $\Omega_{\mathbb{C}}$. The function $J(g, z)$ is called *the universal factor of automorphy.* Finally we define

$$J_\pi(g, z) := \pi(J(g, z)) \quad \text{and} \quad K_\pi(z, w) := \pi(\kappa(z, w))^{-1}\,. \tag{2.5}$$

Then $z, w \mapsto K_\pi(z, w)$ is holomorphic in the first variable and anti-holomorphic in the second variable.

Let $S(\Omega_{\mathbb{C}}) := \left\{g \in G_{\mathbb{C}} \mid g^{-1} \cdot \Omega_{\mathbb{C}} \subset \Omega_{\mathbb{C}}\right\}$. Then $S(\Omega_{\mathbb{C}})$ is a maximal closed semigroup in $G_{\mathbb{C}}$ and there exists a maximal closed and convex G-invariant cone $W_{\max} \subset \mathfrak{g}$ such that $W_{\max} \cap -W_{\max} = \{0\}$ (pointed), $W_{\max} - W_{\max} = \mathfrak{g}$ (generating), and (see [**21, 23, 44**])

$$S(\Omega_{\mathbb{C}}) = G\exp(iW_{\max})\,.$$

Let $W_{\min} = \{X \in \mathfrak{g} \mid \forall Y \in W_{\max} : -B(X, \theta(Y)) \geq 0\}$, where B stands for the Killing form. Then $W_{\min}$ is a minimal G-invariant pointed and generating cone in $\mathfrak{g}$. Define $g^* = \sigma(g)^{-1}$. If $s = g\exp(iW) \in S(\Omega_{\mathbb{C}})$ then

$$s^* = \exp(iW)g^{-1} = g^{-1}\exp(i\mathrm{Ad}(g)W) \in S(\Omega_{\mathbb{C}})\,.$$

We notice the following well known and important lemma.

Lemma 2.2. — *Let $s, s_1, s_2 \in S(\Omega_{\mathbb{C}})$ and let $w, v \in \Omega_{\mathbb{C}}$. Then the following holds:*

1. $(s_1 s_2) \cdot v = s_1 \cdot (s_2 \cdot v)$;
2. $J(s_1 s_2, v) = J(s_1, s_2 \cdot v) J(s_2, v)$;
3. $J_\pi(s_1 s_2, v) = J_\pi(s_1, s_2 \cdot v) J_\pi(s_2, v)$;
4. $\overline{J(s^*, w)} \kappa(s \cdot v, w) J(s, v) = \kappa(v, s^* \cdot w)$;
5. $J_\pi(s, v)^{-1} K_\pi(s \cdot v, w) = K_\pi(v, s^* \cdot w) \left(J_\pi(s^*, w)^*\right)^{-1}$.

Proof. — 1) and 2) We have, with q_1, q_2 and q_3, denoting elements in P^-,

$$\begin{aligned} s_1 s_2 \exp(v) &= s_1 \exp(s_2 \cdot v) k(s_2 \exp v) q_1 \\ &= \exp(s_1 \cdot (s_2 \cdot v)) k(s_1 \exp(s_2 \cdot v)) q_2 k(s_2 \exp v) q_1 \\ &= \exp(s_1 \cdot (s_2 \cdot v)) k(s_1 \exp(s_2 \cdot v)) k(s_2 \exp v) q_3 \end{aligned}$$

where we have used that $K_{\mathbb{C}}$ normalizes $P^{\pm}$.

3) Follows from (2).

4) By the defining relation

$$s^* \exp(w) = \exp(s^* \cdot w) J(s^*, w) q_1$$

we get $\exp(-\overline{s^* \cdot w}) = \overline{q_2} \overline{J(s^*, w)} \exp(-w) s$. Hence

$$\begin{aligned} \exp(-\overline{s^* \cdot w}) \exp(v) &= \overline{q_2} \overline{J(s^*, w)} \exp(-w) s \exp(v) \\ &= \overline{q_2} \overline{J(s^*, w)} \exp(-w) \exp(s \cdot v) J(s, v) q_3 \,. \end{aligned}$$

Hence the claim follows.

5) This follows from (4) using that $\pi(g^*) = \pi(g)^*$. □

We notice that all of those relations can be lifted to $\tilde{G}$. We will use this fact without further comments. Let $(\pi, \mathbf{V})$ be an irreducible representation of $\tilde{K}$. Let $\mathcal{O}(\Omega_{\mathbb{C}}, \mathbf{V})$ be the space of $\mathbf{V}$-valued holomorphic functions on $\Omega_{\mathbb{C}}$. Define a representation ρ_π of $\tilde{G}$ on $\mathcal{O}(\Omega_{\mathbb{C}}, \mathbf{V})$ by

$$\rho_\pi(g) f(z) := J_\pi(g^{-1}, z)^{-1} f(g^{-1} \cdot z) \,.$$

Then ρ_π is a representation of $\tilde{G}$ by Lemma 2.2, part 3. Let $(\rho, \mathbf{H})$ be an irreducible unitary highest weight module with lowest weight π, then we can choose $\mathfrak{p}^+$ such that $\{0\} \neq \mathbf{H}^{\mathfrak{p}^+} \simeq_K \mathbf{V}$. Furthermore there is an injective map $\mathbf{H} \hookrightarrow \mathcal{O}(\Omega_{\mathbb{C}}, \mathbf{H}^{\mathfrak{p}^+})$ intertwining ρ and ρ_π, see [**5**]. Here are the main ideas in the proof. Let $\sigma : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be the conjugation with respect to $\mathfrak{g}$. For $w \in \mathfrak{p}^+$ define $q_w : \mathbf{H}^{\mathfrak{p}^+} \to \mathbf{H}$ by

$$q_w(\mathbf{v}) := \sum_{n=0}^{\infty} \frac{\rho_\pi(\sigma(w))^n \mathbf{v}}{n!} \,.$$

If $\mathbf{v} \neq 0$ the series defining $q_w(\mathbf{v})$ converges if and only if $w \in \Omega_{\mathbb{C}}$. Define $U : \mathbf{H} \to \mathcal{O}(\Omega_{\mathbb{C}}, \mathbf{H}^{\mathfrak{p}^+})$ by

$$(U(\mathbf{w}))(z) := q_z^*(\mathbf{w}) \,.$$

Then U intertwines the representation ρ and ρ_π and gives the geometric realization of ρ. One should remark, that even if the classification of unitary highest weight modules is known, it is still an open problem to describe *analytically* the unitary structure for ρ_π in general.

Let $\lambda_0 \in i\mathfrak{z}(\mathfrak{k})^*$ such that $\lambda_0(-iZ^0) = 1$. Let $\mu(\pi)$ be the highest weight of π. Then $\mu(\pi) = \mu_0 + r\lambda_0$ where $\mu_0 \in i(\mathfrak{k}\cap\mathfrak{k}')^*$ is Δ_c^+ dominant. Consider the affine line $L(\mu_0) = \{\mu_0 + t\lambda_0 | t \in \mathbb{R}\}$. The following is due to Jakobsen, Enright, Howe and Wallach:

Theorem 2.3. — *Let the notation be as above. Then there exists constants $a(\mu_0) \leq 0$ and $c(\mu_0) > 0$ such that the Representation ρ_π is unitary if and only if*

$$\mu(\pi) \in \{\mu_0 + r\lambda_0 \mid r < a(\mu_0)\} \,\dot\cup\, \{\mu_0 + z_j\lambda_0 \mid j = 0, \ldots, n\}$$

with $z_0 = a(\mu_0)$ and $z_{j+1} - z_j = c(\mu_0)$.

Write $U(\mu_0) = \{\mu_0 + r\lambda_0 \mid r < a(\mu_0)\} = \mu_0 + (-\infty, a(\mu_0))\lambda_0$.

Theorem 2.4. — *Assume that $(\rho_\pi, \mathbf{H}(\rho_\pi))$ is unitary. Then all the polynomials on $\mathfrak{p}^+$ are in $\mathbf{H}(\rho_\pi)$ if and only if $r < a(\mu_0)$. In that case the space $\mathbf{H}(\rho_\pi)_K \simeq U(\mathfrak{p}^-) \times_K \mathbf{V}$ is exactly the space of polynomial functions on $\mathfrak{p}^+$.*

If $\mu = t\lambda_0$, i.e., π is a character, then $K_\pi(\cdot, w) \in \mathbf{H}(\rho_\pi)^\omega$ and

$$(f, K_w) = f(w)$$

for all $f \in \mathbf{H}(\rho_\pi)$.

***Theorem 2.5* (Harish-Chandra)**. — *Let $\rho(\Delta^+) = 1/2\sum_{\alpha\in\Delta^+}\alpha$. The representation $(\rho_\pi, \mathbf{H}(\rho_\pi))$ is isomorphic to a direct summand in $L^2(G)$ if and only ρ_π defines a representation of G and*

$$\langle \mu(\pi) + \rho, \alpha\rangle < 0 \quad \forall \alpha \in \Delta_n^+ .$$

The highest weight representations in $L^2(G)$ are called *the holomorphic discrete series of G*. In this case the inner product on $\mathbf{H}(\pi)$ is given by

$$(f, g) = c\int_{\Omega_\mathbb{C}} (f(z), K(z,z)^{-1}g(z))\, d\mu(z)$$

where μ is a G-invariant measure on $\Omega_\mathbb{C}$.

We also have the following, see [**24, 51, 52**]:

Theorem 2.6. — *Let $\mathfrak{a}_q = \mathfrak{t}\cap\mathfrak{q}$ and assume that $\mathfrak{a}_q$ is maximal abelian in $\mathfrak{q}\cap\mathfrak{k}$. Let $\Delta(\mathfrak{p}^+, \mathfrak{a}_q) = \{\alpha|\mathfrak{a}_q \mid \alpha \in \Delta_n^+\}$. Then $(\rho_\pi, \mathbf{H}(\rho_\pi))$ is isomorphic to a direct summand in $L^2(G/H)$ if and only if ρ_π defines a representation of G, $\mathbf{V}^{K\cap H} \neq \{0\}$ (and thus $\mu(\pi) \in i\mathfrak{a}_q^*$), and*

$$\langle \mu(\pi) + \rho|\mathfrak{a}_q, \alpha\rangle < 0\, , \quad \forall \alpha \in \Delta(\mathfrak{p}^+, \mathfrak{a}_q)\, .$$

The highest weight representations in $L^2(G/H)$ are called *the holomorphic discrete series of G/H*.

We notice that there are holomorphic discrete series of G/H that are not in the holomorphic discrete series of G. This is due to the so-called ρ_ℓ-shift, see [**51**] for further discussion and examples.

The highest weight representations are closely related to *holomorphic representations* of semi-groups of the form $G\exp(iW)$, see [**22, 23, 24, 42, 57**]. Let π be a unitary representation of G in a Hilbert space $\mathbf{H}(\pi)$. Let

$$W(\pi) := \{X \in \mathfrak{g} \mid \forall \mathbf{u} \in \mathbf{H}(\pi)^\infty : (id\pi(X)\mathbf{u}, \mathbf{u}) \leq 0\} \; .$$

Then $W(\pi)$ is a G-invariant closed convex cone in $\mathfrak{g}$. If $W(\pi)$ is pointed and generating, then $G\exp(iW(\pi)) = S(W(\pi))$ is a closed semi-group with interior $S(W(\pi))^o = G\exp(iW(\pi)^o) = S(W(\pi)^o)$.

Theorem 2.7. — *Let ρ_π be a unitary highest weight representation. Then $W(\pi)$ is pointed and generating and ρ_π extends to an involutive, holomorphic, and contractive representation of the semi-group $S(W(\pi)) := G\exp(iW(\pi))$. In particular we have for all $s \in S(W(\pi))$:*

1. $\rho_\pi(s)^* = \rho_\pi(s^*)$.
2. $||\rho_\pi(s)|| \leq 1$ *for all $s \in S(W(\pi))$.*

We also notice the following due to K-H. Neeb and Olshanskii, see [**42**]:

Theorem 2.8. — *Let π be a unitary representation of G. If $W(\pi)$ is pointed and generating, then π is a direct integral of highest weight representations.*

We will in the following mainly work with characters, i.e., $d(\pi) = \dim(\mathbf{H}(\pi)) = 1$. Each character corresponds to a element $\mu \in i\mathfrak{z}_\mathfrak{k}$ such that

$$\chi_\mu(\exp Z) = e^{\mu(Z)} \, .$$

If we replace G with the universal covering group $\tilde{G} \simeq \tilde{K} \times \mathfrak{p}$, where $\tilde{K}$ is the universal covering group of K, then each $\mu \in i\mathfrak{z}_\mathfrak{k}$ gives rise to a character of $\tilde{K}$ and its complexification $\tilde{K}_\mathbb{C}$. For K itself we need to assume that $\mu(Z) \in 2\pi i\mathbb{Z}$ for $Z \in \exp^{-1}\{e\}$. If $\pi = \chi_\mu$ is a character, then we write J_μ, K_μ, etc.

3. The Restriction Principle and the Segal-Bargmann Transform

Let us start by recalling the *restriction principle* put forward in [**54**]. Let $M_\mathbb{C}$ be a connected complex manifold and let $M \subset M_\mathbb{C}$ be a *totally real submanifold* thus locally the inclusion $M \hookrightarrow M_\mathbb{C}$ is the same as $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$ and if F is a holomorphic function on $M_\mathbb{C}$ such that the restriction $F|M = 0$ then it follows that $F = 0$. Let $\mathbf{F}$ be a Hilbert space of holomorphic functions $F : M_\mathbb{C} \to \mathbb{C}$. (We can also consider vector valued functions or even sections of a holomorphic vector bundle over $M_\mathbb{C}$.) We assume that $\mathbf{F}$ is a *reproducing Hilbert space*, that is the evaluation maps $F \mapsto F(w)$ are all continuous on $\mathbf{F}$. This implies the existence of a $K_w \in \mathbf{F}$ such that

$$F(w) = (F, K_w)$$

for all $F \in \mathbf{F}$ and all $w \in M_{\mathbb{C}}$. We notice that $K(z,w) := K_w(z)$ is holomorphic in z and anti-holomorphic in w. The function $K(z,w)$ is the *reproducing kernel for* $\mathbf{F}$. We have

1. $K(z,w) = (K_w, K_z) = \overline{(K_z, K_w)} = \overline{K(w,z)}\,.$
2. $||K_w||^2 = K(w,w)$.

Let μ be a measure on M and let $D : M \to \mathbb{C}^*$. Assume that $m \mapsto D(m)F(m)$ is in $L^2(M,\mu)$ for all $F \in \mathbf{F}$. Define the *restriction map* $R : \mathbf{F} \to L^2(M,\mu)$ by

$$[RF](m) := D(m)F(m)\,.$$

Then R is injective. If G is a Lie group and H is a closed subgroup such that G acts on $M_{\mathbb{C}}$, H acts on M, and $\mathbf{F}$ is a unitary G-module, then it is natural to assume that μ is H invariant, so that $L^2(M,\mu)$ is a unitary H-module. We would then determine D such that R is an H-morphism. Assume that $\mathrm{Im}\,(R)$ is dense. Then $R^* : L^2(M,\mu) \to \mathbf{F}$ is injective and

$$\begin{aligned} R^*h(z) &= (R^*h, K_z) \\ &= (h, RK_z) \\ &= \int h(m)\overline{D(m)}K(z,m)\,d\mu\,. \end{aligned}$$

Thus we have

$$RR^*h(x) = \int h(m)D(x)\overline{D(m)}K(x,m)\,d\mu\,. \tag{3.1}$$

Now polarize R^* to get $R^* = U\,|R^*|$. Then $U : L^2(M,\mu) \to \mathbf{F}$ is a unitary isomorphism.

Definition 3.1. — *The map* $U : L^2(M,\mu) \to \mathbf{F}$ *is called the* generalized Segal-Bargmann transform*.*

We have $U^*R^* = |R^*|$ and hence $RU = |R^*|$. Thus for $x \in M$ and $h \in L^2(M,\mu)$ we get

$$Uh(x) = D(x)^{-1}\,|R^*|\,h(x)\,.$$

To find an explicit expression for the Segal-Bargmann transform we need to take the square root of RR^* in (3.1). This can be done in some special cases. Let us start with the classical Segal-Bargmann transform. Let $M_{\mathbb{C}} = \mathbb{C}^n$ and $M = \mathbb{R}^n$. Let $\mathbf{F}$ be the classical Fock-space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$||F||^2 := \pi^{-n}\int |F(z)|^2\,e^{-|z|^2}\,dxdy < \infty\,.$$

Then $\mathbf{F}$ is a reproducing Hilbert space with inner product

$$(F, G) = \pi^{-n} \int F(z)\overline{G(z)}\, e^{-|z|^2}\, dxdy$$

and reproducing kernel $K(z, w) = e^{z\cdot w}$ where $z \cdot w = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. Let $D(x) := e^{-|x|^2/2}$. Then the restriction map R becomes

$$RF(x) = e^{-|x|^2/2} F(x)\,.$$

The holomorphic polynomials $P(z) = \sum a_\alpha z^\alpha$ are dense in $\mathbf{F}$ and obviously $RP \in L^2(\mathbb{R}^n)$. Hence all the Hermite functions $h_\alpha(x) = (-1)^{|\alpha|}\left(D^\alpha e^{-|x|^2}\right) e^{|x|^2/2}$ are in the image of R so $\mathrm{Im}(R)$ is dense. For $z, w \in \mathbb{C}^n$ let $(z, w) = \sum z_j w_j$. We then have

$$\begin{aligned} R^*g(z) &= \int g(y)e^{-|y|^2/2}e^{z\cdot y}\, dy \\ &= e^{(z,z)/2}\int g(y)e^{-(z-y,z-y)/2}\, dy \\ &= e^{(z,z)/2} g * p(z) \end{aligned}$$

where $p(z) = e^{-(z,z)/2}$ is holomorphic. It follows that

$$RR^*g(x) = g * p(x)\,. \tag{3.2}$$

As $p \in L^2(\mathbb{R})$ it follows in particular that $||RR^*|| \leq ||p||_2$, so RR^* is continuous, and

$$(R^*g, R^*g) = (RR^*g, g) \leq ||RR^*||\ ||g||_2^2\ .$$

Thus we have the lemma:

Lemma 3.2. — *R^* is continuous.*

Let $p_t(x) = (2\pi t)^{-n/2}e^{-(x,x)/2t}$ be the heat kernel on $\mathbb{R}^n$. Then $(p_t)_{t>0}$ is a convolution semi-group and $p = (2\pi)^{n/2}p_1$. Hence $\sqrt{RR^*} = (2\pi)^{n/4}p_{1/2}*$ or

$$RUg(x) = |R^*|\, g(x) = (2\pi)^{n/4} p_{1/2} * g(x) = 2^{n/4}\pi^{-n/4}\int g(y)e^{-(x-y,x-y)}\, dy\,.$$

As $RUg(x) = e^{-(x,x)/2}U(g)(x)$ it follows that

$$Ug(x) = (2/\pi)^{n/4}e^{(x,x)/2}\int g(y)e^{-(x-y,x-y)}\, dy$$

for $x \in \mathbb{R}^n$. But the function on the right hand side is holomorphic in x. By analytic continuation we get the following theorem.

Theorem 3.3. — *The map $U : L^2(\mathbb{R}^n) \to \mathbf{F}$ given by*

$$Ug(z) = (2/\pi)^{n/4}\int g(y)\exp(-(y, y) + 2(z, y) - (z, z)/2)\, dy$$

is a unitary isomorphism.

Next we consider the case where $M_{\mathbb{C}} = \Omega_{\mathbb{C}}$ is a bounded symmetric domain of the form G/K and the corresponding real forms $H/H \cap K$ (see [**23, 25, 26, 34, 46, 47**] for classifications, structure theory and further information). Let $\eta : \Omega_{\mathbb{C}} \to \Omega_{\mathbb{C}}$ be an anti-holomorphic involution fixing the point $0 = \{K\} \in \Omega_{\mathbb{C}}$. Then $\Omega = \Omega_{\mathbb{C}}^{\eta}$ is a real form of $\Omega_{\mathbb{C}}$. The group G is locally isomorphic to the group $I(\Omega_{\mathbb{C}})_o$ of holomorphic isomorphism of $\Omega_{\mathbb{C}}$. Define $\tau : I(\Omega_{\mathbb{C}})_o \to I(\Omega_{\mathbb{C}})_o$ by $\tau(f)(z) = \eta(f(\eta(z)))$. Then τ is an involution commuting with the Cartan involution θ with $K = G^{\theta}$. Lift τ to the Lie algebra of $I(\Omega_{\mathbb{C}})_o$ which is isomorphic to $\mathfrak{g}$ and then extend that involution – also denoted by τ – to an involution on $\mathfrak{g}_{\mathbb{C}}$. As $G_{\mathbb{C}}$ is simply connected it follows that τ defines an involution on $G_{\mathbb{C}}$ leaving G invariant. Let $H = G^{\tau} \subset H_{\mathbb{C}} = G_{\mathbb{C}}^{\tau}$ and notice that $H_{\mathbb{C}}$ is connected as $G_{\mathbb{C}}$ is simply connected. We have that $\Omega = H/H \cap K = H_o/H_o \cap K$. The classification of these spaces is given by the following table:

$\mathfrak{g}^c$ with complex structure

$\mathfrak{g}^c = \mathfrak{h}_{\mathbb{C}}$ Dual	$\mathfrak{g} = \mathfrak{h} \times \mathfrak{h}$ Hermitian	$\mathfrak{h}$
$\mathfrak{sl}(p+q, \mathbb{C})$	$\mathfrak{su}(p,q) \times \mathfrak{su}(p,q)$	$\mathfrak{su}(p,q)$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}^*(2n) \times \mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n)$
$\mathfrak{so}(n+2, \mathbb{C})$	$\mathfrak{so}(2,n) \times \mathfrak{so}(2,n)$	$\mathfrak{so}(2,n)$
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R}) \times \mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{e}_6$	$\mathfrak{e}_{6(-14)} \times \mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-14)}$
$\mathfrak{e}_7$	$\mathfrak{e}_{7(-25)} \times \mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$

$\mathfrak{g}^c$ without complex structure

$\mathfrak{g}^c$ Dual	$\mathfrak{g}$ Hermitian	$\mathfrak{h}$
$\mathfrak{sl}(p+q, \mathbb{R})$	$\mathfrak{su}(p,q)$	$\mathfrak{so}(p,q)$
$\mathfrak{su}(n,n)$	$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C}) \times \mathbb{R}$
$\mathfrak{su}^*(2(p+q))$	$\mathfrak{su}(2p,2q)$	$\mathfrak{sp}(p,q)$
$\mathfrak{so}(n,n)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{so}^*(4n)$	$\mathfrak{so}^*(4n)$	$\mathfrak{su}^*(2n) \times \mathbb{R}$
$\mathfrak{so}(p+1,q+1)$	$\mathfrak{so}(2,p+q)$	$\mathfrak{so}(p,1) \times \mathfrak{so}(1,q)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R}) \times \mathbb{R}$
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{e}_{6(6)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{sp}(2,2)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \times \mathbb{R}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{su}^*(8)$

We notice that in the first table the symmetric space Ω is a complex bounded symmetric domain, $\Omega_{\mathbb{C}} = \Omega \times \overline{\Omega}$, where $\overline{}$ stands for "opposite complex structure", and the imbedding of Ω into $\Omega_{\mathbb{C}}$ is the diagonal imbedding $\omega \mapsto (\omega, \overline{\omega})$. The holomorphic functions on $\Omega_{\mathbb{C}}$ are then the functions $f(z,w)$ which are holomorphic in the first variable and anti-holomorphic in the second variable. Some of those cases were treated in detail in [**56**]. We note that all the classical irreducible Riemannian symmetric spaces (with a possible extension by $\mathbb{R}$) show up in the third column in this list. The spaces that are missing are: $\mathrm{E}_{6(2)} / \mathrm{SU}(6) \times \mathrm{SU}(2)$, $\mathrm{E}_{6(6)} / \mathrm{Sp}(4)$, $\mathrm{E}_{7(7)} / \mathrm{SU}(8)$, $\mathrm{E}_{7(-5)} / \mathrm{SO}(12) \times \mathrm{SU}(2)$, $\mathrm{E}_{8(8)} / \mathrm{SO}(16)$, $\mathrm{E}_{8(-24)} / \mathrm{E}_7 \times \mathrm{SU}(2)$, $\mathrm{F}_{4(4)} / \mathrm{Sp}(3) \times \mathrm{SU}(2)$, $\mathrm{G}_{2(2)} / \mathrm{SU}(2) \times \mathrm{SU}(2)$. It is interesting that this list contains all the quaternions exceptional Riemannian symmetric spaces.

Our aim is to use the generalized Bargmann transform to analyze the representation $\rho_\pi | H_o$ where ρ_π is a unitary highest weight representation with minimal K-type π acting on the finite dimensional Hilbert space $\mathbf{V}(\pi)$. In this case we let

$$D_\pi(h) = J_\pi(h,0)^{-1} = \pi(k(h))^{-1}$$

for all $h \in P^+ K_{\mathbb{C}} P^-$. Then $D_\pi(gk) = \pi(k)^{-1} D_\pi(g)$. Assume for the moment that $||D_\pi|| \in L^2(\Omega, dm)$, where dm is the H-invariant measure on Ω given by

$$\int_\Omega f(x) dm(x) = \int_H f(h \cdot 0)\, dh \,.$$

This can be made precise using the root structure of $H/H \cap K$ and G/K. Define

$$RF(h) = D_\pi(h) F(h \cdot 0)\,, \quad F \in \mathbf{H}(\rho_\pi)\,.$$

Let $\mathcal{V}(\pi) \to \Omega$ be the vector bundle

$$H \times_{\pi|H\cap K} \mathbf{V}(\pi) \to \Omega\,.$$

Then $RF \in L^2(\mathcal{V}(\pi))$ for all $F \in \mathbf{H}(\rho_\pi)$.

Lemma 3.4. — *Assume that $D_\pi \in L^2(\Omega)$ and that $\pi \in U(\mu_0)$. Then the restriction map $R : \mathbf{H}(\rho_\pi) \to L^2(\Omega, dm)$ is injective and intertwines $\rho_\pi | H$ and the left regular action λ of H on $L^2(\mathcal{V}(\pi))$. Furthermore $Im(R)$ is dense in $L^2(\mathcal{V}(\pi))$.*

Proof. — Let $h \in H$ and $F \in \mathbf{H}(\rho_\pi)$. Let $a \in H$. Then

$$\begin{aligned} R(\rho_\pi(h)F)(a) &= D_\pi(a)[\rho_\pi(h)F](a) \\ &= D_\pi(a) J_\pi(h^{-1}, a \cdot 0)^{-1} F(h^{-1} a \cdot 0) \\ &= J_\pi(a,0)^{-1} J_\pi(h^{-1}, a \cdot 0)^{-1} F(h^{-1} a \cdot 0) \\ &= J_\pi(h^{-1}a, 0)^{-1} F(h^{-1} a \cdot 0) \\ &= D_\pi(h^{-1}a) F(h^{-1} a \cdot 0) \\ &= \lambda(h) RF(a)\,. \end{aligned}$$

Assume that $RF = 0$. Then $D_\pi(a) F(a \cdot 0) = 0$ for all $a \in H$. As $D_\pi(a)$ is regular it follows that $F(a \cdot 0) = 0$. But then $F|\Omega = 0$ and hence $F = 0$.

Let $f \in L^2(\mathcal{V}(\pi))$ and $\varepsilon > 0$. Let g be a compactly supported section such that $||f-g||_2 < \varepsilon/2$. Then the function $h \mapsto D_\pi(h)^{-1}g(h)$ is $K \cap H$ invariant and can therefore be viewed as a $\mathbf{V}(\pi)$-valued function on the compact set $\mathrm{cl}(\Omega)$, the closure of Ω. Let $p : \Omega \to \mathbf{V}(\pi)$ be a polynomial such that

$$\sup_{z \in \mathrm{cl}(\Omega)} \left|\left|p(x) - D_\pi(x)^{-1}g(x)\right|\right| < \frac{\sqrt{\varepsilon}}{2\,||D_\pi||_2}.$$

Then

$$||f - Rp|| \leq ||f-g|| + ||g - D_\pi p||$$

But

$$\begin{aligned} ||g - D_\pi p||^2 &= \int_H ||g(h) - D_\pi(h)p(h\cdot 0)||^2\, dh \\ &= \int ||D_\pi(h)||^2 \left|\left|D_\pi(h)^{-1}g(h) - p(h\cdot 0)\right|\right|^2 dh \\ &\leq \varepsilon^2/4\,. \end{aligned}$$

Hence $||f - Rp||_2 < \varepsilon$. ☐

Polarizing R^* gives us now a unitary H-isomorphism $U : L^2(\mathcal{V}(\pi)) \to \mathbf{H}(\rho_\pi)$. Hence

Theorem 3.5. — *Assume that $D_\pi \in L^2(\mathcal{V}(\pi))$ and that $\mu \in U(\mu_0)$. Then $\rho_\pi|H$ is unitary equivalent to the representation of H on $L^2(\mathcal{V}(\pi))$.*

Let us remark here, that in the case where $\Omega_{\mathbb{C}} = \Omega \times \bar{\Omega}$, i.e., $G = H \times H$, then each of the highest weight representations ρ_π is of the form $\rho_H \otimes \bar{\rho}_H$ where ρ_H is a highest weight representation of H and $\bar{\rho}_H$ is the conjugate (or dual) representation. The above result therefore tells us, that $\rho_H \otimes \bar{\rho}_H \simeq L^2(\mathcal{V}(\pi_H))$ where π_H is the minimal $K \cap H$-type of ρ_H.

We will now analyze the generalized Segal-Bargmann transform in more detail for the scalar case. In that case $\chi_\mu|H_o \cap K$ is always trivial so D_π, which we will now denote by D_μ, is a function on Ω given by $D_\mu(h\cdot 0) = D_\mu(h)$, $h \in H_o$. The map R^* is given by

$$\begin{aligned} R^*f(z) &= (f, RK_z) \\ &= \int_\Omega f(y)\overline{D_\mu(y)}K(z,y)\,dm(y) \\ &= \int_{H_o} f(h\cdot 0)J_\mu(h^{-1},z)^{-1}\,dh \qquad (3.3) \end{aligned}$$

where the last equation follows from Lemma 2.2. Hence for $x = t \cdot 0$, $t \in H_o$, we get

$$\begin{aligned} RR^*f(x) &= \int_\Omega f(y)D_\mu(x)\overline{D_\mu(y)}K(x,y)\,dm(y) \\ &= \int_H f(h)J_\mu(t,0)^{-1}J_\mu(h^{-1},t\cdot 0)^{-1}\,dh \\ &= \int f(h)\,J_\mu(h^{-1}t,0)^{-1}\,dh \\ &= f * D_\mu(t)\,. \end{aligned}$$

Thus again the result is a convolution operator with an L^2-function and the generalized Segal-Bargmann transform is again given by a convolution with a function φ. Contrary to the classical case, this convolution operator does not result from a convolution semi-group. Hence the task of determining the function φ becomes much more involved. But since D_μ is a $H_o \cap K$-biinvariant function it is determined by its restriction to a maximal vector subgroup of H and can be determined by the spherical Fourier transform on $H_o/H_o \cap K$. Hence one can in "principle" find a $H_o \cap K$-biinvariant function φ on H_o such that the generalized Segal-Bargmann transform is given by:

Theorem 3.6. — *The Segal-Bargmann transform is given by*

$$Uf(z) = D_\mu(z)^{-1} f * \varphi(z)\,.$$

Here are few problems that are still unsolved or have only case by case solutions:

1. Decompose the restriction of ρ_μ for more singular parameters.
2. Find an explicit formula for φ.
3. We have a canonical orthonormal basis $\{p_I\}_I$ for $\mathbf{H}(\rho_\mu)$ given by polynomials. Is there an expression for the corresponding orthonormal bases $\{U^* p_I\}$ for $L^2(\Omega, \mu)$?
4. Work out the case where π is not assumed to be one-dimensional.

4. The Principal Series of G^c and Reflection Positivity

The material in this section follows [**30, 31**] with few modifications. We use [**23**] as standard reference for the structure theory of G^c/H. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the involution from the previous section. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and we define $\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$. Let G^c be the analytic subgroup of $G_\mathbb{C}$ corresponding to $\mathfrak{g}^c$ as before.

Lemma 4.1. — *We have $H = G \cap G^c$.*

Proof. — Let σ be the conjugation with respect to G and let σ^c be the conjugation with respect to G^c. Then $\sigma|G^c = \tau|G^c$ and $\sigma^c|G = \tau|G$. The claim follows directly from this. □

Let $P_{\max} := LAN^- := K_{\mathbb{C}}P^- \cap G^c$. Then $P_{\max}$ is a maximal parabolic subalgebra of G^c. Notice that $LA = K_{\mathbb{C}} \cap G^c$, $A = \exp(\mathbb{R}X^0)$ with $X^0 = iZ^0$, and $N^- = P^- \cap G^c$. We let $N^+ = P^+ \cap G^c = \tau(N^-) = \theta(N^-)$ We remark that things are set up so that

$$\Omega = H/H \cap K = HP_{\max}/P_{\max}\,.$$

Let us recall some facts about parabolic induction that will be used in this and next section. Let $\mathfrak{a}_q$ be the maximal abelian subspace in $\mathfrak{q} \cap \mathfrak{k}$ from Theorem 2.6. Let $\mathfrak{a}^c := i\mathfrak{a} \subset \mathfrak{q}^c \cap \mathfrak{p}^c$. Then $\mathfrak{a}^c$ is maximal abelian in $\mathfrak{q}^c \cap \mathfrak{p}^c$. Let Δ^c be the roots of $\mathfrak{a}^c$ in $\mathfrak{g}^c$ and choose a set of positive roots Δ^{c+}. Let $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}^c) = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}^c)$, and let $\mathfrak{n}_{\min} = \oplus_{\alpha \in \Delta^{c+}} \mathfrak{g}^c_\alpha$. Then $\mathfrak{p}_{\min} = \mathfrak{p}_{\min}(\Delta^{c+}) = \mathfrak{m} \oplus \mathfrak{a}^c \oplus \mathfrak{n}_{\min}$ is a minimal parabolic subalgebra of $\mathfrak{g}^c$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be any parabolic subalgebra containing $\mathfrak{p}_{\min}$ and such that $\mathfrak{a} \subset \mathfrak{a}^c$. Let $P_{\min} = MA^cN = N_G(\mathfrak{p}_{\min}) \subset P =: LAN = N_G(\mathfrak{p})$. Notice that we are using $\mathfrak{p}$ for a moment in two different ways, but the meaning should be clear in each case. Define $\rho(\mathfrak{n}) \in \mathfrak{a}^*$ by $2\rho(\mathfrak{n})(X) := \mathrm{Tr}(\mathrm{ad}(X))|\mathfrak{n}$, $X \in \mathfrak{a}$. Let $\lambda \in \mathfrak{a}^*$ and define

$$\pi_\lambda = \mathrm{Ind}_{LAN}^{G^c} 1 \otimes -\lambda \otimes 1\,.$$

The minus sign has been inserted in order to simplify some formulas later on. The Hilbert space $\mathbf{H}(\lambda)$ for π_λ is the space of measurable function $f : G^c \to \mathbb{C}$ such that

$$f(glan) = a^{\lambda - \rho(\mathfrak{n})} f(g)$$

and $\int_K |f(k)|^2\, dk < \infty$. The inner product in this space is $(f,g) = \int_K f(k)\overline{g(k)}\, dk$. Let $\bar{P} = \tau(P) = LA\bar{N}$. Define $\bar{\pi}_{-\lambda} = \mathrm{Ind}_{\bar{P}}^{G^c} 1 \otimes \lambda \otimes 1$ and $\bar{\mathbf{H}}(-\lambda)$ in the same way by replacing P by $\bar{P}$. The following result is then obtained by a simple calculation.

Lemma 4.2. — *The map $T : \bar{\mathbf{H}}(-\lambda) \to \mathbf{H}(\lambda)$, $f \mapsto f \circ \tau$ defines a G^c-isomorphism between $\bar{\pi}_{-\lambda}$ and $\pi_\lambda \circ \tau$.*

We will now specialize to the situation where $P = P_{\max}$. The set of positive roots is now chosen such that $-\Delta_n^+ \subset \Delta^{c+}$.

***Lemma 4.3* (Matsuki)**. — *The set $HP_{\max}$ is open in G^c.*

We also have:

Lemma 4.4. — *If P is any parabolic subgroup $P_{\min} \subset P \subset P_{\max}$, then $HP = HP_{\max}$.*

Let $\mathbf{K}_0(\lambda) \subset \mathbf{H}(\lambda)$ be the space of smooth functions with $\mathrm{Supp}(f) \subset HP_{\max}$ such that $f|H$ has compact support. Let

$$S(\Omega) = \left\{ g \in G^c \mid g^{-1}HP_{\max} \subset HP_{\max} \right\} = S(\Omega_{\mathbb{C}}) \cap G^c.$$

The cone $C_{\max} = \mathfrak{g}^c \cap iW_{\max}$ is a maximal H-invariant cone in $\mathfrak{q}^c = i\mathfrak{q}$ and $S(\Omega) = H\exp(C_{\max}) = S(C_{\max})$. Notice that

$$\begin{aligned} \tau(S(\Omega)) &= S(-C_{\max}) = \{g \in G^c \mid gHP_{\max} \subset HP_{\max}\} \\ &= \{g \in G^c \mid g \cdot \Omega \subset \Omega\}\,. \end{aligned}$$

Let $f \in \mathbf{K}_0(\lambda)$ and $s \in S(-C_{\max})$. Then $\pi_\lambda(s)f \in \mathbf{K}_0(\lambda)$. Write

$$x = h(x)a_H(x)n(x) \in HP_{\max}\,.$$

Then

$$\pi_\lambda(s)f(h) = a_H(s^{-1}h)^{\lambda-\rho}f(h(s^{-1}h))\,.$$

We have $HP_{\max} \subset N^+P_{\max}$. For $x \in N^+P_{\max}$ write $x = \bar{n}(x)l(x)a_{\bar{N}}(x)n_{\bar{N}}(x)$. As $X^0 = iZ^0$ and $Z_{G_\mathbb{C}}(Z^0) = K_\mathbb{C}$ it follows that our group A is just $\exp(i\mathfrak{z}_\mathfrak{k})$ and hence $a_{\bar{N}}(x)^\mu = \chi_\mu(k_c(x)) = D_\mu(x)$. We will therefore use the notation D_μ in the following. Assume that we have chosen λ such that the unitary highest weight representation $\rho_{\lambda+\rho}$ exists. Then $\Omega \times \Omega \ni (x,y) \mapsto K_{\lambda+\rho}(x,y) \in \mathbb{C}$ is positive definite. By equation (2.2) and (2.5) we get

$$a_{\bar{N}}(\tau(\exp x)^{-1}\exp y)^{-\lambda-\rho} = K_{\lambda+\rho}(x,y) \quad x,y \in \Omega\,. \tag{4.1}$$

Define a map $L_\lambda : \mathbf{K}_0(\lambda) \to \mathbf{H}(-\lambda)$ by

$$\begin{aligned} L_\lambda f(x) &= \int_{N^+} f(\tau(x)\bar{n})\,d\bar{n} \\ &= \int_\Omega f(\exp y)\,K_{\lambda+\rho}(y,x)\,dy \qquad (4.2) \\ &= \int_{H/H\cap K} f(h)a_{\bar{N}}(x^{-1}h)^{-\lambda-\rho}\,dh\,. \qquad (4.3) \end{aligned}$$

Lemma 4.5. — *Let $f \in \mathbf{K}_0(\lambda)$ and $s \in S(\Omega)$. Then $L_\lambda(\pi_\lambda(\tau(s))f) = \pi_{-\lambda}(s)L_\lambda f$.*

Proof. — We have with $s^* = \tau(s^{-1})$:

$$\begin{aligned} L_\lambda(\pi_\lambda(\tau(s))f)(x) &= \int_{N^+} f(s^*y)a_{\bar{N}}(\tau(x)^{-1}y)^{-\lambda-\rho}\,dy \\ &= \int_{N^+} a_{\bar{N}}(s^*y)^{\lambda-\rho}f(\bar{n}(s^*y))a_{\bar{N}}(\tau(x)^{-1}y)^{-\lambda-\rho}\,dy \end{aligned}$$

Notice that $y = \tau(s)s^*y = \tau(s)\bar{n}(s^*y)l(s^*y)a_{\bar{N}}(s^*y)n_{\bar{N}}(s^*y)$. Hence the last integral becomes

$$\begin{aligned} L_\lambda(\pi_\lambda(s^*)f)(x) &= \int_{N^+} f(\bar{n}(s^*y))a_{\bar{N}}(\tau(x)^{-1}\tau(s)\bar{n}(s^*y))^{-\lambda-\rho}a_{\bar{N}}(s^*y)^{-2\rho}\,dy \\ &= \int_{N^+} f(\bar{n})\,a_{\bar{N}}(\tau(s^{-1}x)^{-1}\bar{n})^{-\lambda-\rho}\,d\bar{n} \\ &= \pi_{-\lambda}(s)L_\lambda f(x)\,. \end{aligned}$$

□

Motivated by the fact that the pairing

$$\mathbf{H}(\lambda) \times \mathbf{H}(-\overline{\lambda}) \ni (f,g) \mapsto \int_K f(k)\overline{g(k)}\,dk \in \mathbb{C}$$

is G^c-invariant, we now define a new form on $\mathbf{K}_0(\lambda)$ by

$$(f,g)_\lambda = (f, L_\lambda g)\,.$$

Lemma 4.6. — *Let the notation be as above. Let $f,g \in \mathbf{K}_0(\lambda)$ and let $\lambda \in \mathfrak{a}^*$. Define $F = f \circ \exp$ and $G = g \circ \exp$. Then*

$$\begin{aligned}(f,g)_\lambda &= \int_\Omega \int_\Omega F(x)\overline{G(y)}\, K_{\lambda+\rho}(y,x)\, dydx \\ &= \int_{H_o} \int_{H_o} f(h)\overline{g(k)} a_{\bar{N}}(h^{-1}k)^{-\lambda-\rho}\, dkdh.\end{aligned}$$

In particular $(\cdot,\cdot)_\lambda$ is positive semidefinite if $\lambda + \rho < a(0)$.

Proof. — Let F and G be as before. As $\mathrm{Supp}(F), \mathrm{Supp}(G) \subset \Omega$ it follows by equation (4.2) that

$$\begin{aligned}(f,g)_\lambda &= \int_\Omega f(\exp(x))\overline{L_\lambda g(\exp(x))}\, dx \\ &= \int_\Omega F(x) \overline{\int_\Omega G(y) K_{\lambda+\rho}(y,x)\, dy}\, dx \\ &= \int_\Omega \int_\Omega F(x)\overline{G(y)} K_{\lambda+\rho}(y,x)\ dydx\,.\end{aligned}$$

The second equation follows by equation (4.3). □

Assume that $(\cdot,\cdot)_\lambda$ is positive semidefinite. Let

$$\mathbf{L}(\lambda) = \{u \in \mathbf{K}_0(\lambda) \mid \forall v \in \mathbf{K}_0(\lambda)\ :\ (u,v)_\lambda = 0\}$$

and let $\mathbf{K}(\lambda)$ be the completion of $\mathbf{K}_0(\lambda)/\mathbf{L}(\lambda)$. Then π_λ defines a *involutive* representation of $S(-C_{\max})$ on $\mathbf{K}(\lambda)$:

Lemma 4.7. — *Let $f,g \in \mathbf{K}(\lambda)$. Then for all $s \in S(-C_{\max})$ the relation*

$$(\pi_\lambda(s)f, g)_\lambda = (f, \pi_\lambda(s^*)g)_\lambda$$

follows. Thus $\pi_\lambda \circ \tau$ defines a involutive representation of $S(\Omega)$. In particular $\pi_\lambda(h)$ is unitary for all $h \in H$ and $\pi_\lambda(\exp X)$ is self adjoint for all $X \in -C_{\max}$.

Proof. — As $(\pi_\lambda(s)f, g) = \big(f, \pi_{-\overline{\lambda}}(s^{-1})g\big)$ for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and λ is real it follows that

$$(\pi_\lambda(s)f, L_\lambda g) = (f, \pi_{-\lambda}(s^{-1})L_\lambda g) = (f, L_\lambda(\pi_\lambda(\tau(s^{-1}))g) = (f, \pi_\lambda(s^*)g)_\lambda\,.$$

□

This lemma implies that by starting from the representation π_λ, we have constructed a involutive representation of the semi-group $S(\Omega)$. By the Lüscher-Mack theorem such a representation can be extended to a unitary representation of G. Another way of obtaining such an extension is to use the theory of *local representation* developed by P. Jorgensen [**28, 29**]. We will give a short review of the Lüscher-Mack theory and use reference [**21, 14, 32, 39, 60, 30, 31**].

We have seen that $\pi \simeq \pi_\lambda \circ \tau$ passes to a representation on $\mathbf{K}(\lambda)$ (also denoted by π) such that $\pi(h)$ is unitary and $\pi(\exp(X))$, $X \in C_{\max}$ is self-adjoint. As a result we arrive at self-adjoint operators $d\pi(Y)$ with spectrum in $(-\infty, 0]$ such that for $Y \in C_{\max}$, $\pi(\exp Y) = e^{d\pi(Y)}$ on $\mathbf{K}(\lambda)$. As a consequence of that we notice that

$$t \longmapsto e^{t\, d\pi(Y)}$$

extends to a continuous map on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$ which is holomorphic on the open right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Furthermore,

$$e^{(z+w)\, d\pi(Y)} = e^{z\, d\pi(Y)} e^{w\, d\pi(Y)}\,.$$

The Lüscher-Mack Theorem now states, see [**39**]:

***Theorem 4.8* (Lüscher-Mack).** — *Let $C \subset \mathfrak{q}^c$ be a closed H-invariant pointed and generating convex cone. Let ρ be a strongly continuous contractive representation of $S(C)$ on the Hilbert space $\mathbf{H}$ such that $\rho(s)^* = \rho(s^*)$. Let $\tilde{G}$ be the connected, simply connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then there exists a continuous unitary representation $\tilde{\rho}\colon \tilde{G} \to \mathrm{U}(\mathbf{H})$, extending ρ, such that for the differentiated representations $d\rho$ and $d\tilde{\rho}$:*

1) $d\tilde{\rho}(X) = d\rho(X) \quad \forall X \in \mathfrak{h}$.
2) $d\tilde{\rho}(iY) = i\, d\rho(Y) \quad \forall Y \in C$.

We apply this to our situation and get the following theorem:

***Theorem 4.9*.** — *Let the notation be as before and assume that $\rho_{\lambda+\rho}$ is unitary with $\lambda + \rho < a(0)$. Then the following holds:*

1) *$S(C)$ acts via $s \mapsto \pi(s)$ by contractions on $\mathbf{K}(\lambda)$.*
2) *Let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there exists a unitary representation $\tilde{\pi}$ of $\tilde{G}$ such that $d\tilde{\pi}(X) = d\pi(X)$ for $X \in \mathfrak{h}$ and $i\, d\pi(Y) = d\tilde{\pi}(iY)$ for $Y \in C$.*
3) *The representation $\tilde{\pi}$ is irreducible if and only if π is irreducible.*

We notice the following consequence of 2.8:

***Lemma 4.10*.** — *Let the notation be as before. In particular assume that $\lambda + \rho < a(0)$. Then $\tilde{\pi}$ is a direct integral of highest weight representations.*

We will now identify the representation $\tilde{\pi}$ using the special situation we have here. Let $\mathbf{1}_{\lambda+\rho}$ be the constant function $z \mapsto c_{\lambda+\rho}$ where the constant $c_{\lambda+\rho}$ is determined by

$$||\mathbf{1}_{\lambda+\rho}|| = 1\,.$$

Define a map $U : \mathbf{K}_0(\lambda) \to \mathbf{H}(\rho_{\lambda+\rho})$ by

$$\begin{aligned} Uf(z) &= \int_{H_o} f(h)\,\rho_{\lambda+\rho}(h)\mathbf{1}_{\lambda+\rho}(z)\,dh \\ &= \int_{H_o/H_o\cap K} f(h)\,\rho_{\lambda+\rho}(h)\mathbf{1}_{\lambda+\rho}(z)\,dh \\ &= c_{\lambda+\rho}\int_{H_o} f(h)\,J_{\lambda+\rho}(h^{-1},z)^{-1}\,dh \\ &= c_{\lambda+\rho}\int F(h\cdot 0)J_{\lambda+\rho}(h^{-1},z)^{-1}\,dh \end{aligned}$$

where $F(h\cdot 0) = a_{\bar{N}}(h)^{\lambda-\rho}f(\bar{n}(h))$. Notice the similarity with the map R^* from equation (3.4). So in some sense the map U is just R^* and hence naturally related to the generalized Segal-Bargmann transform. We refer now to [**31**] for the proof of the following Lemma.

Lemma 4.11. — *Let $f, g \in \mathbf{K}_0(\lambda)$ then $(f,g)_\lambda = (Uf, Ug)$.*

Theorem 4.12. — *The map U extends to an unitary isomorphism $U : \widetilde{\mathbf{K}(\lambda)} \to \mathbf{H}(\rho_{\lambda+\rho})$ such that $U(\pi_\lambda(\tau(s))f) = \rho_{\lambda+\rho}(s)U(f)$, $s \in S(\Omega)$. In particular $\widetilde{(\pi_\lambda \circ \tau)}$ is unitarily isomorphic to $\rho_{\lambda+\rho}$*

Proof. — We will prove here the intertwining relation $U(\pi_\lambda(\tau(g))f) = \rho_{\lambda+\rho}(g)U(f)$. For that we need the following transformation rule for the integral over H (see [**30**], Lemma 5.12):

$$\int_{H_o/H_o\cap K} f(h(sh)H\cap K)a_H(sh)^{-2\rho}\,d\dot{h} = \int_{H_o/H_o\cap K} f(hH\cap K)\,d\dot{h}\,.$$

We also notice that if $X \in \mathfrak{p}^+$ then

$$\rho_{\lambda+\rho}(X)\mathbf{1}_{\lambda+\rho} = 0\,.$$

Thus if we decompose s^*h as $s^*h = h_{opp}(s^*h)a_{H,opp}(s^*h)\bar{n} \in HA\bar{N}$, then

$$\rho_\lambda(s^*h)\mathbf{1}_{\lambda+\rho} = a_{H,opp}(s^*h)^{\lambda+\rho}\rho_{\lambda+\rho}(s^*h)\mathbf{1}_{\lambda+\rho}\,.$$

Finally the relation between our usual decomposition according to HAN and the one using $HA\bar{N}$, denoted by the subscript $_{opp}$ in the following, is

$$x = h(x)a_H(x)n \Longleftrightarrow \tau(x) = h(x)a_H(x)^{-1}\tau(n) = h_{opp}(\tau(x))a_{H,opp}(\tau(x))\bar{n}\,.$$

Thus

$$h(x) = h_{opp}(\tau(x)) \quad \text{and} \quad a_{H,opp}(\tau(x)) = a_H(x)^{-1}\,.$$

Using this we get

$$\begin{aligned}
U(\pi_\lambda(\tau(s))f)(z) &= \int_{H_o} f(s^*h)\,\rho_{\lambda+\rho}(h)\mathbf{1}_{\lambda+\rho}\,dh \\
&= \int_{H_o} f(h(s^*h))a_H(s^*h)^{\lambda-\rho}\rho_{\lambda+\rho}(h)\mathbf{1}_{\lambda+\rho}\,dh \\
&= \int_{H_o} f(h(s^*h))a_H(s^*h)^{-2\rho}\,\rho_{\lambda+\rho}(sh(s^*h))\mathbf{1}_{\lambda+\rho}\,dh \\
&= \int_{H_o} f(h)\,\rho_{\lambda+\rho}(sh)\mathbf{1}_{\lambda+\rho}\,dh \\
&= \rho_{\lambda+\rho}(s)Uf(z)\,.
\end{aligned}$$

As this holds for all $s \in S(\Omega)$ it follows that $\tilde{\pi}(s) = \rho_{\lambda+\rho}(s)$ for all $s \in S(\Omega) = S(\Omega_{\mathbb{C}}) \cap G^c$. But then both of them have to agree on $S(W(\rho_{\lambda+\rho}))$. □

Our inner product for the realization of $\rho_{\lambda+\rho}$ has some peculiar properties. Let $U \subset \Omega$ be open and let $\mathbf{K}_{0,U}(\lambda)$ be the set of functions in $\mathbf{K}_0(\lambda)$ such that $\mathrm{Supp}(f) \subset U$ and notice that we can make U arbitrary small. Let $\mathbf{K}_U(\lambda)$ be the projection of $\mathbf{K}_{0,U}(\lambda)$ into $\mathbf{K}(\lambda)$.

Lemma 4.13. — *Let $U \neq \varnothing$ be an open set in Ω. Then $\mathbf{K}_U(\lambda)$ is dense in $\mathbf{K}(\lambda)$.*

The argument proving Lemma 4.13 actually shows that the δ-distribution $f \mapsto f(0)$ is in $\mathbf{K}(\lambda)$. Take δ in the definition of U to get

$$U(\delta)(z) = \mathbf{1}_{\lambda+\rho}(z)$$

or

$$U^*\mathbf{1}_{\lambda+\rho} = \delta\,.$$

Now an orthogonal basis of $\mathbf{H}(\rho_{\lambda+\rho})$ can be constructed applying elements of $U(\mathfrak{p}^-)$ to $\mathbf{1}_{\lambda+\rho}$. This corresponds to applying differential operators to $\mathbf{1}_{\lambda+\rho}$. The above arguments imply the following lemma:

Lemma 4.14. — *The δ distribution is a normalized lowest K-type of the highest weight module $\mathbf{K}(\lambda)$ corresponding to the constant function $z \mapsto c_{\lambda+\rho}$. The K-finite elements of $\mathbf{K}(\lambda)$ are finite linear combinations of derivatives of the δ -distribution.*

We end this section by pointing out one difference in our presentation here and the one in [**31**]. In that paper we identified Ω with a bounded subset in $\mathfrak{p}^- \cap \mathfrak{g}^c$ instead of $\mathfrak{p}^+ \cap \mathfrak{g}^c$. The action of G^c in those two realization is just a twist by τ. Therefore the action of the group on Ω is twisted by τ but there is no twist in the identification of the representation, expressed here by the fact that $\pi_\lambda \circ \tau \simeq \bar{\pi}_{-\lambda}$.

5. The Character Formula

We will now show that the relation between the representations $\pi_\lambda \circ \tau$ and $\rho_{\lambda+\rho}$ can also been made precise using the H-invariant spherical distribution characters. For a moment we will let G stand for any semisimple Lie group and H for a closed subgroup. Let π be a unitary representation of G or one of the induced representations π_λ defined in the last section. The space $\mathbf{H}(\pi)^\infty$ can be made into a complete locally convex, and Hausdorff topological vector space, and G acts continuously on $\mathbf{H}(\pi)^\infty$ by $\pi^\infty(g) = \pi(g)|\mathbf{H}(\pi)^\infty$, see [**66**]. Let $\mathbf{H}(\pi)^{-\infty}$ be the continuous dual of $\mathbf{H}(\pi)^\infty$. Then G acts on $\mathbf{H}(\pi)^{-\infty}$ by

$$\langle \mathbf{u}, \pi^{-\infty}(g)\nu\rangle := \langle \pi^\infty(g^{-1})\mathbf{u}, \nu\rangle\,, \quad \mathbf{u} \in \mathbf{H}(\pi)^\infty,\, \nu \in \mathbf{H}(\pi)^{-\infty}\,.$$

Here $\langle\cdot,\cdot\rangle$ stands for the canonical bilinear pairing between $\mathbf{H}(\pi)^\infty$ and $\mathbf{H}(\pi)^{-\infty}$. A non-zero element $\nu \in \mathbf{H}(\pi)^{-\infty H}$ is called *an H-invariant distribution vector.* Let ν be an H-invariant distribution vector. Then we can imbed $\mathbf{H}(\pi)^\infty$ into $C^\infty(G/H)$ by

$$\mathbf{H}(\pi)^\infty \ni \mathbf{u} \longmapsto \big(gH \mapsto \langle \pi(g^{-1})\mathbf{u}, \nu\rangle\big) \in C^\infty(G/H)\,.$$

Let $\nu \in \mathbf{H}(\pi)^{-\infty H}$ and $\varphi \in C_c^\infty(G/H)$ and define $\pi^{-\infty}(\varphi)\nu$ by

$$\pi^{-\infty}(\varphi)\nu = \int_{G/H} \varphi(\dot g)\, \pi^{-\infty}(g)\nu\, d\dot g\,.$$

Notice that if π is unitary then we have a conjugate linear embedding $\mathbf{H}(\pi) \simeq \mathbf{H}(\pi)^* \subset \mathbf{H}(\pi)^{-\infty}$, and π has an H-invariant distribution vector if and only if the dual representation on $\mathbf{H}(\pi)^*$ has an H-invariant distribution vector, say ν^*. Let $\nu \in \mathbf{H}(\pi)^{-\infty H}$, then $\pi^{-\infty}(\varphi)\nu \in \mathbf{H}(\pi)^{*\infty}$ for all $\varphi \in C_c^\infty(G/H)$ and we can define the H-invariant distribution Θ_π by

$$C_c^\infty(G/H) \ni \varphi \mapsto \Theta_\pi(\varphi) := \langle \pi^{-\infty}(\varphi)\nu, \nu^*\rangle \in \mathbb{C}\,.$$

It is in most cases an unsolved problem to find an explicit formula for the distribution Θ_π.

As an example let us assume that $G = H \times H$. In this case $G/H \simeq H$, $(a,b)H \mapsto ab^{-1}$, such that the action of $H \times H$ on G/H is mapped into $(a,b)\cdot c = acb^{-1}$. An irreducible unitary representation π of G has an H-invariant distribution vector if and only if $\pi = \rho \otimes \bar\rho$, where ρ is an irreducible unitary representation of H and $\bar\rho$ is the representation on $\bar{\mathbf{H}}(\rho) \simeq \mathbf{H}(\rho)^*$. In this case there is – up to scalar – only one H-invariant distribution vector, and the corresponding distribution is a multiple of the usual character

$$\Theta_\pi(\varphi) = \operatorname{Tr}\pi(\varphi)\,.$$

As a second example we take one of the representations $\pi_{-\lambda} = \operatorname{Ind}_{P_{\min}}^{G^c} 1 \otimes \lambda \otimes 1$ where $P_{\min} = MAN$ is a minimal parabolic subgroup of G^c. Define the Weyl groups $W = N_K(A)/Z_K(A)$ and $W_0 = N_{K\cap H}(A)/Z_{K\cap H}(A)$. Then $W_0 \subset W$ and $W_0\backslash W =$

$\dot\bigcup_{j=1}^{r} W_0 m_j$, for some elements $1 = m_1, \dots, m_r \in W$. Then by [**40**] it follows that

$$\dot\bigcup_{j=1}^{r} H m_j MAN \subset G^c$$

is a open and dense. Associated to each of those open orbit is an H -invariant distribution vector [**45**]. We will describe the construction for the set $HP_{\min}$. For $x \in HP_{\min}$ write

$$x = h(x) b_H(x) n(x) \in HAN\,.$$

Define $p_\lambda(x) = b_H(x)^{-\lambda-\rho}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^{c*}$, for $x \in HAN$, and $p_\lambda(x) = 0$ for $x \notin HAN$. The function p_λ is sometimes called *the Poisson kernel of* G^c/H (related to the open orbit $HP_{\min}$). We have $p_\lambda(xman) = a^{-\lambda-\rho} p_\lambda(x)$, and by [**45**] it is known that p_λ is continuous if $\mathrm{Re}\langle \lambda + \rho, \alpha\rangle < 0$ for all positive roots. In that case $p_{-\lambda} \in \mathbf{H}(\lambda) \hookrightarrow \mathbf{H}(-\lambda)^{-\infty}$. Here the pairing is given by

$$\langle f, p_{-\lambda}\rangle = \int_K f(k) p_{-\lambda}(k)\, dk = \int_H f(h)\, dh\,, \quad f \in \mathbf{H}(-\lambda)^\infty\,.$$

By construction it follows that $p_{-\lambda}$ is H-invariant, or $p_{-\lambda} \in \mathbf{H}(-\lambda)^{-\infty H}$. Again by [**45**] it follows that $\lambda \mapsto p_{-\lambda} \in \mathbf{H}(-\lambda)^{-\infty H}$ has a meromorphic continuation to all of $\mathfrak{a}_{\mathbb{C}}^{c*}$. If $\varphi \in C_c^\infty(G^c/H)$ then $\pi_{-\lambda}^{-\infty}(\varphi) p_{-\lambda} \in \mathbf{H}(\lambda)^\infty$. Hence we can apply the distribution vector p_λ to $\pi_{-\lambda}^{-\infty}(\varphi) p_{-\lambda}$ to get the distribution ζ_λ defined by

$$\langle \varphi, \zeta_\lambda\rangle = \langle \pi_{-\lambda}^{-\infty}(\varphi) p_{-\lambda}, p_\lambda\rangle\,.$$

Formally, without bothering about convergence or the use of Fubini's Theorem, we get

$$\begin{aligned}
\langle \varphi, \zeta_\lambda\rangle &= \int_H \pi_{-\lambda}^{-\infty}(\varphi) p_{-\lambda}(h)\, dh \\
&= \int_H \int_{G^c/H} \varphi(\dot g) p_{-\lambda}(g^{-1}h)\, dg dh \\
&= \int_{G^c/H} \varphi(\dot g) \left[\int_H p_{-\lambda}(g^{-1}h)\, dh\right] d\dot g \\
&= \int_{G^c/H} \varphi(\dot g) \varphi_\lambda(\dot g^{-1})\, d\dot g\,.
\end{aligned}$$

It turns out that this calculation is not only formal for our spaces G^c/H and certain spectral parameters λ, but it can be made formally correct and gives rise to the theory of spherical functions, see [**11, 48**].

We will now assume that ρ_π is an unitary highest weight module corresponding to a holomorphic discrete series of G/H. (The same arguments also holds for the universal covering space $\tilde G/H$, but the arguments become more involved because again we would have to discuss the universal covering of the semi-groups $S(W(\pi))$, the lifting of $g \mapsto k(g)$ to $\tilde G$ and $\tilde K$, etc.) We refer to [**4, 24, 35, 36, 51, 52, 55**] for the theory of spherical highest weight modules, Hardy spaces and the holomorphic discrete series. We use the notation from section 2. In particular $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{g}$ such that $\mathfrak{a}_q = \mathfrak{t} \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{q}$. We use Δ, etc for the set of roots of $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, whereas $\tilde{\Delta}$ stands for the set of roots of $\mathfrak{a}_{q\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. We choose compatible sets of roots, i.e., $\tilde{\Delta}^+ = \{\tilde{\alpha} = \alpha|\mathfrak{a}_q \mid \alpha \in \Delta\}$. In this case $\tilde{\rho} = \rho|\mathfrak{a}_q$.

Let W be a G-invariant cone in $\mathfrak{g}$ such that $W_{\min} \subset W \subset W_{\max}$ for some choice of ordering. Let $S(W) = G\exp(iW)$, $S(W^o) = G\exp(iW^o) = S(W)^o$, and define $\Xi \subset G_{\mathbb{C}}/H_{\mathbb{C}}$ by

$$\Xi = \Xi(W^o) = S(-W^o)x_o, \quad x_o = \{H_{\mathbb{C}}\} \in G_{\mathbb{C}}/H_{\mathbb{C}}\,.$$

Then Ξ is an open submanifold of $G_{\mathbb{C}}/H_{\mathbb{C}}$. In particular Ξ is complex. We notice that $G/H \subset \mathrm{cl}(\Xi)\backslash\Xi$ and that $\gamma^{-1}(G/H) \subset \Xi$ for all $\gamma \in S(W^o)$. Let $\mathbf{H}_2(W)$ be the *Hardy space* of all holomorphic functions on Ξ such that $\gamma \cdot F : G/H \ni m \mapsto F(\gamma^{-1}m) \in \mathbb{C}$, is in $L^2(G/H)$ and the L^2-limit

$$\beta(F) := \lim_{\gamma\to 1} \gamma \cdot F \in L^2(G/H)$$

exists. Define an inner product on $\mathbf{H}_2(W)$ by

$$(F, G) = (\beta(F), \beta(G))\,.$$

Then $\mathbf{H}_2(W)$ is a Hilbert space carrying an involutive representation of $S(W)$ defined by $\gamma \cdot F(z) = T(\gamma)F(z) = F(\gamma^{-1} \cdot z)$. This representation decomposes discretely and with multiplicity one into a direct sum of holomorphic discrete series representations

$$\mathbf{H}_2(W) = \bigoplus_{\pi} \mathbf{E}(\pi)$$

where the sum is taken over all the holomorphic discrete series representations that extend to holomorphic representations of $S(W)$. Here $\mathbf{E}(\pi)$ stands for the realization of ρ_π in $\mathbf{H}_2(W)$. This realization can be made explicit in the following way. Let $0 \neq \mathbf{u}_0$ be a $H \cap K$-invariant vector in $\mathbf{V}_\pi$, the representation space for the minimal K-type π. Let $s \in S(W)$. Then $s \in H_{\mathbb{C}}K_{\mathbb{C}}P^+$. Write $s = hk_H(s)p$. For $\mathbf{u} \in \mathbf{V}_\pi$ let

$$\varphi_{\pi,\mathbf{u}}(s) := (\pi(k_H(s^{-1}))\mathbf{u}, \mathbf{u}_0)\,.$$

Then $\varphi_{\pi,\mathbf{u}} \in \mathbf{H}_2(W)$ if and only if $\langle \mu(\pi) + \tilde{\rho}, \tilde{\alpha}\rangle < 0$ for all $\tilde{\alpha} \in \tilde{\Delta}_n^+$ and in that case the closed G-invariant subspace generated by $\{\varphi_{\pi,\mathbf{u}} \mid \mathbf{u} \in \mathbf{V}_\pi\}$ is isomorphic to $\mathbf{E}(\pi)$. We normalize $\mathbf{u}_0$ so that this map is an unitary isomorphism of $\mathbf{V}_\pi$ into $L^2(G/H)$. Notice that the map $\mathbf{V}_\pi \ni \mathbf{u} \mapsto \varphi_{\pi,\mathbf{u}}$ is K-equivariant.

The space $\mathbf{H}_2(W)$ is a reproducing Hilbert space. Let $K(z, w)$ be the corresponding reproducing kernel. For $z \in \Xi$ the map $w \mapsto K(w, z) = K_z(w)$ extends to a smooth function on $S(-W)x_o$. In particular $K(z,\cdot) = \overline{\beta(K_z)}$ is a well defined, smooth L^2-function on G/H. For $F \in \mathbf{H}_2(W)$ and $\gamma \in S(-W^o)$ we have

$$\begin{aligned} F(\gamma \cdot z) &= (F, K_{\gamma\cdot z}) \\ &= [T(\gamma^{-1})F](z) \\ &= (T(\gamma^{-1})F, K_z) \\ &= (F, T(\overline{\gamma})K_z)\,. \end{aligned}$$

Thus

$$K(\gamma \cdot z, w) = K(z, \gamma^* \cdot w)\,. \tag{5.1}$$

It follows that K is determined by the function $k \in \mathcal{O}(\Xi)^H$, $k(z) = K(z, x_0)$, and $K(\gamma_1 \cdot x_0, \gamma_2 \cdot x_0) = k(\gamma_2^*\gamma_1)$ (by abuse of notation, viewing k as a H-biinvariant function). Finally we have

$$\begin{aligned} \beta^{-1}(f)(z) = F(z) &= \int_{G/H} f(m)\, K(z,m)\, dm \\ &= (f, K_z) \\ &= \int_{G/H} f(\dot g)\, k(\dot g^{-1} \cdot z)\, d\dot g \end{aligned}$$

Lemma 5.1. — *Let $F \in \mathbf{H}_2(W)^\infty$. Then*

$$\beta(F)(m) = \lim_{0<t\to 0} F(\exp(itZ^0)\cdot m)\,. \tag{5.2}$$

Proof. — Decompose f into K-types $f = \sum_{\delta\in\hat K} f_\delta$. The Fourier series $\sum f_\delta$ converges to f in the C^∞-topology, a fact due to Harish-Chandra, see [**66**], Lemma 4.4.2.1. Denote the highest weight of δ by $\mu(\delta)$. Then, using the notation from Theorem 2.3, we have $\mu(\delta) = \mu_0 + r_\delta\lambda_0$. Each $\mu(\delta)$ is a weight in one of the spaces $\mathbf{E}(\pi)$ and we have $r_\pi < 0$. Furthermore — as $\mathbf{E}(\pi)_K = U(\mathfrak{p}^-)\mathbf{V}_\pi$ — we get

$$\mu(\delta) = \mu(\pi) - \sum_{\alpha\in\Delta^+} n_\alpha \alpha\,, \qquad n_\alpha \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\ . \tag{5.3}$$

Hence $r_\delta \le r_\pi < 0$. Furthermore

$$\delta(\exp(-itZ^0)) = e^{r_\delta t}\,\mathrm{Id}\,.$$

Notice that $0 < e^{r_\delta t} < 1$ for all $t > 0$. Hence we have for $z \in S(-W)\cdot x_0$:

$$\begin{aligned} f(\exp(itZ^0)\cdot z) &= \sum f_\delta(\exp(itZ^0)\cdot z) \\ &= \sum e^{r_\delta t} f_\delta(z)\,. \end{aligned}$$

It follows that the series $\sum_\delta \left|f_\delta(\exp(itZ^0)\cdot z)\right|$ is uniformly dominated by the series $\sum_\delta |f_\delta(z)|$, and the claim follows. ◻

Let $\mathrm{pr}_\pi : L^2(G/H) \to \mathbf{E}(\pi)$ be the orthogonal projection and define $T_\pi := \beta^{-1} \circ \mathrm{pr}_\pi : L^2(G/H) \to \mathbf{H}_2(W)$. Then

$$T_\pi(f)(w) := \beta^{-1}(\mathrm{pr}_\pi(f))(w) = \int \mathrm{pr}_\pi(f) K(w,m)\, dm\,.$$

As $f \mapsto T_\pi f(w)$ is continuous it comes from a function $\Theta_\pi(\cdot, w) \in \mathbf{E}(\pi) \subset \mathbf{H}_2(W)$. Thus

$$\beta^{-1}(\mathrm{pr}_\pi(f))(w) = \int f(m)\overline{\Theta_\pi(m,w)}\, dm\,. \tag{5.4}$$

The function $m \mapsto \Theta_\pi(m,w)$ extends to Ξ such that $\Theta_\pi(z,w)$ is holomorphic in the first variable, anti-holomorphic in the second variable, and $\Theta_\pi(z,w) = \overline{\Theta_\pi(w,z)}$. As the projection $L^2(G/H) \to \mathbf{H}_2(W)$ is given by $f \mapsto (w \mapsto (f, K_w))$ and $\mathbf{H}_2(W) = \oplus_\pi \mathbf{E}(\pi)$ it follows that

$$K(z,w) = \sum_\pi \Theta_\pi(z,w)\,.$$

Lemma 5.2. — *Let $\{\varphi_\nu\}_{\nu\in\mathbb{N}}$ be an orthonormal basis for $\mathbf{E}(\pi)$. Then $\Theta_\pi(z,w) = \sum_{\nu=1}^\infty \varphi_\nu(z)\overline{\varphi_\nu(w)}$.*

Proof. — This is well known, but let us recall the proof here. As $z \mapsto \Theta_\pi(z,w)$ is in $\mathbf{E}(\pi)$ it follows that

$$\Theta_\pi(z,w) = \sum a_\nu(w)\varphi_\nu(z)\,.$$

But

$$\varphi_\nu(w) = T_\pi(\beta\varphi_\nu)(w) = (\varphi_\nu, \Theta_\pi(\cdot, w)) = \overline{a_\nu(w)}\,. \qquad \square$$

Since pr_π and β^{-1} are intertwining it follows that

$$\Theta_\pi(s\cdot z, w) = \Theta_\pi(z, s^*\cdot w)\,, \quad s \in S(-W)\,.$$

Hence Θ_π is determined by a function θ_π:

$$\theta_\pi(z) = \Theta_\pi(z, x_0) = \overline{\Theta_\pi(x_0, z)}\,. \tag{5.5}$$

By construction we have $\Theta_\pi(\gamma_1\cdot x_0, \gamma_2\cdot x_0) = \theta_\pi(\gamma_2^*\gamma_1)$ and

$$T_\pi f(\gamma\cdot x_0) = \int_{G/H} f(\dot g)\theta_\pi(\dot g^{-1}\gamma)\, d\dot g = \int_{G/H} f(m)\,\overline{\theta_\pi(\gamma^* m)}\, dm\,.$$

Furthermore we have the Lemma:

Lemma 5.3. — *The reproducing kernel has a decomposition in the form $k = \sum_\pi \theta_\pi$.*

Let Θ_π be the spherical distribution defined by $f \mapsto \mathrm{pr}_\pi(f)(x_0)$. We can now realize the spherical distribution character as a *hyperfunction* on G/H in the following way:

Lemma 5.4. — *Let $f \in C_c^\infty(G/H)$. Then*

$$\begin{aligned}\Theta_\pi(f) &= \lim_{0<t\to 0}\int_{G/H} f(m)\Theta_\pi(\exp(itZ^0)\cdot x_0, m)\, dm\\ &= \lim_{0<t\to 0}\int_{G/H} f(m)\overline{\theta_\pi(\exp(itZ^0)\cdot m)}\, dm.\end{aligned}$$

Furthermore we have

1. *The distribution Θ_π extends to a holomorphic function on $\Xi = S(-W^o)\cdot x_0$ given by $\Theta_\pi(z) = \theta_\pi(z) = \sum \overline{\varphi_\nu(x_0)}\varphi_\nu(z)$;*
2. *There exists a character $\chi : \mathbb{D}(G/H) \to \mathbb{C}$ such that $D\Theta_\pi = \chi(D)\Theta_\pi$.*

Proof. — It is clear that Θ_π is an H-invariant distribution. The first part now follows by (5.2), (5.4), (5.5), and Lemma 5.2. Let $D \in \mathbb{D}(G/H)$. Then $D : \beta\mathbf{E}(\pi)_K \to L^2(G/H)$. But the multiplicity of ρ_π in $L^2(G/H)$ is one. Hence $D(\beta\mathbf{E}(\pi)_K) = \mathbf{E}(\pi)_K$. But $\mathbf{E}(\pi)$ is irreducible, so D has to be a scalar on $\mathbf{E}(\pi)$. □

Finding the *character formula* for the holomorphic discrete series now becomes the problem of determining the function θ_π. Let $\mu = \mu(\pi)$ denote the highest weight of π, let $\mathbf{u} = \mathbf{u}_{\mu(\pi)} \in \mathbf{V}_\pi$ be a highest weight vector. We assume that $\varphi_1(z) = \varphi_{\pi,\mathbf{u}}$. Furthermore $\|\mathbf{u}\| = 1$ as $\|\varphi_1\| = 1$. We have

$$\varphi_1(t^{-1}z) = t^\mu \varphi_1(z)\,, \qquad \forall t \in \exp(\mathfrak{t}_\mathbb{C}) \cap S(W)\,.$$

We assume that the other functions φ_ν are weight vectors of weight say μ_ν. Since $\mathbf{H}(\rho_\pi) = U(\mathfrak{p}^-)\cdot \mathbf{V}_\pi$ it follows that

$$\mu_\nu = \mu - \sum_{\alpha\in\Delta^+} n_\alpha \alpha\,, \quad n_\alpha \in \mathbb{N}\cup\{0\} =: \mathbb{N}_0\,.$$

This, $\varphi_1(x_0) = (\mathbf{u}, \mathbf{u}_0)$, and Lemma 5.4 imply the following.

Theorem 5.5. — *Let $\theta_\pi = \sum \overline{\varphi_\nu(x_0)}\varphi_\nu$ be as above. Then θ_π is a H-invariant function on Ξ such that for all $D \in \mathbb{D}(G/H) = \mathbb{D}(G^c/H)$*

$$D\theta_\pi = \chi_\pi(D)\theta_\pi$$

where χ_π is a character on $\mathbb{D}(G/H)$. Furthermore

$$\lim_{t\to\infty} e^{-t\mu(\pi)(X)}\theta_\pi(\exp -tX\cdot x_0) = |(\mathbf{u}, \mathbf{u}_0)|^2$$

for all $X \in i\mathfrak{t}$, such that $\alpha(X) > 0$ for all $\alpha \in \Delta_c^+ \cup \Delta_n^+$.

Theorem 5.5 implies that $\theta_\pi|G^c/H\cap\Xi$ is a H -invariant eigenfunction on $S(-C^o)/H$, $C = W\cap\mathfrak{q}^c$. We will therefore turn our attention to *spherical functions* on G^c/H. We use [**11, 48**] as standard reference.

Definition 5.6. — *A continuous H-invariant function $\varphi : S(-C^o)\cdot x_0 \to \mathbb{C}$, $\varphi \neq 0$, is called a* spherical function *if there exists a character $\chi : \mathbb{D}(G^c/H) \to \mathbb{C}$ such that (in the sense of distributions)*

$$D\varphi = \chi(D)\varphi\,, \quad \forall D \in \mathbb{D}(G^c/H)\,.$$

To construct spherical functions let $\mathfrak{a}^c = i\mathfrak{a}$, and $P_{\min} = MAN$ be the minimal parabolic subgroup of G corresponding to the positive system $\tilde{\Delta}^+$. Then $N^+ := P^+ \cap G^c \subset N$, $N = N_0 N^+$, and $\bar{N} = \bar{N}_0 N^-$. Let $LA = K_{\mathbb{C}} \cap G^c$ as before. Let now $X^0 = -iZ^0$, and define $\Delta_+ = \{\alpha \in \Delta(\mathfrak{g}^c, \mathfrak{a}^c) \mid \alpha(X^0) = 1\}$. For $\alpha \in \Delta_+$ let $H_\alpha \in \mathfrak{a}^c$ be determined by $H_\alpha = [X, \tau(X)]$ for some $X \in \mathfrak{g}^c_\alpha$ and $\alpha(H_\alpha) = 2$. Let $\Delta_0 = \{\alpha \in \Delta \mid \alpha(X^0) = 0\}$ and $\Delta_0^+ = \Delta_0 \cap \Delta^+$.

Lemma 5.7. — *Let $\lambda \in \mathfrak{a}^*_{p\mathbb{C}}$ and $s \in S(-C^o_{\max})$ Then the following conditions are equivalent.*

1. *The function $H \ni h \mapsto p_{-\lambda}(sh) \in \mathbb{C}$ is integrable;*
2. *The function $\Omega \ni \omega \mapsto p_\lambda(\omega) \in \mathbb{C}$ is integrable,*
3. $\lambda \in \mathcal{E} := \{\lambda \in \mathfrak{a}^*_{p\mathbb{C}} \mid \forall \alpha \in \Delta_+ \,:\, Re\langle \lambda, H_\alpha\rangle < 2 - m_\alpha\}$.

The equivalence of (1) and (2) was first proved in [**11**]. The implication (3)$\Longrightarrow$(2) was proved in [**48**], and finally (2)$\Longrightarrow$(3) was proved in [**37**].

Assume that $\lambda \in \mathcal{E}$. Define

$$\varphi_\lambda(s) := \int_H p_{-\lambda}(sh)\, dh$$

and

$$c_\Omega(\lambda) = \int_\Omega p_\lambda(\exp X)\, dX\,.$$

Let $c_0(\lambda)$ be the Harish-Chandra c-function for the Riemannian symmetric space $L/H \cap K$. Thus

$$c_0(\lambda) = \int_{\bar{N}\cap L} p_\lambda(\bar{n})\, d\bar{n}$$

wherever the integral converges. Finally we let

$$c(\lambda) := \int_{\bar{N}\cap HP_{\min}} p_\lambda(\bar{n})\, d\bar{n} = c_\Omega(\lambda) c_0(\lambda)\,.$$

There is a well known product formula for the c-function for a Riemannian symmetric space in terms of the c-function $c_\alpha(\lambda_\alpha)$ for rank one symmetric spaces constructed from the roots α, see [**18**]. For the space G^c/H this was an open problem for a long time. The general case was solved by B. Krötz and the author in [**37**]. We refer to that article and [**50**] for further references.

***Theorem 5.8* (KÓ99).** — *The following product formula for $c_\Omega(\lambda)$ is valid*

$$c_\Omega(\lambda) = d \prod_{\alpha\in\Delta_+} B\left(\frac{m_\alpha}{2}, -\langle\lambda, \frac{1}{2}H_\alpha\rangle - \frac{m_\alpha}{2} + 1\right) \tag{5.6}$$

$$= d \prod_{\alpha\in\Delta_+} \frac{\Gamma\left(\frac{m_\alpha}{2}\right)\Gamma\left(-\frac{\lambda(H_\alpha)+m_\alpha}{2}+1\right)}{\Gamma\left(-\frac{\lambda(H_\alpha)}{2}+1\right)} \tag{5.7}$$

for some constante d. In particular the c-function for G/H is a product over the c-functions for the rank one subsymmetric spaces $G(\alpha)/H\cap G(\alpha)$, $\alpha\in\Delta^+$:

$$c(\lambda) = \prod_{\alpha\in\Delta^+} c_\alpha(\lambda_\alpha)\,.$$

Here $G(\alpha)$ is the analytic subgroup of G^c corresponding to the Lie algebra $\mathfrak{g}(\alpha) = \mathfrak{g}_\alpha + [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] + \mathfrak{g}_{-\alpha}$.

For the function φ_λ we have according to [**11, 48**]:

***Theorem 5.9*.** — *The function φ_λ is a spherical function for $\lambda\in\mathcal{E}$. It has a meromorphic continuation as a spherical function for $\lambda\in\mathfrak{a}_\mathbb{C}^{c*}$. Furthermore*

$$\lim_{0<t\to\infty} e^{-t\langle\lambda-\tilde\rho,X\rangle}\varphi_\lambda(\exp(tX)) = c(\lambda)$$

for all $X\in(-C^o_{\max})\cap\left\{Y\in\mathfrak{a}^c \mid \forall\alpha\in\Delta_0^+ : \alpha(Y)>0\right\}$ and $\lambda\in\mathcal{E}$.

Recall that if π is a irreducible representation of K with highest weight μ, and such that $\mathbf{V}_\pi^{H\cap K}\neq\{0\}$, then $\mu\in i\mathfrak{a}_q^* = \mathfrak{a}^{c*}$.

***Corollary 5.10*.** — *Let the notation be as before. In particular ρ_π is a holomorphic discrete series representation of G/H. Denote the highest weight of π by μ. Let $\lambda := \mu+\tilde\rho\in\mathfrak{a}^{c*}$. Assume that $\lambda\in\mathcal{E}$. Then*

$$\theta_\pi(s\cdot x_0) = \frac{|(\mathbf{u},\mathbf{u}_0)|^2}{c(\lambda)}\varphi_\lambda(s\cdot x_0)\,.$$

Define the coefficients $\Gamma_\mu(\lambda)$ by recursion as in [**18**], p. 427, and set for $a\in\exp(-C^o_{\max}\cap\left\{Y\in\mathfrak{a}_p \mid \forall\alpha\in\Delta_0^+ : \alpha(Y)>0\right\})$:

$$\Phi_\lambda(a) = a^{\lambda-\rho}\sum_{\mu\in\mathbb{N}_0\Delta^+}\Gamma_\mu(\lambda)a^{-\mu}\ .$$

***Theorem 5.11* (Ó97).** — *For generic $\lambda\in\mathfrak{a}_\mathbb{C}^*$ we have*

$$\varphi_\lambda = \sum_{w\in W_0} c(w\cdot\lambda)\Phi_{w\cdot\lambda} = c_\Omega(\lambda)\sum_{w\in W_0} c_0(w\cdot\lambda)\Phi_{w\cdot\lambda}.$$

For the evaluation of $(\mathbf{u},\mathbf{u}_0)$ we recall some facts from [**33**]. As before we let $\mu=\mu(\pi)$ and $\lambda=\mu+\tilde{\rho}$. Let

$$d(\mu)^G=\frac{\prod_{\alpha\in\Delta^+}(\mu+\rho,\alpha)}{\prod_{\alpha\in\Delta^+}(\rho,\alpha)}\,.$$

If ρ_π is in the holomorphic discrete series of G, then $d(\mu)^G$ is the formal dimension of ρ_π, [**17**].

Lemma 5.12. — *Let the notation be as above. Then*

$$|(\mathbf{u},\mathbf{u}_0)|^2=d(\mu)^G c(\lambda)\,.$$

Proof. — By definition

$$\langle\varphi_1,\rho_\pi^{-\infty}(g)\Theta_\pi\rangle=\varphi_1(g\cdot x_0)\,.$$

We also have $\|\varphi_1\|=1$. Assume first that ρ_π belongs to the holomorphic discrete series of G and that $\lambda\in\mathcal{E}$. Then $|(\mathbf{u},\mathbf{u}_0)|^2=d(\mu)^G c(\lambda)$ follows from Definition III.3 and Theorem III.4 in [**33**]. The general statement follows now by analytic continuation, see Theorem IV.15 in [**33**]. □

Using the same analytic continuation arguments as in the proof of Theorem IV.15, [**33**] combined with Corollary 5.10 and Theorem 5.11 we arrive at the following character formula for the holomorphic discrete series representation ρ_π.

Theorem 5.13. — *Assume that ρ_π corresponds to a holomorphic discrete series representation of G/H. Let μ be the highest weight of π and let $\lambda=\mu+\tilde{\rho}$. Then*

$$\theta_\pi=d(\mu)^G\varphi_\lambda\,.$$

In particular

$$\theta_\pi=d(\mu)^G c_\Omega(\lambda)\sum_{w\in W_0}c_0(w\cdot\lambda)\Phi_{w\cdot\lambda}$$

for generic μ.

This gives us the *character formula* for ρ_π. Assume now that π is a character $\chi_{\lambda+\rho}$ related to the representation $\bar{\pi}_{-\lambda}$ as before. Then this does not relate the H-spherical character of π directly with that of $\bar{\pi}_{-\lambda}$. But write $\rho(\Delta^+)=\rho_0+\rho_+$ where ρ_0 is the sum over the compact roots and ρ_+ is the sum over the non-compact roots. Then $\mathbf{H}(-\lambda)$ can be embedded into $\bar{\mathbf{H}}_{P_{\min}}(-\lambda-\rho_0)$ where the subscript $_{P_{\min}}$ indicates that we are inducing from the minimal parabolic subgroup. It is easy to check that the restriction of $p_{-\lambda}$ to $\bar{\mathbf{H}}(-\lambda)$ is non-zero and corresponds to the corresponding Poisson kernel. Thus by restriction φ_λ can be viewed as the character of $\bar{\pi}_{-\lambda}$ (related to the open orbit $HP_{\min}=HP_{\max}$).

As a final remark we would like to mention, that *most* of the compactly causal symmetric spaces G/H can be compactified as the Sylov boundary $G_1/P_{1\max}$ of a bounded symmetric domain G_1/K_1 of tube type, see [**3, 55**] for details and list.

In short there exists a connected semisimple Lie group G_1 such that G_1/K_1 is a bounded symmetric domain which can also be realized as a tube type domain. The Sylov boundary of G_1/K_1 is a compact symmetric space $S_1 = K_1/L_1 = G_1/P_{1\max}$, where $P_{1\max}$ is a maximal parabolic subgroup of G_1. Furthermore there exists an injective G-map $F : G/H \to S_1$ with open dense image. Using F one can identify $L^2(G/H)$ with $L^2(S_1)$. The map F can be extended to a holomorphic isomorphism of Ξ into a open dense subset of G_1/K_1. This can be used to compare the Hardy space on G/H (or a covering in some cases) with the classical Hardy space. We refer to [**4, 55**] for details. It is still an open question how to use the orbit G-structure of S_1 to analyze $L^2(G/H)$.

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