Polynomial Approximation and $\omega_\varphi^r(f,t)$ Twenty Years Later

Z. Ditzian

16 September 2007

Abstract

About twenty years ago the measure of smoothness $\omega_\varphi^r(f,t)$ was introduced and related to the rate of polynomial approximation. In this article we survey developments about this and related concepts since that time.

MSC: 41A10, 41A17, 41A25, 41A27, 41A30, 41A36, 41A40, 41A50, 41A63, 26A15, 26B35, 26B05, 42C05, 26A51, 26A33, 46E35

keywords: Moduli of smoothness, K-functionals, realization functionals, polynomial approximation, direct and converse inequalities, Bernstein, Jackson, Marchaud, Nikol'skii and Ul'yanov type inequalities.

Contents

1 Introduction **107**

2 Jackson-type estimates **108**

3 K-functionals **110**

4 K-functionals (second approach) **111**

5 Realization **113**

6 Sharp Marchaud and sharp converse inequalities **115**

7 Moduli of smoothness of functions and of their derivatives **116**

8 Relations with Bernstein polynomial approximation and other linear operators **118**

9 Weighted moduli of smoothness, doubling weights **122**

Surveys in Approximation Theory
Volume 3, 2007. pp. 106–151.

ISSN 1555-578X

10 Weighted moduli, Jacobi-type weights **125**

11 Weighted moduli, Freud weights **129**

12 Multivariate polynomial approximation **131**

13 Ul'yanov-type result **136**

14 $\omega^r_{\varphi^\lambda}(f,t)_\infty$, $0 \le \lambda \le 1$, **filling the gap** **138**

15 Shape-preserving polynomial approximation **139**

16 Average moduli of smoothness (Ivanov's moduli) **141**

17 Algebraic addition (Felten's moduli) **142**

18 Generalized translations **143**

19 Lipschitz-type and Besov-type spaces **144**

20 Other methods **145**

21 Epilogue **146**

22 Appendix **146**

References **147**

1 Introduction

It was observed long ago (see [Ni]) that for investigating the rate of algebraic polynomial approximation the ordinary moduli of smoothness are not completely satisfactory. For $C[-1,1]$ it was shown that near the boundary the rate of pointwise approximation was better for a given degree of smoothness than at other points such as those further away from the boundary. The model of the relation between the ordinary moduli of smoothness and the rate of best trigonometric approximation (i.e. direct and weak converse inequalities) could not be followed. Characterization of the class of functions for which the rate of best polynomial approximation is prescribed cannot be described by the ordinary moduli of smoothness.

About twenty years ago the moduli $\omega^r_\varphi(f,t)$ were introduced (see [Di-To,87]) to deal with this problem. There were other attempts made, the most notable being the works of K. Ivanov (see [Iv] for additional references) on the average moduli of smoothness. The measure of smoothness $\omega^r_\varphi(f,t)_p$ on $[-1,1]$ (for example) is given by

$$\omega^r_\varphi(f,t)_p = \sup_{|h|\le t} \|\Delta^r_{h\varphi} f\|_{L_p[-1,1]} \tag{1.1}$$

where

$$\Delta_{h\varphi}^r f(x) = \begin{cases} \sum\limits_{k=0}^{r} (-1)^k \binom{r}{k} f\big(x + (\frac{r}{2} - k)h\varphi(x)\big), \\ \qquad\qquad \text{if} \quad [x - \frac{r}{2}\, h\varphi(x), x + \frac{r}{2}\, h\varphi(x)] \subset [-1,1] \\ \\ 0 \qquad\qquad \text{otherwise,} \end{cases} \tag{1.2}$$

$\varphi(x)^2 = 1 - x^2$ and

$$\|g\|_{L_p[a,b]} = \Big\{ \int_a^b |g(x)|^p dx \Big\}^{1/p}, \quad p < \infty; \qquad \|g\|_{L_\infty[a,b]} = \operatorname*{ess\,sup}_{a \le x \le b} |g(x)|.$$

Many properties of $\omega_\varphi^r(f,t)_p$ and related measures were studied in [Di-To,87] as well as the basic relation with polynomial approximation. In the last two decades numerous articles were written using $\omega_\varphi^r(f,t)$ or competing with it. In this paper I will give a survey of what I believe to be the main advances made in the last twenty years connecting the rate of approximation of functions by algebraic polynomials with measures of smoothness of these functions. In [Di-To,87] the "step weight" function φ was just a function satisfying very mild conditions. Here φ will be a function that is directly used in applications to approximation and in particular to polynomial approximation and to some common linear processes. Unless otherwise specified, when we write $\omega_\varphi^r(f,t)_p$, we assume the definition in (1.1) and (1.2) on $[-1,1]$ but we will deal also with related concepts as well as other domains and "step weights" φ.

We will be discussing relations among different concepts of smoothness which include $\omega_\varphi^r(f,t)$, various K-functionals, realization functionals, rate of best approximation, strong converse inequalities as well as the τ modulus by Ivanov, moduli given by generalized translations and others. Results on the rate of weighted and multivariate polynomial approximation in relation to various measures of smoothness will also be described.

The topics are itemized in the Contents (at the beginning); however, inevitably some remarks relating to one topic may appear in a section dedicated to another. In particular, when a concept or result is introduced in some section, its relation to items in later sections will be presented in those sections.

2 Jackson-type estimates

It is well-known that for $L_p(T)$, where T is the "circle" $[-\pi,\pi]$ and $0 < p \le \infty$,

$$E_n^*(f)_p \equiv E_n^*(f)_{L_p(T)} \le C\omega^r(f,1/n)_{L_p(T)} \tag{2.1}$$

where

$$E_n^*(f)_{L_p(T)} = \inf(\|f - T_n\|_{L_p(T)} : T_n \in \boldsymbol{T}_n), \tag{2.2}$$

$\boldsymbol{T}_n \equiv \operatorname{span}\{e^{ikx} : |k| < n\}$ is the set of trigonometric polynomials of degree less than n for $n = 1,2,\ldots$, and

$$\begin{aligned} \omega^r(f,t)_{L_p(T)} &= \sup_{|h| \le t} \|\Delta_h^r f\|_{L_p(T)}, \\ \Delta_h^r f(x) &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\Big(x + \big(\frac{r}{2} - k\big)h\Big), \end{aligned} \tag{2.3}$$

for $r = 0, 1, 2, \ldots$ are the ordinary L_p moduli of smoothness. In fact (2.1) is valid also if $L_p(T)$ is replaced by a Banach space B of functions on T satisfying

$$\|f(\cdot + a)\|_B = \|f(\cdot)\|_B \qquad \forall\, a \in \mathbb{R} \tag{2.4}$$

and

$$\|f(\cdot + h) - f(\cdot)\|_B = o(1), \quad h \to 0; \tag{2.5}$$

that is,

$$E_n^*(f)_B = \inf(\|f - T_n\|_B : T_n \in \boldsymbol{T}_n) \le C\omega^r(f, 1/n)_B \tag{2.1$'$}$$

where $E_n^*(f)_B$ and $\omega^r(f, 1/n)_B$ are given by (2.2) and (2.3) with B replacing $L_p(T)$. (See Appendix for a proof of (2.1)$'$.)

For $L_p[-1, 1]$, $1 \le p \le \infty$, it was proved in [Di-To,87, Theorem 7.2.1] that

$$E_n(f)_p \equiv E_n(f)_{L_p[-1,1]} \le C\omega_\varphi^r(f, 1/n)_{L_p[-1,1]} \tag{2.6}$$

where

$$E_n(f)_{L_p[-1,1]} = \inf(\|f - P_n\|_{L_p[-1,1]} : P_n \in \Pi_n), \tag{2.7}$$

$\Pi_n \equiv \operatorname{span}(1, x, \ldots, x^{n-1})$ is the set of algebraic polynomials of degree at most $n - 1$ and $\omega_\varphi^r(f, t)_p$ is given by (1.1) and (1.2).

DeVore, Leviatan and Yu [De-Le-Yu, Theorem 1.1] showed that (2.6) is valid for $0 < p < 1$ as well. The method of their proof uses a Whitney-type estimate by polynomials of degree $r - 1$ and "patching" them up by polynomials of degree n that form a partition of unity, (see also the remark in [Di-Hr-Iv, p. 74] about the necessity of Lemma 5.2 there for their proof). This type of argument is used in [De-Lo] to prove the result for $1 \le p \le \infty$ as well.

For $L_p[-1, 1]$ and other spaces a Jackson-type estimate using a measure of smoothness given by a K-functional which is not always equivalent to $\omega_\varphi^r(f, t)$ but is still optimal (in the same sense) will be discussed in Section 4. However, (2.6) was not extended to a form which follows (2.1)$'$. That is, we do not have (2.6) with B (satisfying some general conditions) replacing $L_p[-1, 1]$.

It was proved by M. Timan [Ti,M,58] that for trigonometric polynomials a sharper (than (2.1)) Jackson-type inequality holds, i.e. for $E_k^*(f)_p$ of (2.2)

$$\begin{gathered} n^{-r}\Big\{\sum_{k=1}^{n} k^{sr-1} E_k^*(f)_p^s\Big\}^{1/s} \le C(r, s, p)\omega^r(f, n^{-1})_p, \\ s = \max(p, 2), \quad 1 < p < \infty. \end{gathered} \tag{2.8}$$

This result, which is best possible for $1 < p < \infty$, has rarely been cited in literature in the English language and I could find it only in a text by Trigub and Belinsky [Tr-Be, p. 191, 4.8.8] (and there without proof and with n^{-r} missing on the left of (2.8)).

Recently, an analogue of this result was proved in [Da-Di-Ti], that is

$$\begin{gathered} n^{-r}\Big\{\sum_{k=r}^{n} k^{sr-1} E_k(f)_p^s\Big\}^{1/s} \le C(r, s, p)\omega_\varphi^r(f, n^{-1})_p, \\ s = \max(p, 2), \quad 1 < p < \infty \end{gathered} \tag{2.9}$$

where $\omega_\varphi^r(f, t)_p$ and $E_k(f)_p$ are given in (1.1) and (2.7).

We note that in [Da-Di-Ti] (2.9) is just one of many related formulae and the treatment in [Da-Di-Ti] uses best approximation by various systems of functions and various measures of smoothness.

We also note that for $1 < p < \infty$ (2.9) was shown in [Da-Di-Ti] to be equivalent to

$$t^r \left\{ \int_t^{1/2} \frac{\omega_\varphi^{r+1}(f,u)_p^s}{u^{sr+1}}\, du \right\}^{1/s} \le C\omega_\varphi^r(f,t)_p, \tag{2.10}$$

for $1 < p < \infty$ and $s = \max(p,2)$.

Examples were given in [Da-Di-Ti, Section 10] to show that (as far as s is concerned) the inequalities (2.9) and (2.10) are optimal for $1 < p < \infty$.

The inequality (2.10) is sharper than the inequality

$$\omega_\varphi^{r+1}(f,t)_p \le C\omega_\varphi^r(f,t)_p \tag{2.11}$$

for the range $1 < p < \infty$. The inequality (2.11), however, is valid for the bigger range $0 < p \le \infty$ (see [Di-To,87, Chapter 7] and [Di-Hr-Iv]).

3 K-functionals

As an alternative to $\omega_\varphi^r(f,t)$ one can measure smoothness using K-functionals.

It was shown in [Di-To,87, Theorem 2.1.1] (not just for the case $\varphi(x)^2 = 1 - x^2$) that

$$K_{r,\varphi}(f,t^r)_p \approx \omega_\varphi^r(f,t)_p, \quad 1 \le p \le \infty, \tag{3.1}$$

that is

$$C^{-1}K_{r,\varphi}(f,t^r)_p \le \omega_\varphi^r(f,t)_p \le CK_{r,\varphi}(f,t^r)_p, \quad 1 \le p \le \infty, \tag{3.2}$$

where

$$K_{r,\varphi}(f,t^r)_p = \inf \left(\|f-g\|_{L_p[-1,1]} + t^r \|\varphi^r g^{(r)}\|_{L_p[-1,1]} : g,\dots,g^{(r-1)} \in A.C._{\ell\text{oc}} \right). \tag{3.3}$$

In fact, it is known that in (3.3) $g,\dots,g^{(r-1)} \in A.C._{\ell\text{oc}}$ can be further restricted using instead $g \in C^r[-1,1]$ or even $g \in C^\infty[-1,1]$ without any effect on (3.1). One could have observed that $g \in C^r[-1,1]$ is sufficient already from the proof in [Di-To,87]. That it is sufficient to consider g in the class $C^\infty[-1,1]$ follows from the realization results mentioned in Section 5. We note also that for $p = \infty$ the result is of significance only when $f \in C[-1,1]$ as otherwise neither side of (3.1) is small when t is.

For the well-studied analogue on the circle T one has

$$\omega^r(f,t)_B \approx \inf \left(\|f-g\|_B + t^r \|g^{(r)}\|_B : g^{(r)} \in B \right) = K_r(f,t^r)_B \tag{3.4}$$

where B is any Banach space of functions on T in which translations are continuous isometries, that is translations satisfy (2.5) and (2.4) respectively. The notation $g^{(r)} \in B$ means that the r-th derivative in $\mathcal{S}'$ (the space of tempered distributions) is in B.

We will often use the notation $A(t) \approx B(t)$ and, following (3.2), we mean $C^{-1}B(t) \le A(t) \le CB(t)$ for all relevant t.

We do not have

$$K_{r,\varphi}(f,t^r)_B \approx \omega_\varphi^r(f,t)_B \tag{3.5}$$

where $\|f\|_{L_p[-1,1]}$ is replaced by $\|f\|_B$ for a "general" Banach space on $[-1,1]$.

For an Orlicz space of functions on $[-1,1]$ this was done in [Wa] in his thesis in Chinese (and I believe also earlier). Not being able to read that work, I cannot describe it. I learned about it from its extension to the multivariate situation in [Zh-Ca-Xu] where the univariate case is taken for granted.

In the next section different but related K-functionals will be described for which the treatment for various spaces is given.

For $L_p[-1,1]$ when $0<p<1$ it was shown in [Di-Hr-Iv] that for all f in $L_p[-1,1]$, $0<p<1$,

$$K_{r,\varphi}(f,t^r)_p = 0 \tag{3.6}$$

where $K_{r,\varphi}(f,t^r)_p$ is defined by (3.3) with the quasinorm $\|\cdot\|_{L_p[-1,1]}$. The proof in [Di-Hr-Iv] is univariate and local and applies to the circle T as well, that is

$$f \in L_p(T) \quad \text{implies} \quad K_r(f,t^r)_p = 0 \quad \text{for} \quad 0<p<1. \tag{3.6$'$}$$

The identity (3.6) implies that we cannot have (3.2) for $0<p<1$ as $\omega_\varphi^r(f,t)_p$ is not always zero. (Clearly, $|x| \in L_p[-1,1]$ and $\omega_\varphi(f,t)_p \equiv \omega_\varphi^1(f,t)_p \neq 0$.) Even before (3.6) was proved, it was clear that $\omega_\varphi^r(f,t)_p$ cannot be equivalent to $K_{r,\varphi}(f,t^r)_p$ when $0<p<1$, as the saturation rate of $\omega_\varphi^r(f,t)_p$ is $O(t^{r-1+\frac{1}{p}})$ for that range and $K_{r,\varphi}(f,t^r)_p$ as a K-functional cannot tend to zero at a rate faster than t^r unless it equals 0.

4 K-functionals (second approach)

For a Banach space B of functions on domain $\mathcal{D}$ and a differential operator $P_r(D)$ of degree r we define the K-functional

$$K_{rm}\big(f, P_r(D)^m, t^{rm}\big)_B \equiv \inf\big(\|f-g\|_B + t^{rm}\|P_r(D)^m g\|_B : P_r(D)^m g \in B\big). \tag{4.1}$$

One can assume $P_r(D)^m g$ is defined as a distributional derivative, and in most cases we deal with we may assume $g \in C^{rm}(\mathcal{D})$ without changing the asymptotic behaviour of $K_{rm}\big(f, P_r(D)^m, t^{km}\big)_B$ given in (4.1). The K-functional $K_{r,\varphi}(f,t^r)_p$ of (3.3) is $K_r\big(f, P_r(D), t^r\big)_p$ with $P_r(D) = \varphi^r\big(\frac{d}{dx}\big)^r$ on $L_p[-1,1]$. In relation to polynomials on $[-1,1]$ it is natural to study the K-functional given in (4.1) with $P_2(D) = \frac{d}{dx}(1-x^2)\frac{d}{dx}$. It was essentially shown in [Ch-Di,94, Theorem 5.1], using a maximal function estimate, that

$$K_{2,\varphi}(f,t^2)_{L_p[-1,1]} \le C K_2\Big(f, \frac{d}{dx}(1-x^2)\frac{d}{dx}, t^2\Big)_{L_p[-1,1]} \quad \text{for} \quad 1<p\le\infty. \tag{4.2}$$

It follows from [Di-To,87, Chapter 9, 135-6] which uses the Hardy inequality, that for $1 \le p < \infty$

$$K_{2r}\Big(f, \big(\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^r, t^{2r}\Big)_{L_p[-1,1]} \le C K_{2r,\varphi}(f,t^{2r})_p + t^{2r}E_1(f)_p\,. \tag{4.3}$$

It can easily be deduced from [Da-Di,05, Theorem 7.1] that for $1<p<\infty$

$$K_{2r}\Big(f, \big(\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^r, t^{2r}\Big)_{L_p[-1,1]} \approx K_{2r,\varphi}(f,t^{2r})_p + t^{2r}E_1(f)_p\,. \tag{4.4}$$

For $p = 1$ and $p = \infty$ (4.4) does not hold, as shown in [Da-Di,05, Remark 7.9, p.88].

We observe that for $r = 1$ (4.4) is a corollary of (4.2) and (4.3), whose proof is more elementary. (It does not use the Muckenhoupt transplantation theorem nor the Hörmander-type multiplier theorem used for the proof of [Da-Di,05, Theorem7.1].) It would be nice if we had a proof for (4.2) with 2 replaced by $2r$ and could deduce (4.4) directly from it and (4.3).

For an orthonormal sequence of functions $\{\varphi_n\}$ on some set D the Cesàro summability of order ℓ is given by

$$C_n^\ell(f,x) = \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) \cdots \Big(1 - \frac{k}{n+\ell}\Big) P_k(f,x) \tag{4.5}$$

where the (L_2 type) projection $P_k f$ is given by

$$P_k(f,x) = \varphi_k(x) \int_D \varphi_k(y) f(y) dy. \tag{4.6}$$

Here $D = [-1,1]$ and $\varphi_k(x)$ are the eigenvectors of $\frac{d}{dx}(1-x^2)\frac{d}{dx}$ satisfying

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}\varphi_k(x) = -k(k+1)\varphi_k(x), \qquad \int_{-1}^{1} \varphi_k(x)\varphi_\ell(x)dx = \begin{cases} 0, & k \neq \ell, \\ 1, & k = \ell. \end{cases} \tag{4.7}$$

In later sections we deal with weights in (4.6) and (4.7) when we discuss progress made for measures of smoothness and polynomial approximation in weighted L_p and in other related Banach spaces. Furthermore, it will be crucial to examine (4.5) when the projection is on a finite dimensional orthonormal space which is needed for the multivariate situation (and has the precedent of projection on span $(\sin kx, \cos kx)$).

The Legendre operator $\frac{d}{dx}(1-x^2)\frac{d}{dx}$ has as eigenvectors the Legendre orthogonal polynomials. It was shown in [Ch-Di,97, Theorem 4.1 and (6.13)] and [Di,98] that for B a Banach space of functions on $[-1,1]$ for which

$$\|C_n^\ell(f,\cdot)\|_B \le C\|f\|_B \tag{4.8}$$

is satisfied for some ℓ, one has

$$E_n(f)_B = \inf_{P\in\Pi_n} \|f-P\|_B \le CK_{2r}\Big(f, \big(\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^r, t^{2r}\Big)_B. \tag{4.9}$$

It is known that $B = L_p[-1,1]$ satisfies (4.8) (see for discussion and references of more general results [Ch-Di,97, Theorem A, page 190]) and perhaps this should be an incentive to investigate for which class of Banach spaces (4.8) is valid (with respect to eigenfunctions of $\frac{d}{dx}(1-x^2)\frac{d}{dx}$), and hence imply (4.9) which is a Jackson-type result for a different measure of smoothness.

For $\alpha > 0$, the operator $\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha g$ is defined by

$$\Big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\Big)^\alpha g \sim \sum_{k=1}^{\infty} \lambda(k)^\alpha P_k g, \quad \lambda(k) = k(k+1) \tag{4.10}$$

and we say $\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha g \in B$ if there exists a function $G_\alpha \in B$ which satisfies $P_k G_\alpha = \lambda(k)^\alpha P_k g$. We may define the K-functional (see [Di,98, p. 324]) by

$$K_{2\alpha}\Big(f, \big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha, t^{2\alpha}\Big)_B = \inf \Big(\|f-g\|_B + t^{2\alpha}\Big\|\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha g\Big\|_B\Big) \tag{4.11}$$

where the infimum is taken on g such that $g \in B$ and $\left(-\frac{d}{dx}(1-x^2)\frac{d}{dx}g\right)^\alpha \in B$. For integer $\alpha = r$, (4.11) and (4.1) with $P_r(D) = \left(\frac{d}{dx}\,(1-x^2)\,\frac{d}{dx}\right)^r$ are the same concept. In [Di,98, Theorem 6.1] it was shown for B such that (4.8) is satisfied that

$$E_n(f)_B \le CK_{2\alpha}\Big(f, \big(-\frac{d}{dx}\,(1-x^2)\,\frac{d}{dx}\big)^\alpha, 1/n^{2\alpha}\Big)_B. \tag{4.12}$$

5 Realization

Realization functionals were introduced by Hristov and Ivanov [Hr-Iv] in order to characterize K-functionals. As it happened, this concept gained in usefulness when it was observed that certain K-functionals are always equal to zero for $0 < p < 1$ (see (3.6) or (3.6)$'$), and one needs an expression that will replace the K-functional and will yield a meaningful measure of smoothness for all $0 < p \le \infty$. Realization functionals were shown in [Di-Hr-Iv] to be such a concept. It is a mistake, however, to think that realizations are useful only for $0 < p < 1$. Many articles, starting with [Hr-Iv], utilized properties of realizations for various applications. We will present here realization-functionals that are measures of smoothness related to polynomial approximation and $\omega_\varphi^r(f,t)_p$.

The most common realization related to $\omega_\varphi^r(f,t)_p$ is

$$R_{r,\varphi}(f, n^{-r})_p = \|f - P_n\|_{L_p[-1,1]} + n^{-r}\|\varphi^r P_n^{(r)}\|_{L_p[-1,1]} \tag{5.1}$$

where $P_n \in \Pi_n$ is the best polynomial approximant from Π_n to f in L_p, that is

$$E_n(f)_p = \inf_{P\in\Pi_n} \|f - P\|_{L_p[-1,1]} = \|f - P_n\|_{L_p[-1,1]}, \qquad P_n \in \Pi_n \tag{5.2}$$

or a near best polynomial approximant

$$\|f - P_n\|_{L_p[-1,1]} \le AE_n(f)_p, \qquad P_n \in \Pi_n \tag{5.3}$$

with A independent of n and f. Sometimes it is convenient to use P_n as a polynomial of degree mn which satisfies (5.3). A particularly convenient polynomial of this nature for $1 \le p \le \infty$ is the de la Vallée Poussin-type operator on f given by

$$\eta_n f = \sum_{k=0}^{2n} \eta\Big(\frac{k}{n}\Big) P_k f \tag{5.4}$$

where $P_k f$ is given by (4.6) and (4.7), $\eta(y) \in C^\infty[0,\infty)$, $\eta(y) = 1$ for $y \le 1$ and $\eta(y) = 0$ for $y \ge 2$. Clearly, $\eta_n f \in \Pi_{2n}$, $\eta_n P = P$ for $P \in \Pi_n$, and it is known that $\|\eta_n f\|_p \le C\|f\|_p$ for $1 \le p \le \infty$. The inequality $\|\eta_n f\|_B \le C\|f\|_B$ for $B = L_p[-1,1]$ (and in fact for any B satisfying (4.8)) follows the same method used in [Ch-Di,97, p. 192] and [Di,98, p. 326–327] (using the Abel tranformation and $P_k f = \overleftarrow{\Delta}^{\ell+1} \binom{k+\ell}{\ell} C_k^\ell f$ where $\overleftarrow{\Delta} a_k = a_k - a_{k-1}$ and $\overleftarrow{\Delta}^m a_k = \overleftarrow{\Delta}(\overleftarrow{\Delta}^{m-1} a_k)$). Other de la Vallée Poussin-type operators (or delayed means) were also used for realizations (see for instance [Ch-Di,97] and [Di,98]. The advantage of using a de la Vallée Poussin-type operator (in some form) over using the best approximant is threefold: it is given by a linear operator, it is often independent of $1 \le p \le \infty$, and it commutes with the differential operator $\frac{d}{dx}\,(1-x^2)\,\frac{d}{dx}$.

We can also define (as was originally done)

$$R^*_{r,\varphi}(f,n^{-r})_p = \inf_{P\in\Pi_n} \big(\|f-P\|_{L_p[-1,1]} + n^{-r}\|\varphi^r P^{(r)}\|_{L_p[-1,1]}\big). \tag{5.5}$$

It is known and easy to show that (5.1) with P_n of (5.2) or (5.3) and (5.5) are equivalent for $0 < p \le \infty$, that is, $R^*_{r,\varphi}(f,n^{-1})_p \approx R_{r,\varphi}(f,n^{-1})_p$. If we use $\eta_n f$ of (5.4) in (5.1) for P_n, the equivalence holds only for $1 \le p \le \infty$. ((5.4) is not defined for $0 < p < 1$.)

It was proved in [Di-Hr-Iv] that

$$R^*_{r,\varphi}(f,n^{-r})_p \approx \omega^r_\varphi(f,n^{-1})_p \tag{5.6}$$

for $0 < p \le \infty$, and hence $R_{r,\varphi}(f,n^{-r})_p \approx \omega^r_\varphi(f,n^{-1})_p$ for $0 < p \le \infty$ if P_n is given by (5.2) or (5.3), and for $1 \le p \le \infty$ if for P_n we write $\eta_n f$ given in (5.4).

We note (see [Di-Hr-Iv]) that an analogous result to (5.6) is known for $L_p(T)$ where T_n, an n-th degree trigonometric polynomial, replaces P_n, and $\omega^r(f,t)_p$ replaces $\omega^r_\varphi(f,t)_p$. The equivalence (5.6) was also extended to other realizations and measures of smoothness.

For $L_p[-1,1]$, $1 \le p \le \infty$ and other Banach spaces some sequences of linear operators $A_n f$ other than $\eta_n f$ given in (5.4) were used for defining the realization

$$\widetilde{R}_{r,\varphi}(f,n^{-r})_{L_p[-1,1]} = \|f-A_n f\|_{L_p[-1,1]} + n^{-r}\|\varphi^r (A_n f)^{(r)}\|_{L_p[-1,1]} \tag{5.7}$$

(see, for instance, [Ch-Di,97] and [Di,98]).

Of course for $\widetilde{R}_{r,\varphi}(f,n^{-r})_{L_p[-1,1]}$, A_n may depend on r. We will encounter some natural expressions of the form (5.7) in this survey. In most situations here when dealing with (5.7) either the choice (5.1) where $P_n = \eta_n f$ with $\eta_n f$ of (5.4) (which is a near best approximant) is more useful or we have a linear approximation process $A_n f$ which satisfies a relation with $\omega^r_\varphi(f,t)_p$ that is superior to $\widetilde{R}_{r,\varphi}(f)_p \approx \omega^r_\varphi(f,n^{-r})_p$ (see Section 8). The conditions that P_n satisfies, (5.2), (5.3) or (5.4), are independent of r and this fact has proved useful in many applications. We note that in the expression $R^*_{r,\varphi}(f,n^{-r})_p$, P_n depends on r and hence in applications it is sometimes more advantageous to use the equivalent form $R_{r,\varphi}(f,n^{-1})_p$.

For a general Banach space B on $[-1,1]$ it is convenient to deal with

$$R_{2\alpha}\big(f,P(D)^\alpha,n^{-2\alpha}\big)_B = \|f-P_n\|_B + \frac{1}{n^{2\alpha}}\,\big\|\big(P(D)\big)^\alpha P_n\big\|_B \tag{5.8}$$

where $P(D)$ is the Legendre operator $P(D) = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$, P_n is given by (5.2), (5.3) or (5.4), and $\big(P(D)\big)^\alpha$ is given by (4.9). We have (see [Da-Di,05])

$$R_{2r}\big(f,P(D)^r,n^{-2r}\big)_p \approx R_{2r,\varphi}(f,n^{-2r})_p + n^{-2r}E_1(f)_p \quad \text{for} \quad 1 < p < \infty. \tag{5.9}$$

However, (5.9) is not valid for $p=1$ and $p=\infty$ since

$$R_{2r}\big(f,P(D)^r,n^{-2r}\big)_p \approx K_{2r}\big(f,P(D)^r,n^{-2r}\big)_p, \quad 1 \le p \le \infty, \tag{5.10}$$

and also since (4.4) is not valid for $p=1$ and $p=\infty$. In fact, for any Banach space B for which (4.8) is satisfied we have

$$R_{2\alpha}\big(f,P(D)^\alpha,n^{-2\alpha}\big)_B \approx K_{2\alpha}\big(f,P(D)^\alpha,n^{-2\alpha}\big)_B \tag{5.11}$$

(see also [Di,98, Theorem 6.2]).

In the following sections we will mention the results for which realization functionals were used. We will also present extensions to weighted spaces and to spaces of multivariate functions.

Like most interesting concepts, realizations were discussed before the concept was introduced formally. For instance, the equivalence

$$R_{r,\varphi}(f, n^{-r})_p \approx \omega_\varphi^r(f, 1/n)_p \qquad 1 \le p \le \infty \tag{5.12}$$

with P_n of (5.2) or (5.3) was shown already in [Di-To,87] and its trigonometric analogue much earlier. This should not diminish the significance of the systematic treatment of realizations and their importance for various spaces and applications (not only in relation to algebraic polynomial approximation).

6 Sharp Marchaud and sharp converse inequalities

The converse inequality of (2.6) is given by

$$\omega_\varphi^r(f, t)_p \le M(r)t^r \sum_{n=1}^{\lfloor \frac{1}{t} \rfloor} n^{r-1} E_n(f)_p, \quad 1 \le p \le \infty \tag{6.1}$$

with $E_n(f)_p$ given in (2.7) was proved in [Di-To,87, Theorem 7.2.4]. (Note that we write here n instead of $n+1$ in [Di-To,87] as here $\Pi_n = \operatorname{span}(1, \dots, x^{n-1})$.) The Marchaud inequality

$$\omega_\varphi^r(f, t)_p \le Ct^r \Big\{ \int_t^c \frac{\omega_\varphi^r(f, u)_p}{u^{r+1}}\, du + \|f\|_p \Big\}, \quad 1 \le p \le \infty \tag{6.2}$$

was proved in [Di-To,87, Theorem 4.3.1] for a general class of step weights $\varphi(x)$. (Not just $\varphi(x) = \sqrt{1-x^2}$.)

For trigonometric polynomials A. Zygmund [Zy] and M. Timan [Ti,M,58] proved

$$\omega^r(f, t)_{L_p(T)} \le M(r)t^r \Big\{ \sum_{n=1}^{\lfloor \frac{1}{t} \rfloor} n^{rq-1} E_n^*(f)_p^q \Big\}^{1/q}, \quad 1 \le p < \infty, \quad q = \min(p, 2) \tag{6.3}$$

where $\omega^r(f, t)_p$ and $E_n^*(f)_p$ are given by (2.3) and (2.4) respectively. In addition, it was shown in [Zy] and [Ti,M,58] that for $1 \le p < \infty$

$$\omega^r(f, t)_{L_p(T)} \le Ct^r \Big[\Big\{ \int_t^c \frac{\omega^{r+1}(f, u)_{L_p(T)}^q}{u^{qr+1}}\, du \Big\}^{1/q} + \|f\|_{L_p(T)} \Big], \quad q = \min(p, 2). \tag{6.4}$$

(The term $\|f\|_{L_p(T)}$ in (6.4) is redundant.) The classic converse and Marchaud inequalities, i.e. (6.3) and (6.4) for $1 \le p \le \infty$ when $q = 1$ replaces $q = \min(p, 2)$, are clearly weaker for $1 < p < \infty$ than (6.3) and (6.4) with $q = \min(p, 2)$. Moreover, $q = \min(p, 2)$ is the optimal power in (6.3) and (6.4) for $1 \le p < \infty$. Using partially a new proof and extension of (6.4) given in [Di,88], Totik proved in [To,88] for $1 < p < \infty$ that

$$\omega_\varphi^r(f, t)_p \le Ct^r \Big[\Big\{ \int_t^c \frac{\omega_\varphi^{r+1}(f, u)_p^q}{u^{rq+1}}\, du \Big\}^{1/q} + \|f\|_p \Big] \quad \text{where} \quad q = \min(p, 2) \tag{6.5}$$

(here $\|f\|_p$ can be replaced by $E_{r-1}(f)_p$ but not eliminated), and he deduced from it for $1 < p < \infty$

$$\omega_\varphi^r(f,t)_p \le M(r)t^r\Big[\sum_{n=1}^{\lfloor\frac{1}{t}\rfloor} n^{rq-1}E_n(f)_p^q\Big]^{1/q} \quad \text{where} \quad q = \min(p,2). \tag{6.6}$$

In Totik's paper (see [To,88]) (6.5) is given for $1 < p \le 2$ with a more general step weight φ. For $2 < p < \infty$ he gave (6.5) and (6.6) only for $\varphi(x) = \sqrt{1-x^2}$.

Examples were given in [Da-Di-Ti, Section 10] to show that the power q in (6.5) and (6.6) is optimal for $1 < p < 4$. (The power q is probably optimal in (6.5) and (6.6) for all $1 < p < \infty$.)

Later it was shown in [Di-Ji-Le, Theorem 1.1] that (6.6) is valid for $0 < p < 1$ as well. Using (2.6) which was proved in [De-Le-Yu, Theorem 1.1] for $0 < p < 1$ and applying it to $k+1$ (instead of k), one has (6.5) also for $0 < p < 1$.

Recently, (see [Da-Di,05, Theorem 6.2]) it was shown that for $\alpha < \beta$, $1 \le p < \infty$, $q = \min(p,2)$ and $P(D) = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ (among other operators) one has

$$K_{2\alpha}\big(f, P(D)^\alpha, t^{2\alpha}\big)_p \le Ct^{2\alpha}\Big\{\int_t^c \frac{K_{2\beta}\big(f, P(D)^\beta, u^{2\beta}\big)_p^q}{u^{2\alpha q+1}}\,du\Big\}^{1/q} \tag{6.7}$$

with $K_{2\alpha}\big(f, P(D)^\alpha, t^{2\alpha}\big)_p$ given in (4.10).

As we have for $1 \le p \le \infty$ and all $\gamma > 0$

$$K_{2\gamma}\big(f, P(D)^\gamma, n^{-2\gamma}\big)_p \le Cn^{-2\gamma}\sum_{k=1}^n k^{2\gamma-1}E_k(f)_p, \tag{6.8}$$

the inequality (6.7) used for $\gamma = \beta$ implies

$$K_{2\alpha}\big(f, P(D)^\alpha, t^{2\alpha}\big)_p \le Ct^{2\alpha}\Big\{\sum_{1\le k\le 1/t} k^{2\alpha q-1}E_k(f)_p^q\Big\}^{1/q}. \tag{6.9}$$

For $1 < p < \infty$ and $2\ell = r$ we have

$$K_{2\ell}\big(f, P(D)^\ell, t^{2\ell}\big)_p \approx \omega_\varphi^{2\ell}(f,t)_p$$

(see [Da-Di,05, Theorem 7.1] and [Di-To,87, Chapter 9]). In fact, one can use (6.8) and (6.9) to obtain (6.5) and (6.6). However, in my opinion, the main advantage of the technique in [Da-Di,05] for polynomial approximation is not its applicability to fractional α but that this method is applicable to $L_p[-1,1]$ with Jacobi-type weights (see Section 10).

7 Moduli of smoothness of functions and of their derivatives

For $f, f^{(k)} \in L_p(T)$, $1 \le p \le \infty$, it is well-known (see [De-Lo, p. 46]) that

$$\omega^r(f,t)_p \le Ct^k\omega^{r-k}\big(f^{(k)},t\big)_p \quad \text{where} \quad 1 \le k \le r \tag{7.1}$$

and that (see [De-Lo, p. 178])

$$\omega^{r-k}\big(f^{(k)},t\big)_p \le C\int_0^t \frac{\omega^r(f,u)_p}{u^{k+1}}\,du \quad \text{where} \quad 1 \le k < r. \tag{7.2}$$

It was shown recently (see [Di-Ti,07]) that the converse-type inequality (7.2) can be improved for $1 < p < \infty$ and has an analogue for $0 < p < 1$.

For $\omega_\varphi^r(f,u)_{L_p[-1,1]}$ it was proved in [Di-To,87, Theorem 6.2.2 and Theorem 6.3.1] that

$$\Omega_\varphi^r\big(f,t\big)_p \le Ct^k\omega_\varphi^{r-k}\big(f^{(k)},t\big)_{p,\varphi^k} \quad \text{for} \quad 1\le p\le\infty \quad \text{and} \quad r>k \tag{7.3}$$

and

$$\Omega_\varphi^{r-k}\big(f^{(k)},t\big)_{p,\varphi^k} \le C\int_0^t \frac{\Omega_\varphi^r(f,u)_p}{u^{k+1}}\,du \quad \text{for} \quad 1\le p\le\infty \quad \text{and} \quad r>k \tag{7.4}$$

where

$$\Omega_\varphi^\ell(g,t)_{p,\varphi^m} = \sup_{|h|<t}\|\Delta_{h\varphi}^\ell g\|_{L_{p,\varphi^m}[I(h,\ell)]}, \tag{7.5}$$

$$I(h,\ell) = [-1+2h^2\ell^2, 1-2h^2\ell^2] \tag{7.6}$$

and

$$\|F\|_{L_{p,w}(D)} = \Big\{\int_D |F(x)|^p w(x)dx\Big\}^{1/p} \tag{7.7}$$

(and $\Delta_{h\varphi}^\ell f(x)$ is still defined by (1.2) with the underlying interval $[-1,1]$). In [Di-Ti,07, Section 5], generalization of (7.4) was achieved, i.e. for $0<p<\infty$ and $r>k$

$$\Omega_\varphi^{r-k}\big(f^{(k)},t\big)_{p,\varphi^k} \le C\Big\{\int_0^t \frac{\omega_\varphi^r(f,u)_p^q}{u^{qk+1}}\,du\Big\}^{1/q} \tag{7.8}$$

where $q=\min(p,2)$. Simple examples can be given to show that (7.3) does not hold for $0<p<1$. It can be noted that a best approximation version of (7.8) follows from the proof in [Di-Ti,07, Section 5], that is,

$$\Omega_\varphi^{r-k}\big(f^{(k)},t\big)_{p,\varphi^k} \le C\Big\{\sum_{\ell\ge\lfloor 1/t\rfloor} \ell^{qk-1}E_\ell(f)_p^q\Big\}^{1/q} \tag{7.9}$$

where $f\in L_p[-1,1]$, $r>k$, $0<p<\infty$ and $q=\min(p,2)$. In [Di-Ti,07, Section 5] it was shown that

$$\Omega_\varphi^{r-k}\big(f^{(k)},t\big)_{p,\varphi^k} \le C\Big\{\sum_{2^m\ge\lfloor 1/t\rfloor} 2^{mkq}E_{2^m}(f)_p^q\Big\}^{1/q}, \tag*{(7.9)$'$}$$

which is equivalent to (7.9).

Another approach to this question which is applicable to Banach spaces satisfying (4.8) (see (4.5), (4.6) and (4.7)) is implied by the results in [Di,98, Sections 6 and 7]. We note that the result below applies to $L_p[-1,1]$ with $1\le p\le\infty$ but not with $0<p<1$.

We have for $P(D) = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ and B satisfying (4.8)

$$\begin{gathered} E_{2\lambda}(f)_B \le \|f-\eta_\lambda f\|_B \le CE_\lambda(f)_B, \quad \text{and} \\ E_{2\lambda}\big(P(D)^\alpha f\big)_B \le \|P(D)^\alpha f - \eta_\lambda\big(P(D)^\alpha f\big)\|_B \le CE_\lambda\big(P(D)^\alpha f\big)_B \end{gathered} \tag{7.10}$$

where η_λ is the de la Vallée Poussin-type operator defined by (5.4) using (4.6) and (4.7). (Other de la Vallée Poussin-type operators will yield a result similar to (7.10).)

Using the realization theorem (see [Di,98, Theorem 7.1]) given by

$$K_{2\alpha}\Big(f,\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha,\frac{1}{n^{2\alpha}}\Big)_B \approx \|f-V_nf\|_B + \frac{1}{n^{2\alpha}}\Big\|\Big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\Big)^\alpha V_nf\Big\|_B, \tag{7.11}$$

with $V_n f = \eta_{1/n} f$ or other de la Vallée Poussin-type operators, one has the following result:

For $\alpha < \beta$ and $\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha f \in B$ we have, using [Di,98, Theorem 7, (7.10) and (5.11)],

$$\begin{aligned} &K_{2\beta}\Big(f,\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\beta,t^{2\beta}\Big)_B \\ &\quad\le Ct^{2(\beta-\alpha)}K_{2(\beta-\alpha)}\Big(\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^\alpha f,\big(-\frac{d}{dx}(1-x^2)\frac{d}{dx}\big)^{\beta-\alpha},t^{2(\beta-\alpha)}\Big)_B. \end{aligned} \tag{7.12}$$

For $\alpha < \beta$ and $P(D) = -\frac{d}{dx}(1-x^2)\frac{d}{dx}$ we also have, using [Di,98, Theorem 7.1],

$$K_{2(\beta-\alpha)}\Big(P(D)^\alpha f,\big(P(D)\big)^{\beta-\alpha},t^{2(\beta-\alpha)}\Big)_B \le C\int_0^t \frac{K_\beta\big(f,P(D)^\beta,u^{2\beta}\big)_B}{u^{2\alpha+1}}\,du. \tag{7.13}$$

For $B = L_p[-1,1]$, $1 < p < \infty$ one can follow [Da-Di,05] and obtain a sharper version of (7.13), that is

$$\begin{aligned} &K_{2(\beta-\alpha)}\big(P(D)^\alpha f,P(D)^{\beta-\alpha},t^{2(\beta-\alpha)}\big)_{L_p[-1,1]} \\ &\quad\le C\Big\{\int_0^t \frac{K_\beta\big(f,P(D)^\beta,u^{2\beta}\big)^q_{L_p[-1,1]}}{u^{2\alpha q+1}}\Big\}^{1/q}, \quad q=\min(p,2). \end{aligned} \tag{7.14}$$

For the rate of best approximation $E_n(f)_B$ given in (2.7) or (4.9) (when (4.8) is satisfied), (7.13) and (7.14) take the forms

$$K_{2(\beta-\alpha)}\Big(P(D)^\alpha f,\big(P(D)\big)^{\beta-\alpha},t^{2(\beta-\alpha)}\Big)_B \le C\sum_{\ell\ge\lfloor 1/t\rfloor}\ell^{2\alpha-1}E_\ell(f)_B, \tag{7.15}$$

and for $1 < p < \infty$

$$\begin{aligned} &K_{2(\beta-\alpha)}\Big(P(D)^\alpha f,P(D)^{\beta-\alpha},t^{2(\beta-\alpha)}\Big)_{L_p[-1,1]} \\ &\quad\le C\Big\{\sum_{\ell\ge\lfloor 1/t\rfloor}\ell^{2\alpha q-1}E_\ell(f)^q_{L_p[-1,1]}\Big\}^{1/q} \quad \text{where} \quad q=\min(p,2) \end{aligned} \tag{7.16}$$

respectively.

8 Relations with Bernstein polynomial approximation and other linear operators

Chapters 9 and 10 of [Di-To,87] were dedicated to relations between $\omega^r_\varphi(f,t)_p$ (with appropriate φ and domain) and the rate of convergence of Bernstein, Szasz and Baskakov operators (including appropriate combinations and modifications).

We remind the reader that the Bernstein operator is given by

$$B_n(f,x) = \sum_{k=0}^n \binom{n}{k}x^k(1-x)^{n-k}f\Big(\frac{k}{n}\Big) \equiv \sum_{k=0}^n P_{n,k}(x)f\Big(\frac{k}{n}\Big) \quad \text{for} \quad x\in[0,1]. \tag{8.1}$$

Perhaps the first real progress in the last twenty years was the general group of concepts called strong converse inequalities S.C.I. (see [Di-Iv]). In [Di-Iv, Section 8] it was shown as one of the applications of the general method given in [Di-Iv, Section 3] that

$$\omega^2_\varphi(f,n^{-1/2})_{C[0,1]} \le C\Big(\|B_nf-f\|_{C[0,1]} + \|B_{An}f-f\|_{C[0,1]}\Big) \tag{8.2}$$

for some $A > 1$ where $\omega_\varphi^2(f,t)_{C[0,1]}$ (in relation to Bernstein polynomials) is a copy of $\omega_\varphi^2(f,t)_{C[-1,1]}$ given by (1.1) and (1.2) in which $[0,1]$ replaces $[-1,1]$ and $\varphi(x) = \sqrt{x(1-x)}$ replaces $\sqrt{1-x^2}$. The inequality (8.2) is a strong converse inequality of type B (with two terms on the right hand side) and is called "strong" as it matches the direct result (see [Di-To,87]) given by

$$\|B_n f - f\|_{C[0,1]} \le C\omega_\varphi^2\Big(f, n^{-1/2}\Big)_{C[0,1]}. \tag{8.3}$$

One observes that (8.2) implies

$$\omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \le C \sup_{k\ge n} \|B_k f - f\|_{C[0,1]}, \tag{8.2$'$}$$

which is a strong converse inequality of type D in the terminology of [Di-Iv]. Combining (8.2)$'$ with (8.3), one has

$$\omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \approx \sup_{k\ge n} \|B_n f - f\|_{C[0,1]}\,.$$

In [Di-Iv, Remark 8.6] it was conjectured that the superior strong converse inequality of type A is also valid, that is, that

$$\omega_\varphi^2(f, n^{-1/2})_{C[0,1]} \le C\|B_n f - f\|_{C[0,1]} \tag{8.4}$$

which, together with (8.3), implies

$$\|B_n f - f\|_{C[0,1]} \approx \omega_\varphi^2\Big(f, n^{-1/2}\Big)_{C[0,1]}. \tag{8.5}$$

In a remarkable paper (see [To,94]) V. Totik gave the first proof of (8.4). He used an intricate modification of the parabola technique. Totik's method is applicable to Bernstein, Szasz and Baskakov operators. Explicitly, Totik treated the Szasz-Mirakian operator given by

$$S_n(f,x) = \sum_{k=0}^{\infty} e^{-nx}\,\frac{(nx)^k}{k!}\,f\big(\frac{k}{n}\big), \tag{8.6}$$

for which he showed

$$\|S_n(f,x) - f(x)\|_{C[0,\infty)} \approx \omega_\varphi^2\Big(f, n^{-1/2}\Big)_{C[0,\infty)} \tag{8.7}$$

where $\omega_\varphi^2(f,t)_{C[0,\infty)}$ is defined on $[0,\infty)$ (instead of $[-1,1]$) and $\varphi(x) = \sqrt{x}$ (instead of $\sqrt{x(1-x)}$ or $\sqrt{1-x^2}$). The proof of (8.7) is neater than that of (8.4) as $[0,\infty)$ has only one finite endpoint and $\sqrt{x}$ is simpler than $\sqrt{x(1-x)}$. Totik stated that the proof in the case of Bernstein and Baskakov operators is essentially the same. To prove (8.4) directly would be just a bit longer, more cluttered and would perhaps obscure the idea.

The second proof of (8.4) was given by Knopp and Zhou (see [Kn-Zh,94]), who used the fact that

$$\frac{1}{n}\,\Big\|\varphi^2\Big(\frac{d}{dx}\Big)^2 B_n^m f\Big\|_{C[0,1]} \le C(m)\|f\|_{C[0,1]} \tag{8.8}$$

with $C(m)$ small enough for some m (independent of n and f) being sufficient. ($B_n^m f = B_n B_n^{m-1} f$ and $B_n^1 f = B_n f$.) It was shown in [Di-Iv, Section 4] for a large class of operators O_n and an appropriate differential operator $P(D)$ that a condition like $\|P(D)O_n^m f\|_B \le C(m)\|f\|_B$ would be sufficient for proving S.C.I. of type A provided that $C(m)$ is small enough. In [Kn-Zh,95] (which precedes [Kn-Zh,94]) a general ingenious method was given to show that under some conditions $C(m) \to 0$ as $m \to \infty$ for many operators. This technique is useful and the conditions necessary are easy to verify when the various operators treated commute, and it is applicable to many spaces (not just L_∞). However, as $B_n B_m f \neq B_m B_n f$ and $\varphi^2 \big(\frac{d}{dx}\big)^2 B_n(f,x) \neq B_n(\varphi^2 f'', x)$ even for very smooth functions, the proof in [Kn-Zh,94] becomes extremely complicated. I note that in papers of X. Zhou with Knoop and others strong converse inequalities are called lower estimate (to match the direct result like (8.3) which Zhou et al. call the upper estimate). Besides this linguistic innovation, and their new idea to show $C(m) = o(1)$ as $m \to \infty$, they also repeated the arguments of [Di-Iv, Sections 3-4], perhaps because they felt they could explain things better.

The third proof of (8.4), given by C. Sanguesa (see [Sa]), uses probabilistic ideas to show that $C(m)$ of (8.8) is sufficiently small for $m = 3$. The ideas of [Sa] can be translated from probabilistic to classical analytic.

While S.C.I. of type B are now quite easy to prove and yield most results about the relation between the K-functional and $\|O_n f - f\|$, S.C.I. of type A are much more elegant and hence more desirable. (They are also more amenable to iterations.) I still would like to see a new simple proof of (8.4) which I am sure will have implications for other operators. One wonders what condition on the sequence of operators (not just the Bernstein polynomials), which is easy to verify, is sufficient to guarantee that a S.C.I. of type B implies a S.C.I. of type A.

As the Bernstein operators are not defined on $L_p[0,1]$ for $1 \le p < \infty$, their Kantorovich modification given by

$$K_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \Big[(n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(u)du\Big] \tag{8.9}$$

was extensively used. (Similar extensions were given to Szasz and Baskakov operators.)

In [Go-Zh] the following S.C.I. of type A is claimed for $1 \le p \le \infty$:

$$\|K_n f - f\|_{L_p[0,1]} \approx \inf \Big(\|f-g\|_{L_p[0,1]} + \frac{1}{n} \Big\| \frac{d}{dx}\, x(1-x)\, \frac{d}{dx}\, g \Big\|_{L_p[0,1]} \Big). \tag{8.10}$$

One recalls that the affine transformation $[-1,1] \to [0,1]$ and (4.2), (4.3) and (4.4) here imply for $1 < p < \infty$

$$\omega_\varphi^2\Big(f, \frac{1}{\sqrt{n}}\Big)_{L_p[0,1]} + \frac{1}{n}\, \|f\|_{L_p[0,1]} \approx \inf \Big(\|f-g\|_{L_p[0,1]} + \frac{1}{n} \Big\| \frac{d}{dx}\, x(1-x)\, \frac{d}{dx}\, g \Big\|_{L_p[0,1]} \Big). \tag{8.11}$$

For $p = 1$ and $p = \infty$ (8.11) is not valid (see [Da-Di,05, p. 88]).

Most of the (multitude of) papers on Bernstein-type operators deal with:

(a) Combinations (for higher levels of smoothness).

(b) Weighted approximation of the operators (see also Sections 10 and 14).

(c) Different step-weights (see also Section 14).

(d) Multivariate analogues (see also Section 12).

(e) Simultaneous approximation.

(f) Shape-preserving properties (see also Section 15).

(g) Other modifications and generalizations.

If I describe all related results on the subject, I will exhaust both myself and the reader (who is probably tired already), and therefore I will try to be somewhat more selective in this survey. Even after remarks in the following sections, the treatment is by no means complete and many, perhaps most, results on the topics (a) – (g) are not described.

The Bernstein polynomial operator preserves many properties. Its rate of convergence is equivalent to $\omega_\varphi^2\big(f,n^{-1/2}\big)_{C[0,1]}$. Realization results using it are valid (and weaker than (8.5)). Moreover, the Bernstein polynomial operator is a model for many other operators, mostly yielding similar or weaker results for $C[0,1]$. Therefore, it was a surprise that a modification emerged that had many "nice" properties, some different from those of B_nf, yet extremely useful. Such an operator, introduced by Durrmeyer (see [Du] and [De,81]), is now called the Durrmeyer-Bernstein polynomial operator and is given by

$$M_n(f,x)=\sum_{k=0}^{n}P_{n,k}(x)(n+1)\int_0^1 P_{n,k}(y)f(y)dy,\quad P_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}. \tag{8.12}$$

Among the properties of $M_n(f,x)$ we state:

I. $M_nf=M_n(f,x):L_p[0,1]\to\Pi_{n+1}$ for $1\le p\le\infty$.

II. $\|M_nf\|_{L_p[0,1]}\le\|f\|_{L_p[0,1]}$ for $1\le p\le\infty$.

III. $\langle M_nf,g\rangle=\langle f,M_ng\rangle$ where $\langle F,G\rangle=\int_0^1 F(x)G(x)dx$.

IV. For $f\sim\sum\limits_{k=0}^{\infty}P_kf,\quad M_nf\sim\sum\limits_{k=0}^{n}a_kP_kf$ where P_kf is given by (4.6) with $\mathcal{D}=[0,1]$ and (4.7) is replaced by

$$\frac{d}{dx}\,x(1-x)\,\frac{d}{dx}\,\varphi_k(x)=-k(k+1)\varphi_k(x),\qquad\int_0^1\varphi_k(x)\varphi_\ell(x)dx=\begin{cases}0 & k\ne\ell,\\ 1 & k=\ell.\end{cases} \tag{8.13}$$

As a result of IV one has:

V. $M_nM_kf=M_kM_nf$.

VI. $\frac{d}{dx}\big(x(1-x)\big)\frac{d}{dx}M_nf=M_n\big(\frac{d}{dx}\,x(1-x)\,\frac{d}{dx}\,f\big)$ for f smooth enough.

VII. $M_nf-f=\sum\limits_{k=n+1}^{\infty}\frac{1}{k(k+1)}\frac{d}{dx}\big(x(1-x)\big)\frac{d}{dx}M_kf$.

Using all these properties, it was shown in [Ch-Di-Iv, Theorem 6.3] for $1\le p\le\infty$ that

$$\|M_nf-f\|_p\approx\inf\Big(\|f-g\|_p+\frac{1}{n}\Big\|\frac{d}{dx}\big(x(1-x)\big)\frac{d}{dx}\,g\Big\|_p\Big). \tag{8.14}$$

Many properties of $M_n f$ were investigated and the proof did not always use the obvious advantages enumerated above (by I $\to$ VII).

Other multiplier-type polynomial approximation processes are the Cesàro means $C_n^\ell(f,x)$ given in (4.5) with $P_k f = P_k(f,x)$ given in (4.6) and $\varphi_k(x)$ given above (in (8.11)) and the Riesz means

$$R_{\lambda,\alpha,\ell} f = \sum_{\lambda(k)<\lambda} \Big(1 - (\frac{\lambda(k)}{\lambda})^\alpha\Big)^\ell P_k f, \quad \lambda(k) = k(k+1). \tag{8.15}$$

F. Dai proved in [Da,03] for $\ell \ge 1$ and $1 \le p \le \infty$ that

$$\|C_n^\ell f - f\|_{L_p[0,1]} \approx \inf\Big(\|f-g\|_{L_p[0,1]} + \frac{1}{n}\big\|\big(P(D)\big)^{1/2} g\big\|_{L_p[0,1]}\Big) \tag{8.16}$$

and

$$\|R_{n^2,\alpha,\ell} f - f\|_{L_p[0,1]} \approx \inf\Big(\|f-g\|_{L_p[0,1]} + \frac{1}{n^{2\alpha}}\big\|\big(P(D)\big)^\alpha g\big\|\Big)_{L_p[0,1]} \tag{8.17}$$

with $P(D) = -\frac{d}{dx}\big(x(1-x)\big)\frac{d}{dx}$.

We note that in this section we use linear operators and S.C.I. which, when applicable, are more powerful than results on K-functionals or realizations.

9 Weighted moduli of smoothness, doubling weights

In a series of articles Mastroianni and Totik introduced the concept of doubling weights and showed that many results about trigonometric polynomials on T and about algebraic polynomials on $[-1,1]$ can be extended (or modified) to include weighted L_p versions with such weights. I will deal here only with results for algebraic polynomials on $[-1,1]$.

Mastroianni and Totik also gave results related to earlier concepts such as the Muckenhoupt A_p condition and others. Let me now briefly describe the concepts involved.

A doubling weight on $[-1,1]$ is a non-negative measurable function $w(x)$ satisfying

$$w(2I) \equiv \int_{2I\cap[-1,1]} w(t)dt \le L\int_I w(t)dt \equiv Lw(I) \tag{9.1}$$

where $I \subset [-1,1]$, $2I$ is the interval with the same midpoint and twice the length of I, and L is the doubling constant. In [Ma-To,00, Lemma 2.1] many definitions equivalent to (9.1) were given.

A non-negative measurable function $w(x)$ is a weight satisfying the A_∞ condition if for any set E, $E \subset I \subset [-1,1]$ with $m(E) \equiv |E| \ge \alpha|I|$

$$w(E) \equiv \int_E w(t)dt \ge \beta w(I) \tag{9.2}$$

with $\beta = \beta(\alpha)$.

A non-negative measurable weight function $w(x)$ satisfies the A_p condition, for some p, $1 \le p < \infty$, if for $q = p/(p-1)$

$$\Big(\frac{1}{|I|}\int_I w(t)dt\Big)\Big(\frac{1}{|I|}\int_I w(t)^{-q/p}dt\Big)^{p/q} \le A \tag{9.3}$$

for all $I \subset [-1,1]$.

A non-negative measurable weight function $w(x)$ satisfies the A^* condition if

$$w(x) \le L\, \frac{1}{|I|} \int_I w(t)dt \tag{9.4}$$

for $x \in I \subset [-1,1]$ and L independent of x and I. (Note that satisfying the A^* condition implies that $w(x)$ is bounded.)

Clearly, the conditions are ordered in increasing strength and the doubling weight condition is the most general (weakest).

In the next section we will describe results for the Jacobi weights which are not known, not valid, or just not applicable to the classes of weights mentioned above. We note that except for A^* the above-mentioned classes of weights contain the Jacobi weights treated in Section 10, and A^* contains the bounded Jacobi weights. Furthermore, we note that, for example, the weight

$$w(t) = h(t) \prod_{j=1}^{k} |t - x_j|^{\gamma_j},$$

with $\gamma_j > -1$, $x_j \in [-1,1]$, $x_j < x_{j+1}$ and $h(t)$ a positive measurable function satisfying $0 < A \le h(t) \le B < \infty$, is a doubling weight.

One defines $w_n(x)$ by

$$w_n(x) = \frac{1}{\Delta_n(x)} \int_{\left(x-\Delta_n(x),x+\Delta_n(x)\right)\cap[-1,1]} w(u)du, \quad \Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \tag{9.5}$$

and notes that $w_n(x)$ is a doubling weight whenever $w(x)$ is.

We denote (as usual)

$$\|f\|_{L_p(w)} = \|f\|_{w,p} = \left\{ \int_{-1}^{1} |f(x)|^p w(x)dx \right\}^{1/p}, \quad 0 < p < \infty, \tag{9.6}$$

$$\|f\|_{L_\infty(w)} = \|f\|_{w,\infty} = \operatorname*{ess\ sup}_{x\in[-1,1]} |f(x)w(x)|, \tag*{(9.6)$'$}$$

$$E_n(f)_{w,p} \equiv \inf_{P_n\in\Pi_n} \|f - P_n\|_{w,p}, \quad \Pi_n = \operatorname{span}\left(1,\dots,x^{n-1}\right), \tag{9.7}$$

and

$$\omega_\varphi^r(f,t)_{w,p} = \sup_{|h|\le t} \|\Delta_{h\varphi}^r f\|_{w,p} \tag{9.8}$$

where $\Delta_{h\varphi}^r f$ is given in (1.2).

A Jackson-type result for general doubling weights and $1 \le p < \infty$ was given (see [Ma-To,98, Theorem 3.2]) by

$$E_n(f)_{w,p} \le \frac{C}{n^r} \|f^{(r)}\varphi_n^r\|_{w_n,p}\,, \quad \varphi_n(x) = \sqrt{1-x^2} + \frac{1}{n} \tag{9.9}$$

where w_n is given in (9.5) and $f,\dots,f^{(r-1)} \in$ A.C.$_{\ell oc}$. For a weight w satisfying the A_p condition w can replace w_n in (9.9) (see [Ma-To,98, Theorem 3.4]), and when $w(x) \approx w_n(x)$ for $x \in [-1+\frac{1}{n^2}, 1-\frac{1}{n^2}]$, both w_n and φ_n can be replaced by w and φ in (9.9) (see [Ma-To,98, Theorem 3.6]). For $p = \infty$ a Jackson-type result was given by

$$E_n(f)_{w_n,\infty} \le \frac{C}{n^r} \|f^{(r)}\varphi^r\|_{w_n,\infty} \tag*{(9.9)$'$}$$

(see [Ma-To,99, Theorem 1.1]).

Clearly, (9.9) and (9.9)′ imply

$$E_n(f)_{w,p} \le C \inf_g (\|f-g\|_{w_n,p} + n^{-r}\|g^{(r)}\varphi_n^r\|_{w_n,p}) \equiv CK_{r,\varphi_n}(f,n^{-r})_{w_n,p} \tag{9.10}$$

for $1 \le p < \infty$, and

$$E_n(f)_{w_n,\infty} \le C \inf_g (\|f-g\|_{w_n,\infty} + n^{-r}\|g^{(r)}\varphi^r\|_{w_n,\infty}) \equiv CK_{n,\varphi}(f,n^{-r})_{w_n,\infty}. \tag{9.10$'$}$$

We note that the price for dealing with such general weights as the doubling weight is that in (9.10) and (9.10)′ we do not have one K-functional but a sequence of (somewhat) different ones which depend on n. (Recall that for $w = 1$, (2.6) and (3.2) imply $E_n(f)_p \le CK_{r,\varphi}(f,n^{-r})_p$ for $1 \le p \le \infty$.) Using [Ma-To,98, Theorem 3.6] and [Ma-To,99, Theorem 1.2], we also have one K-functional when $w_n(x) \approx w(x)$for $x \in [-1+\frac{1}{n^2}, 1-\frac{1}{n^2}]$ and for the class of weights given by A^* as in these cases w_n and φ_n are replaced by w and φ in (9.9).

Following the proof in [Di-To,87, Theorem 2.1.1], one has

$$\omega_\varphi^r(f,t)_{w_n,\infty} \approx K_{r,\varphi}(f,t^r)_{w_n,\infty} \quad \text{and} \quad \omega_{\varphi_n}^r(f,t)_{w_n,p} \approx K_{r,\varphi_n}(f,t^r)_{w_n,p}$$

(see [Ma-To,01, p. 188]).

The converse result

$$\omega_\varphi^{r+2}\Big(f,\frac{1}{n}\Big)_{w_n,\infty} \le Cn^{-r}\sum_{k=1}^n k^{r-1}E_k(f)_{w_k,\infty} \tag{9.11}$$

was proved in [Ma-To,01, (1.8)] where it was shown that in general $r+2$ on the left of (9.11) cannot be improved. For w satisfying the A^* condition, $\omega_\varphi^r(f,1/n)_{w_n,\infty}$ can replace $\omega_\varphi^{r+2}(f,1/n)_{w_n,\infty}$ in (9.11).

Some questions such as: estimating $E_n(f)_{w_n,p}$ by $\omega_{\varphi_n}^r(f,t)_{w_n,p}$ for $0 < p < 1$, the connection between $\omega_{\varphi_n}^r(f,t)_{w_n,p}$ and appropriate realizations, and whether $r+2$ on the left of (9.11) is still necessary for $1 \le p < \infty$, were not considered as far as I know.

A wealth of results about inequalities concerning polynomials on $[-1,1]$ in weighted L_p norms were given in the series of papers mentioned and in particular in [Ma-To,00]. These inequalities will be crucial for further investigations.

For a doubling weight w and for $w_n(x)$ given by (9.5) it was shown [Ma-To,00, Theorem 7.2] that for $P_n \in \Pi_n$ and $1 \le p < \infty$ one has

$$\frac{1}{C}\int_{-1}^1 |P_n|^p w \le \int_{-1}^1 |P_n|^p w_n \le C\int_{-1}^1 |P_n|^p w. \tag{9.12}$$

The Bernstein and Markov inequalities for a doubling weight (see [Ma-To,00, Theorem 7.3 and 7.4]) were given for $1 \le p < \infty$ and $P_n \in \Pi_n$ by

$$\int_{-1}^1 \varphi^p |P_n'|^p w \le Cn^p \int_{-1}^1 |P_n|^p w \tag{9.13}$$

and

$$\int_{-1}^1 |P_n'|^p w \le Cn^{2p}\int_{-1}^1 |P_n|^p w, \quad 1 \le p < \infty \tag{9.14}$$

respectively.

The Nikol'skii inequality was given in two different forms for $1 \le p < q < \infty$ and $P_n \in \Pi_n$ (see [Ma-To,00, p. 67]) by

$$\Big(\int_{-1}^{1} |P_n|^q w\Big)^{1/q} \le C n^{\frac{2}{p}-\frac{2}{q}} \Big(\int_{-1}^{1} |P_n|^p w^{p/q}\Big)^{1/p} \tag{9.15}$$

and by

$$\Big(\int_{-1}^{1} |P_n|^q w\Big)^{1/q} \le C n^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{-1}^{1} |P_n|^p w^{p/q} \varphi^{\frac{p}{q}-1}\Big)^{1/p}. \tag{9.16}$$

Note that for the special case of Jacobi weights a third different form will be presented in the next section, and while (9.15) and (9.16) are best possible of their type, the third form (for Jacobi weights) will also be best possible. I find it amusing to see three different Nikol'skii-type inequalities for algebraic polynomials, all best possible in their way, which treat the weight on the right hand side differently.

For w satisfying the A^* condition one has ([Ma-To,00, p. 69]) the Bernstein inequality

$$\|\varphi P_n' w\|_{L_\infty[-1,1]} \le Cn \|P_n w\|_{L_\infty[-1,1]}, \tag{9.17}$$

the Markov-Bernstein inequality

$$\|P_n' w\|_{L_\infty[-1,1]} \le Cn^2 \|P_n w\|_{L_\infty[-1,1]}, \tag{9.18}$$

and the Nikol'skii-type inequality

$$\|P_n w\|_{L_\infty[-1,1]} \le Cn^{2/p} \|P_n w\|_{L_p[-1,1]}, \quad p < \infty. \tag{9.19}$$

For Jacobi-type weights (9.19) is improved on in the next section, but as applicable to all w satisfying the A^* condition, the inequality (9.19) is best possible as well.

The multivariate analogues were not considered for algebraic polynomials. (For the multivariate situation on the sphere results using spherical harmonic polynomials are treated in [Da,06].) The case $0 < p < 1$ for algebraic polynomials was not considered explicitly. For trigonometric polynomials analogues of Bernstein, Marcinkiewicz, Nikol'skii and Schur type inequalities (but not the Jackson-type inequality) are given for $0 < p$ and doubling weights or A^* weights in [Er]. These results can probably be extended to algebraic polynomials with appropriate modifications but Erdélyi states "For technical reasons we discuss only the trigonometric cases".

10 Weighted moduli, Jacobi-type weights

The Jacobi weights given by

$$w(x) = w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta, \quad \alpha > -1, \quad \beta > -1, \tag{10.1}$$

are doubling weights, (that is, they satisfy (9.1)), and for $\alpha \ge 0$, $\beta \ge 0$ they are also A^* type weights (i.e. satisfying (9.4)). Moreover, for $x \in [-1 + \frac{1}{n^2}, 1 - \frac{1}{n^2}]$ they satisfy $w(x) \approx w_n(x)$, and hence the discussion in the last section implies for $1 \le p < \infty$ and $w(x)$ (see [Ma-To,98, Theorem 3.6])

$$\begin{aligned} E_n(f)_{w,p} &\le C \inf_g \big(\|f-g\|_{w,p} + n^{-r} \|g^{(r)} \varphi^r\|_{w,p}\big) \\ &\equiv C K_{r,\varphi}(f, n^{-r})_{w,p}. \end{aligned} \tag{10.2}$$

For $\alpha \ge 0$, $\beta \ge 0$ (10.2) follows for $p=\infty$ as well (see [Ma-To,99, Theorem 1.2].

For the Jacobi weights different K-functionals (see [Ch-Di,97], [Di,98] and [Da-Di,05]), which are given for $\alpha > -1$, $\beta > -1$ by

$$K_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, t^{2\gamma}\big)_{w_{\alpha,\beta},p} = \inf\big(\|f-g\|_{w_{\alpha,\beta},p} + t^{2\gamma}\|P_{\alpha,\beta}(D)^\gamma g\|_{w_{\alpha,\beta},p}\big) \tag{10.3}$$

where

$$P_{\alpha,\beta}(D) = -w_{\alpha,\beta}(x)^{-1}\,\frac{d}{dx}\,w_{\alpha,\beta}(x)(1-x^2)\,\frac{d}{dx} \tag{10.4}$$

were shown to be useful.

We note that for integer γ the differential operator $\big(P_{\alpha,\beta}(D)\big)^\gamma$ is defined in the usual way, and we describe it below for other γ in a manner similar to the way $P(D)^\gamma$ was described in (4.9) (for the special case $\alpha=\beta=0$). First, we recall the normalized Jacobi polynomial φ_n given by

$$P_{\alpha,\beta}(D)\varphi_n = n(n+\alpha+\beta+1)\varphi_n, \qquad \int_{-1}^{1}\varphi_n(x)\varphi_k(x)w_{\alpha,\beta}(x)dx = \begin{cases}1, & n=k\\ 0, & n\ne k\end{cases} \tag{10.5}$$

and the expansion of f given by

$$f(x) \sim \sum_{k=0}^{\infty} a_k\varphi_k \quad \text{where} \quad a_k = \int_{-1}^{1}\varphi_k(y)f(y)w_{\alpha,\beta}(y)dy, \quad P_kf \equiv P_k^{(\alpha,\beta)}f \equiv a_k\varphi_k\,. \tag{10.6}$$

We now define $P_{\alpha,\beta}(D)^\gamma$ by

$$P_{\alpha,\beta}(D)^\gamma f \sim \sum_{k=1}^{\infty}\big(k(k+\alpha+\beta+1)\big)^\gamma P_kf, \tag{10.7}$$

with P_kf given in (10.6) and $P_{\alpha,\beta}(D)^\gamma f \in B$ whenever there exists $g \in B$ which satisfies $P_k^{(\alpha,\beta)}g \equiv P_kg = \big(k(k+\alpha+\beta+1)\big)^\gamma P_kf$.

In [Da-Di,05, Theorem 7.1] it was shown for $1<p<\infty$ that

$$\|\varphi^r g^{(r)}\|_{w_{\alpha,\beta},p} \approx \big\|P_{(\alpha,\beta)}(D)^{r/2}\big(g - S_{r-1}^{(\alpha,\beta)}g\big)\big\|_{w_{\alpha,\beta},p} \tag{10.8}$$

where $S_{r-1}^{(\alpha,\beta)}g = \sum\limits_{k=0}^{r-1}P_k^{(\alpha,\beta)}g$.

Clearly, for $1<p<\infty$ (10.8) implies

$$\begin{aligned}\big[\inf\big(\|f-g\|_{w_{\alpha,\beta},p} + t^r\|\varphi^r g^{(r)}\|_{w_{\alpha,\beta},p}\big)\big] + t^r E_1(f)_{w_{\alpha,\beta},p}&\\ \approx \inf\big(\|f-g\|_{w_{\alpha,\beta},p} + t^r\|P_{(\alpha,\beta)}(D)^{r/2}g\|_{w_{\alpha,\beta},p}\big)&.\end{aligned} \tag{10.9}$$

For $p=1$ and $p=\infty$ (10.9) is not valid in the case $\alpha=\beta=0$.

For smoothness given by $K_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, t^{2\gamma}\big)_{L_p(w)}$ many results related to approximation were proved. (Some are valid for all Banach spaces of functions for which the Cesàro summability of some order of the Jacobi expansion is bounded.) The boundedness of the Cesàro summability of order r, $r > \max\big(\alpha+\frac12, \beta+\frac12\big)$ for $L_{p,w}[-1,1]$ (with $w = w_{\alpha,\beta}$) was given in [Ch-Di,97, p. 190] as a corollary of earlier results (see also [Du-Xu] for the multivariate case), that is

$$\|C_n^r f\|_{w_{\alpha,\beta},p} \le C\|f\|_{w_{\alpha,\beta},p}, \tag{10.10}$$

(where $C_n^r f \equiv C_n^r(f,x)$ is given by (4.5) with φ_k of (10.5)). Therefore, many theorems on polynomial approximation are valid.

The inequality (10.10) for $\alpha > -1$, $\beta > -1$ and $1 \le p \le \infty$ yields the following results:

(A) A Bernstein-type inequality given by

$$\|P_{\alpha,\beta}(D)^\gamma P_n\|_{w_{\alpha,\beta},p} \le Cn^{2\gamma}\|P_n\|_{w_{\alpha,\beta},p} \tag{10.11}$$

where $\gamma > 0$ (see [Ch-Di,97, (1.9)] and [Di,98, (3.5)]).

(B) A direct or Jackson-type result given by

$$E_n(f)_{w_{\alpha,\beta},p} \le CK_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, n^{-2\gamma}\big)_{w_{\alpha,\beta},p} \tag{10.12}$$

where $\gamma > 0$ (see [Ch-Di,97, (4.2)] and [Di,98, (5.8) and (5.22)]).

(C) A realization result given by

$$K_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, n^{-2\gamma}\big)_{w_{\alpha,\beta},p} \approx \|f - P_n\|_{w_{\alpha,\beta},p} + n^{-2\gamma}\|P_{\alpha,\beta}(D)^\gamma P_n\|_{w_{\alpha,\beta},p} \tag{10.13}$$

where P_n is the best approximant to f (i.e. satisfies $E_n(f)_{w_{\alpha,\beta},p} = \|f - P_n\|_{w_{\alpha,\beta},p}$) or $P_n = V_n f$ where V_n is a de la Vallée Poussin-type operator (see [Di,98, (7.2)]). We remark that there are many operators of the de la Vallée Poussin type and we may choose

$$V_n f \sim \sum_{k=0}^\infty \eta\Big(\frac{k}{n}\Big)P_k f \quad \text{for} \quad f \sim \sum_{k=0}^\infty P_k f$$

where $\eta(t) \in C^\infty[0,\infty)$, $\eta(t) = 1$ for $t \le 1$, and $\eta(t) = 0$ for $t \ge 2$ for example.

(D) The Marchaud-type inequality given by

$$K_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, t^{2\gamma}\big)_{w_{\alpha,\beta},p} \le Ct^{2\gamma}\int_t^1 \frac{K_\eta\big(f, P_{\alpha,\beta}(D)^\eta, u^{2\eta}\big)_{w_{\alpha,\beta},p}}{u^{2\gamma+1}}\,du, \quad \eta > \gamma > 0 \tag{10.14}$$

(see [Ch-Di,97, (5.25)] and [Di,98, (6.7)]).

(E) The converse-type inequality given by

$$K_\gamma\Big(f, P_{\alpha,\beta}(D)^\gamma, \frac{1}{n^{2\gamma}}\Big)_{w_{\alpha,\beta},p} \le Cn^{-2\gamma}\sum_{k=1}^n k^{2\gamma-1}E_k(f)_{w_{\alpha,\beta},p} \tag{10.15}$$

(see [Ch-Di,97, (5.23)] and [Di,98, (6.6)]).

(F) As a result of simultaneous approximation, one has

$$E_n(f)_{w_{\alpha,\beta},p} \le Cn^{-2\gamma}E_n\big(P_{\alpha,\beta}(D)^\gamma f\big)_{w_{\alpha,\beta},p} \tag{10.16}$$

whenever $P_{\alpha,\beta}(D)^\gamma f \in L_{w_{\alpha,\beta},p}$ (see [Di,98, (7.3)]).

In addition, we have the sharp Marchaud-type inequality

$$K_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, t^{2\gamma}\big)_{w_{\alpha,\beta},p} \le Ct^{2\gamma}\Big\{\int_t^1 \frac{K_\eta\big(f, P_{\alpha,\beta}(D)^\eta, u^{2\eta}\big)^q_{w_{\alpha,\beta},p}}{u^{2\gamma q+1}}\,du\Big\}^{1/q} \tag{10.17}$$

for $0 < \gamma < \eta$, $1 < p < \infty$ and $q = \min(p,2)$, which was proved in [Da-Di,05, (6.8)] and we also have the corresponding sharp converse result

$$K_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, t^{2\gamma}\big)_{w_{\alpha,\beta},p} \le Ct^{2\gamma}\Big(\sum_{1\le n<1/t} n^{2\gamma q-1}E_n(f)^q_{w_{\alpha,\beta},p}\Big)^{1/q} \tag{10.18}$$

for $0 < \gamma$, $1 < p < \infty$ and $q = \min(p,2)$ (see [Da-Di,05, (6.10)]).

Another inequality related to Jacobi weights is the Nikol'skii-type inequality

$$\|P_n\|_{w_{\alpha,\beta},q} \le Cn^{\gamma(\frac{1}{p}-\frac{1}{q})}\|P_n\|_{w_{\alpha,\beta},p} \quad \text{for} \quad P_n \in \Pi_n, \tag{10.19}$$

where $\gamma = \max\big(2 + 2\max(\alpha,\beta), 1\big)$ and $0 < p < q \le \infty$ (see [Di-Ti,05, Theorem 6.6] for a somewhat more general result and a simple proof). The inequality (10.19) is also best possible (like (9.15) and (9.16)) as equality holds for $p = 2$ and $q = \infty$.

Recently (see [Da-Di-Ti, Theorem 6.1]) a sharp version of the Jackson inequality (10.12) for $1 < p < \infty$ ($p \ne 1, \infty$) was given by

$$2^{-2n\gamma}\Big\{\sum_{j=1}^{n} 2^{2j\gamma s}E_{2^j}(f)^s_{L_{p,w_{\alpha,\beta}}[-1,1]}\Big\}^{1/s} \le CK_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, 2^{-2n\gamma}\big)_{L_{p,w_{\alpha,\beta}}[-1,1]} \tag{10.20}$$

where $s = \max(p,2)$ and $\gamma > 0$. Similarly, a form equivalent to (10.20) comparing K-functionals and extending (10.14) for $1 < p < \infty$ was achieved (see [Da-Di-Ti, (6.3)]) and is given for $\zeta > \gamma$ and $s = \max(p,2)$ by

$$\begin{aligned} 2^{-nr}\Big\{\sum_{j=1}^{n} 2^{2j\gamma s}K_\zeta\big(f, P_{\alpha,\beta}(D)^\zeta, 2^{-2j\zeta}\big)^s_{L_{p,w_{\alpha,\beta}}[-1,1]}\Big\}^{1/s} \\ \le CK_\gamma\big(f, P_{\alpha,\beta}(D)^\gamma, 2^{-2n\gamma}\big)_{L_{p,w_{\alpha,\beta}}[-1,1]}. \end{aligned} \tag{10.21}$$

For Jacobi weights the Durrmeyer operator

$$M_n^{(\alpha,\beta)}f \equiv M_n^{(\alpha,\beta)}(f,x) = \sum_{k=1}^{n}\big(A_{n,k}^{(\alpha,\beta)}\big)^{-1}P_{n,k}(x)\int_0^1 P_{n,k}(y)f(y)w_{\alpha,\beta}(y)dy \tag{10.22}$$

where $P_{n,k}$ is given in (8.1) and $A_{n,k} = \int_0^1 P_{n,k}(y)w_{\alpha,\beta}(y)dy$ satisfies a strong converse inequality with the K-functional given in (10.3). That is,

$$\|f - M_n^{(\alpha,\beta)}f\|_p \approx K_1\Big(f, P_{\alpha,\beta}(D), \frac{1}{n}\Big)_p \tag{10.23}$$

where $K_1\big(f, P_{\alpha,\beta}(D), \frac{1}{n}\big)_p$ is given in (10.3) (with $\gamma = 1$ and $t^2 = n^{-1}$). This will be discussed, together with its multivariate analogues, in Section 12.

For more information on weighted approximation with Jacobi weights see Section 18.

11 Weighted moduli, Freud weights

To approximate functions on $\mathbb{R}$ by polynomials, one needs to consider weighted approximation. A detailed discussion of this problem appears in the major survey on that topic by Lubinsky [Lu]. Other important sources on the subject are the books by Levin and Lubinsky [Le-Lu] and by Mhaskar [Mh]. Here we just briefly outline the results related to polynomial approximation and put them in the context of this survey.

To investigate the rate of approximation by polynomials to a function in $L_p(W,\mathbb{R})$ given by the norm or quasinorm $\|Wf\|_{L_p(\mathbb{R})}$, one must first ascertain for which type of weights $W(x)$ polynomials are dense in $L_p(W,\mathbb{R})$. Necessary and sufficient conditions on $W(x)$ were given by Akhieser, Carleson, Mergelian and Pollard. (For a more detailed discussion see [Lu, Section 1]).

We deal here with Freud weights (see [Fr], [Lu] and [Mh]) which are given by $W(x) = \exp\big(-Q(x)\big)$ with $Q(x)$ an even continuous function, with $Q''(x)$ continuous, $Q'(x)$ positive in $(0,\infty)$, and for some $a,b>0$

$$a \le \frac{xQ''(x)}{Q'(x)} \le b \quad \text{for} \quad x\in(0,\infty). \tag{11.1}$$

In fact, the results are valid for somewhat more general $Q(x)$, but the most prominent cases, that is, when $Q(x)=|x|^\alpha$, $\alpha>1$ already satisfy the above conditions. To define moduli of smoothness, K-functionals, and realization functionals for the spaces $L_{p,W}(\mathbb{R})$ of functions satisfying $Wf\in L_p(\mathbb{R})$, one needs to define the Mhaskar-Rahmanov-Saff number a_n which is the root of

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)dt}{\sqrt{1-t^2}}\,, \quad n>0. \tag{11.2}$$

The number a_n gives rise to the crucial Remez-type inequalities

$$\|PW\|_{L_\infty(\mathbb{R})} \le \|PW\|_{L_\infty(-a_n,a_n)} \quad \text{for} \quad P\in\Pi_n, \tag{11.3}$$

and for $0<p<\infty$

$$\|PW\|_{L_p(\mathbb{R})} \le (1+e^{-Cn})\|PW\|_{L_p(|x|<a_n+\varepsilon)} \quad \text{for} \quad P\in\Pi_n \tag{11.4}$$

where $C=C(p,\varepsilon,W)$ does not depend on n.

The modulus of smoothness is given by

$$\omega^r(f,W,t)_p = \sup_{0<h\le t}\|W\Delta_h^r f\|_{L_p[-\sigma(h),\sigma(h)]} + \inf_{P\in\Pi_r}\|W(f-P)\|_{L_p[|x|\ge\sigma(t)]} \tag{11.5}$$

where

$$\sigma(h) = \inf\left\{a_n : \frac{a_n}{n}\le h\right\} \tag{11.6}$$

and

$$\Delta_h^r f(x) = \sum_{i=0}^r \binom{r}{i}(-1)^i f\big(x+\frac{rh}{2}-ih\big).$$

We observe that for $W_\alpha(x)=e^{-|x|^\alpha}$ with $\alpha>1$, $a_n\approx n^{1/\alpha}$ and $\sigma\big(\frac{a_n}{n}\big)\approx a_n$. We also note that, following [Di-Lu], the second term on the right of (11.5) is different from that in [Di-To,87, Chapter 11, (11.2.2)] to accommodate the space $L_{p,W}(\mathrm{I\!R})$ of functions for which $Wf \in L_p(\mathrm{I\!R})$ with $0 < p < 1$. (For $1 \le p \le \infty$ the two forms are equivalent.)

The rate of best polynomial approximation is given by

$$E_n(f)_{W,p} = \inf\left(\|W(f-P)\|_{L_p(R)} : P \in \Pi_n\right). \tag{11.7}$$

The K-functional is given by

$$K_r(f, W, t^r)_p = \inf_g \left(\|(f-g)W\|_{L_p(\mathrm{I\!R})} + t^r \|g^{(r)}W\|_{L_p(\mathrm{I\!R})}\right), \tag{11.8}$$

which, following the technique in [Di-Hr-Iv], satisfies

$$K_r(f, W, t^r)_p = 0 \quad \text{for} \quad Wf \in L_p(\mathrm{I\!R}) \quad \text{when} \quad 0 < p < 1. \tag{11.9}$$

The realization functional $\widetilde{R}_r(f, W, t^r)_p$ is given by

$$\widetilde{R}_r(f, W, t^r)_p = \inf\left\{\|(f-P)W\|_{L_p(\mathrm{I\!R})} + t^r\|P^{(r)}W\|_{L_p(\mathrm{I\!R})} : P \in \Pi_n,\ n = \inf\left(k : \frac{a_k}{k} \le t\right)\right\} \tag{11.10}$$

(see [Di-Lu]), or by its equivalent form

$$R_r(f, W, t^r)_p = \|(f - P_n)W\|_{L_p(\mathrm{I\!R})} + t^r \|P_n^{(r)}W\|_{L_p(\mathrm{I\!R})} \tag{11.11}$$

where $n = \inf\left(k : \frac{a_k}{k} \le t\right)$ for P_n satisfying $E_n(f)_{W,p} = \|(f-P_n)W\|_p$, and $P_n \in \Pi_n$.

For $1 \le p \le \infty$ a de la Vallée Poussin-type expression $V_n f$ can replace P_n in (11.11). We note that because of the boundedness of the Cesàro summability of order 1, V_n may be given by the classical form

$$V_n(f, x) = 2C_{2n}(f, x) - C_n(f, x), \quad C_n(f, x) = C_n^1(f, x) \tag{11.12}$$

where $C_n^{\ell}(f, x)$ is given by (4.5) $\left(C_n(f, x) = C_n^1(f, x)\right)$ with $\varphi_k \in \Pi_{k+1}$, orthonormal polynomials with respect to W, i.e.

$$\int_{\mathrm{I\!R}} \varphi_k \varphi_\ell W^2 = \begin{cases} 1, & k = \ell \\ 0, & k \ne \ell. \end{cases}$$

The properties $V_n f \in \Pi_{2n}$, $V_n P = P$ for $P \in \Pi_n$ and $\|WV_n f\|_p \le A\|Wf\|_p$ are clear and given for instance in [Mh, p. 70]. We note that $\eta_n f$ given by (5.4) is also a de la Vallée Poussin-type operator, and $\|W\eta_n f\|_p \le C\|Wf\|_p$ follows from $\|WC_n f\|_p \le A_1\|Wf\|_p$ and summation by parts (Abel transformation).

We define the realization with V_n by

$$R_r^*(f, W, t^r)_p = \|(f - V_n f)W\|_{L_p(\mathrm{I\!R})} + t^r \left\|W\left(\frac{d}{dx}\right)^r V_n f\right\|_{L_p(\mathrm{I\!R})}, \tag{11.13}$$

where $n = \inf\left(k : \frac{a_k}{k} \le t\right)$ and we note that as $V_n f$ and R_r^* are not defined for $0 < p < 1$, the equivalence $R_r^*(f, W, t^r)_p \approx R_r(f, W, t^r)_p$ is valid only for $1 \le p \le \infty$.

For $0 < p \le \infty$ one has

$$\omega^r(f, W, t)_p \approx R_r(f, W, t^r)_p \approx \widetilde{R}_r(f, W, t^r)_p\,. \tag{11.14}$$

For $1 \le p \le \infty$ one has

$$\omega^r(f, W, t)_p \approx K_r(f, W, t^r)_p \approx R_r^*(f, W, t^r)_p\,. \tag{11.15}$$

The Markov-Bernstein inequality is given by

$$\|P'W\|_{L_p(\mathbb{R})} \le C\,\frac{n}{a_n}\,\|PW\|_{L_p(\mathbb{R})}, \quad P \in \Pi_n, \quad 0 < p \le \infty \tag{11.16}$$

where $C = C(p, W)$ does not depend on n or P (see the discussion in [Le-Lu]). The Jackson inequality (see [Di-Lu, p. 102]) is given by

$$E_n(f)_p \le C_1 \omega^r\big(f, W, C_2(a_n/n)\big)_p, \quad 0 < p \le \infty, \tag{11.17}$$

where C_i are independent of n and f. Also C_2 can be replaced by 1 for $1 \le p \le \infty$ (see [Di-Lu, p. 104]). Furthermore, we have the converse (to (11.17)) result (see [Di-Lu, p. 105]) given by

$$\omega^r(f, W, t)_p^q \le C\Big(\frac{a_n}{n}\Big)^{rq} \sum_{j=0}^{\ell} \Big(\frac{2^j}{a_{2^j}}\Big)^{rq} E_{2^j}(f)_{w,p}^q, \quad 0 < p \le \infty, \quad q = \min(p, 1) \tag{11.18}$$

for t small enough and $n = \inf\big(k : \frac{a_k}{k} \le t\big)$. As a corollary of (11.17) and (11.18), we have the Marchaud-type inequality (see [Di-Lu, p. 105]) for $0 < p \le \infty$, $q = \min(p, 1)$ given by

$$\omega^r(f, W, t)_p \le C_1 t^r \Big\{ \int_t^1 \frac{\omega^{r+1}(f, W, t)_p^q}{u^{rq+1}}\, du + \|fW\|_p^q \Big\}^{1/q}. \tag{11.19}$$

For $1 \le p \le \infty$ one has

$$\omega^r(f, W, t)_p \le Ct^r \|f^{(r)} W\|_{L_p(\mathbb{R})}\,. \tag{11.20}$$

As the saturation class of $\omega^r(f, W, t)_p$ for $0 < p < 1$ is $O(t^\gamma)$ with $\gamma > r$, (11.20) is not useful for that range.

I conjecture that for $1 < p < \infty$ a sharp Marchaud and a sharp Jackson inequality will eventually be established. That is, for $1 < p < \infty$, (11.18) and (11.19) will be proved with $q = \min(p, 2)$ rather than with $q = \min(p, 1)$, and an analogue of (2.8) with $s = \max(p, 2)$ will replace (11.17).

I will deal with Nikol'skii and Ul'yanov-type inequalities in Section 13 and with multivariate analogues (essentially the lack thereof) in Section 12.

12 Multivariate polynomial approximation

The space of polynomials of total degree smaller than n, Π_n is given by

$$\Pi_n = \operatorname{span}\Big\{x_1^{\alpha_1} \cdots x_d^{\alpha_d} : \alpha_i = 0, 1, 2, \ldots, \alpha_1 + \cdots + \alpha_d < n\Big\}. \tag{12.1}$$

It is a natural question to ask for what spaces of functions and on what domains one can extend the Bernstein, Jackson, Marchaud and other inequalities. In this section we will outline the progress made after the text [Di-To,87] appeared.

For a convex set S in $\mathbb{R}^d$ it was shown in [Di,92, Theorem2.1] that the Bernstein inequality on the interval can be copied to read for $0 < p \le \infty$ and $r = 1, 2, \ldots$

$$\Big\|\widetilde{d}(\boldsymbol{x}, \boldsymbol{\xi})^{r/2} \Big(\frac{\partial}{\partial \boldsymbol{\xi}}\Big)^r P_n(\boldsymbol{x})\Big\|_{L_p(S)} \le Cn^r \|P_n\|_{L_p(S)} \quad \text{for} \quad P_n \in \Pi_n \tag{12.2}$$

with C independent of $\boldsymbol{\xi}, n, P_n$ and S and with $\widetilde{d}(\boldsymbol{x},\boldsymbol{\xi})$ which was introduced in [Di,92] and given by

$$\varphi_{\boldsymbol{\xi}}(\boldsymbol{x})^2 \equiv \widetilde{d}(\boldsymbol{x},\boldsymbol{\xi}) \equiv \widetilde{d}_S(\boldsymbol{x},\boldsymbol{\xi}) = \sup_{\boldsymbol{x}+\lambda\boldsymbol{\xi}\in S} \lambda \sup_{\boldsymbol{x}-\mu\boldsymbol{\xi}\in S} \mu\,, \quad \boldsymbol{x}\in S, \quad |\boldsymbol{\xi}|=1. \tag{12.3}$$

We note that when S is unbounded, (12.2) is meaningless, and the same is true when the interior of S is empty and $p<\infty$, so we may as well consider only bounded convex sets S with non-empty interior. The introduction of $\varphi_{\boldsymbol{\xi}}(\boldsymbol{x})^2 = \widetilde{d}(\boldsymbol{x},\boldsymbol{\xi})$ in [Di,92], which in (12.2) yields a constant independent of S, is natural since for $S=[-1,1]\subset \mathbb{R}$, (for which only ξ is equal to $\pm e$ where $e=(0,1)$ is possible), $\varphi(x) = \widetilde{d}_S(x,\xi)^{1/2} = \sqrt{1-x^2}\,$.

The Markov-type inequality for a bounded convex set with non-empty interior $S\subset R^d$ was given by (see [Di,92, Theorem 4.1])

$$\Big\| \Big(\frac{\partial}{\partial\boldsymbol{\xi}}\Big)^r P_n \Big\|_{L_p(S)} \le Cn^{2r}\|P_n\|_{L_p(S)}, \quad P_n\in\Pi_n \tag{12.4}$$

where C depends on S and $\boldsymbol{\xi}$ but not on n or P_n.

The Remez-type inequality (see [Di,92, Theorem 3.1]) for a bounded convex set S with non-empty interior is given by

$$\|P_n\|_{L_p(S)} \le C(p,L,S)\|P_n\|_{L_p(S(L,n))}, \tag{12.5}$$

where $S(L,n) = \{\boldsymbol{u} : B(\boldsymbol{u}, L/n^2)\subset S\}$ and $B(\boldsymbol{x},r)$ is the ball of center $\boldsymbol{x}$ and radius r.

Inequalities like (12.4) were studied extensively and for various more general multivariate domains, but the polynomial approximation and its relations to concepts of smoothness generalizing $\omega^r_\varphi(f,t)$ were not tackled even if one assumes that we are dealing with a general bounded convex set with non-empty interior.

In [Di-To,87, Chapter 12] the direct and the weak converse inequalities were proved for $L_p(S)$ (when $1\le p\le\infty$) and where S is a simple polytope. We recall that a polytope is the convex hull of finitely many points in $\mathbb{R}^d$, and a simple polytope is a polytope all of whose vertices are joined to other vertices by exactly d edges. A simplex and a box or a cube are perhaps the most familiar simple polytopes. The Egyptian pyramid is not a simple polytope.

The moduli of smoothness on a polytope S can be given by

$$\widetilde{\omega}^r_S(f,t)_{L_p(S)} = \sup_{\substack{|h|<t\\ \boldsymbol{u}\in E(S)\\ \boldsymbol{v}=\boldsymbol{u}/|\boldsymbol{u}|}} \|\Delta^r_{h\widetilde{d}(\boldsymbol{x},\boldsymbol{v})^{1/2}\boldsymbol{v}} f(\boldsymbol{x})\|_{L_p(S)} \tag{12.6}$$

where $E(S)$ is the set of edges of S and

$$\Delta^r_{h\widetilde{d}(\boldsymbol{x},\boldsymbol{v})^{1/2}\boldsymbol{v}} f(\boldsymbol{x}) = \begin{cases} \sum\limits_{k=0}^{r}(-1)^k\binom{r}{k} f\Big(\boldsymbol{x} + \big(\frac{r}{2}-k\big)h\widetilde{d}(\boldsymbol{x},\boldsymbol{v})^{1/2}\boldsymbol{v}\Big), & \boldsymbol{x}\pm\frac{r}{2}h\widetilde{d}(\boldsymbol{x},\boldsymbol{v})^{1/2}\boldsymbol{v}\in S\\ 0\,, & \text{otherwise.}\end{cases} \tag{12.7}$$

We may also define $\omega^r_S(f,t)_p$ as

$$\omega^r_S(f,t)_{L_p(S)} = \sup_{\substack{|h|\le t\\ |\boldsymbol{v}|=1\ \boldsymbol{v}\in\mathbb{R}^d}} \|\Delta^r_{h\widetilde{d}(\boldsymbol{x},\boldsymbol{v})^{1/2}\boldsymbol{v}} f(\boldsymbol{x})\|_{L_p(S)}\,, \tag{12.8}$$

which is defined for all convex sets S. It is known that for $p=1$ and $p=\infty$ and for a simple polytope S, $\widetilde{\omega}_S(f,t)_p$ is not equivalent to $\omega_S(f,t)_p$ (see [Di-To,87, Remark 12.2.1]). I expect that for $1 < p < \infty$ and a simple polytope S, $\omega_S(f,t)_p \approx \widetilde{\omega}_S(f,t)_p$, though this was not yet proved. The definition of the moduli of smoothness in (12.6) and (12.8) are different from those in [Di-To,87] only in style, using here the somewhat more convenient concept $\widetilde{d}(\boldsymbol{x},\boldsymbol{\xi})$ given in (12.3). For a simple polytope S the Jackson inequality, given by

$$E_n(f)_{L_p(S)} \equiv \inf_{P\in\Pi_n} \|f-P\|_{L_p(S)} \le C\widetilde{\omega}_S^r\Big(f,\frac{1}{n}\Big)_{L_p(S)}, \quad 0<p\le\infty \tag{12.9}$$

was proved in [Di-To,87, Chapter 12] for $1 \le p \le \infty$ and in [Di,96, Theorem 1.1] for $0 < p < 1$.

If the boundary effect is ignored, a Jackson-type estimate is possible for a much more general domain $\mathcal{D}$. We define the modulus $\omega^r(f,t)_{L_p(D)}$ by

$$\omega^r(f,t)_{L_p(\mathcal{D})} = \sup_{|\boldsymbol{h}|\le t} \|\Delta_{\boldsymbol{h}}^r f\|_{L_p(\mathcal{D})} \tag{12.10}$$

where

$$\Delta_{\boldsymbol{h}}^r f(\boldsymbol{x}) = \begin{cases} \sum\limits_{k=0}^{r}(-1)^k\binom{r}{k} f\Big(\boldsymbol{x}+\boldsymbol{h}\big(\frac{r}{2}-k\big)\Big), & \Big\{\boldsymbol{x}+\tau\boldsymbol{h} : |\tau|\le\frac{r}{2}\Big\}\subset\mathcal{D} \\ 0, & \text{otherwise.}\end{cases} \tag{12.11}$$

If one can extend f to be defined on a cube Q in such a way that $g = Ef$ defined on Q satisfies

$$\omega^r(g,t)_{L_p(Q)} \le C\omega^r(f,t)_{L_p(D)} \quad \text{and} \quad f(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x}\in\mathcal{D}, \tag{12.12}$$

and if $Q \supset Q^* \supset D$ where $Q^* + \boldsymbol{\xi} \subset Q$ for $|\boldsymbol{\xi}| \le 1$, then (12.9) clearly implies

$$E_n(f)_{L_p(\mathcal{D})} \le C\omega^r(f,t)_{L_p(\mathcal{D})}. \tag{12.13}$$

Such an extension of f on $\mathcal{D}$ is discussed for instance in [De-Sh], and the results there are given for all p and for many bounded domains.

However, formulae like (12.13) are a departure from the topic of this survey, as the effect of being near the boundary is neglected. Moreover, there is no hope of having a matching converse result to (12.13) as the results described below will imply.

For a simple polytope S the converse result (see [Di-To,87, Theorem 12.2.3] for $1 \le p \le \infty$ and [Di,96] for $0 < p < 1$) is given by

$$\begin{aligned}\widetilde{\omega}_S^r(f,t)_{L_p(S)} &\le \omega_S^r(f,t)_{L_p(S)} \\ &\le Mt^r\Big\{\sum_{1\le k\le\frac{1}{t}} k^{rq-1}E_k(f)^q_{L_p(S)}\Big\}^{1/q}\end{aligned} \tag{12.14}$$

with $q = \min(p,1)$.

The Marchaud-type variation of (12.14) given by

$$\omega_S^r(f,t)_{L_p(S)} \le C\Big\{\int_t^1 \frac{\omega_S^{r+1}(f,u)_p^q}{u^{rq+1}}\,du + \|f\|^q_{L_p(S)}\Big\}^{1/q} \tag{12.15}$$

for $0 < p \le \infty$, $q = \min(p,1)$ and $C = C(r,p,S)$ independent of f and t was proved for a simple polytope S and $1 \le p \le \infty$ in [Di-To,87] and for $0 < p < 1$ in [Di,96, Theorem 5.1].

We note that one does not have the sharp versions of the Jackson, Marchaud and the (weak) converse inequality, that is, the analogues of (2.8), (6.6) and (6.4) to replace (for simple polytopes and $1 < p < \infty$) the inequalities (12.9), (12.14) and (12.15) respectively. I believe that such results will eventually be proved. For the simplex (see [Da-Di,07]) (12.10) was extended, and that result will be described later in this section.

For $1 \le p \le \infty$ and a simple polytope S, $\widetilde{\omega}^r_S(f,t)_{L_p(S)}$ and $\omega^r_S(f,t)_{L_p(S)}$ are equivalent to the K-functionals $\widetilde{K}_{r,S}(f,t^r)_{L_p(S)}$ and $K_{r,S}(f,t^r)_{L_p(S)}$ respectively as stated in the following formulae:

$$\begin{aligned}\widetilde{K}_{r,S}(f,t^r)_{L_p(S)} &\equiv \inf_g \Big(\|f-g\|_{L_p(S)} + t^r \sup_{\boldsymbol{\xi}\in E(S)} \Big\|\varphi^r_{\boldsymbol{\xi}}\Big(\frac{\partial}{\partial\boldsymbol{\xi}}\Big)^r g\Big\|_{L_p(S)}\Big) \\ &\approx \widetilde{\omega}^r_S(f,t)_{L_p(S)}\end{aligned} \tag{12.16}$$

and

$$\begin{aligned}K_{r,S}(f,t^r)_{L_p(S)} &\equiv \inf_g \Big(\|f-g\|_{L_p(S)} + t^r \sup_{\boldsymbol{v}} \Big\|\varphi^r_{\boldsymbol{v}}\Big(\frac{\partial}{\partial\boldsymbol{v}}\Big)^r g\Big\|_{L_p(S)}\Big) \\ &\approx \omega^r_S(f,t)_{L_p(S)}\end{aligned} \tag{12.16$'$}$$

where $\varphi_{\boldsymbol{\xi}}$ is given by (12.3).

While it was not shown explicitly in [Di-Hr-Iv], the method there for $f \in L_p(S)$ with $0 < p < 1$ implies the equality

$$K_{r,S}(f,t^r)_{L_p(S)} = \widetilde{K}_{r,S}(f,t^r)_{L_p(S)} = 0. \tag{12.17}$$

This just adds to the interest in the realization concept.

It was shown in [Di,96, (4.4)] that

$$\widetilde{\omega}^r_S\big(f,\frac{1}{n}\big)_{L_p(S)} \approx \widetilde{R}_{r,S}(f,n^{-r})_{L_p(S)} \equiv \|f-P_n\| + n^{-r} \sup_{\boldsymbol{\xi}\in E_S} \Big\|\varphi^r_{\boldsymbol{\xi}}\Big(\frac{\partial}{\partial\boldsymbol{\xi}}\Big)^r P_n\Big\|_{L_p(S)} \tag{12.18}$$

where $0 < p \le \infty$, $r = 1,2,\dots$, S is a simple polytope and P_n a near best n-th degree polynomial approximant of f in $L_p(S)$. (A similar result holds for $\omega^r_S\big(f,\frac{1}{n}\big)_{L_p(S)}$.)

The multivariate Bernstein polynomials on a simplex $S \subset \mathbb{R}^d$ where

$$S = \Big\{(x_1,\dots,x_d) : 0 \le x_i,\ 0 \le x_0 = 1 - \sum_{i=1}^d x_i\Big\} \tag{12.19}$$

is given by

$$B_n(f,\boldsymbol{x}) = \sum_{|\boldsymbol{k}|\le n} P_{n,\boldsymbol{k}}(\boldsymbol{x}) f\Big(\frac{\boldsymbol{k}}{n}\Big) \tag{12.20}$$

where $\boldsymbol{k} = (k_1,\dots,k_d)$, $|\boldsymbol{k}| = \sum_{i=1}^d k_i$, $k_0 = 1 - |\boldsymbol{k}|$, $x_0 = 1 - |\boldsymbol{x}|$, and

$$P_{n,\boldsymbol{k}}(\boldsymbol{x}) = \frac{n!}{k_0!k_1!\dots k_d!} x_0^{k_0} x_1^{k_1} \dots x_d^{k_d}. \tag{12.21}$$

The Bernstein polynomial is a contraction operator on $C(S)$ with the Voronovskaja

$$n\big(B_n f(\boldsymbol{x}) - f(\boldsymbol{x})\big) \to \frac{1}{2} \sum_{\boldsymbol{\xi}\in E_S,\, \boldsymbol{\xi}=\boldsymbol{\zeta}/|\boldsymbol{\zeta}|} \widetilde{d}_S(\boldsymbol{x},\boldsymbol{\xi}) \Big(\frac{\partial}{\partial\boldsymbol{\xi}}\Big)^2 f(\boldsymbol{x}) \equiv P_S(D). \tag{12.22}$$

In earlier texts a long form of (12.22) was discussed, but the use of (12.3) yields the compact expression (12.22), which also demonstrates the intrinsic symmetry among the edges.

The strong converse inequality

$$\|B_n f - f\|_{C(S)} \approx K\big(f, P_S(D), n^{-1}\big)_{C(S)} = \inf_g \big(\|f-g\|_{C(S)} + n^{-1}\|P_S(D)g\|_{C(S)}\big) \tag{12.23}$$

for S, $B_n f \equiv B_n(f,x)$ and $P_S(D)$ given in (12.19), (12.20) and (12.22) respectively was claimed in [Zh].

For $\alpha < 2$ one has

$$E_n(f)_{C(S)} = O\Big(\frac{1}{n^\alpha}\Big) \Longleftrightarrow \|B_n f - f\|_{C(S)} = O\Big(\frac{1}{n^{\alpha/2}}\Big),$$

which was known earlier.

The multivariate Durrmeyer-Bernstein polynomial approximation on the simplex S (given by (12.19)) with the Jacobi weight $w_{\boldsymbol{\alpha}}(\boldsymbol{x})$ given by

$$w_{\boldsymbol{\alpha}}(\boldsymbol{x}) = \big(1-|\boldsymbol{x}|\big)^{\alpha_0} x_1^{\alpha_1}\dots x_d^{\alpha_d}, \quad \alpha_i > -1, \quad \boldsymbol{x}\in S \tag{12.24}$$

for $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_d)$, $\boldsymbol{x} = (x_1, \dots, x_d)$ and $x_0 = 1 - |\boldsymbol{x}|$ is defined by

$$M_{n,\boldsymbol{\alpha}}(f,\boldsymbol{x}) = \sum_{|\boldsymbol{k}|\le n} P_{n,\boldsymbol{k}}(\boldsymbol{x}) A^{-1}_{n,\boldsymbol{k},\boldsymbol{\alpha}} \int_S P_{n,\boldsymbol{k}}(\boldsymbol{y}) f(\boldsymbol{y}) w_{\boldsymbol{\alpha}}(\boldsymbol{y}) d\boldsymbol{y} \tag{12.25}$$

where $\int_S P_{n,\boldsymbol{k}}(\boldsymbol{y}) w_\alpha(\boldsymbol{y}) dy = A_{n,\boldsymbol{k},\boldsymbol{\alpha}}$.

The behaviour of $M_{n,\boldsymbol{\alpha}}(f,\boldsymbol{x})$ and its rate of approximation in $L_{p,W_{\boldsymbol{\alpha}}}(S)$ were studied in many articles (see [Ch-Di-Iv], [De,85], [Di,95], [Zh] and others).

The Voronovskaja of $M_{n,\boldsymbol{\alpha}} f$ is given by

$$\begin{aligned} n\big(M_{n,\boldsymbol{\alpha}} f(\boldsymbol{x}) - f(\boldsymbol{x})\big) &\to \frac{1}{2}\sum_{\xi\in E_S} \frac{1}{w_{\boldsymbol{\alpha}}(\boldsymbol{x})} \frac{\partial}{\partial\boldsymbol{\xi}} \widetilde{d}_S(\boldsymbol{x},\boldsymbol{\xi}) w_{\boldsymbol{\alpha}}(\boldsymbol{x}) \frac{\partial}{\partial\boldsymbol{\xi}} f(\boldsymbol{x}) \\ &= \frac{1}{2}\widetilde{P}_{S,\boldsymbol{\alpha}}(D), \end{aligned} \tag{12.26}$$

which clearly exhibits both its self-adjointness and dependence on $w_{\boldsymbol{\alpha}}(\boldsymbol{x})$. In [Di,95] the technique of Knoop and Zhou is used (and modified) to obtain the strong converse inequality

$$\begin{aligned} \|M_{n,\boldsymbol{\alpha}} f - f\|_{L_{w_{\boldsymbol{\alpha}},p}(S)} &\approx \inf\big(\|f-g\|_{L_{w_{\boldsymbol{\alpha}},p}(S)} + \frac{1}{n}\|\widetilde{P}_{S,\boldsymbol{\alpha}}(D)g\|_{L_{w_{\boldsymbol{\alpha}},p}(S)}\big) \\ &\equiv K\big(f, \widetilde{P}_{S,\boldsymbol{\alpha}}(D), n^{-1}\big)_{L_{w_{\boldsymbol{\alpha}},p}(S)} \end{aligned} \tag{12.27}$$

where $\widetilde{P}_{S,\boldsymbol{\alpha}}$ is given by (12.25). For $w_{\boldsymbol{\alpha}}(\boldsymbol{x}) = 1$ we denote $M_{n,\boldsymbol{\alpha}} \equiv M_n$ and $\widetilde{P}_{S,\alpha}(D) \equiv \widetilde{P}_S(D)$. Berens et al. (see [Be-Sc-Xu]) conjectured that for $\boldsymbol{\xi} \in E_S$

$$\Big\|\frac{\partial}{\partial\boldsymbol{\xi}} \widetilde{d}_S(\boldsymbol{x},\boldsymbol{\xi}) \frac{\partial}{\partial\boldsymbol{\xi}} f\Big\|_{L_p(S)} \le C\|\widetilde{P}_S(D) f\|_{L_p(S)} \quad \text{for} \quad 1<p<\infty \tag{12.28}$$

and proved (12.28) for $p=2$. In fact, for $p=2$ (12.28) was proved for the weighted case as well (see [Ch-Di,93]). I believe that (12.28) is valid for $1<p<\infty$ even for the weighted case. The inequality

(12.28) would have some worthwhile applications if proved. (For instance, the equivalence between the K-functional in (12.27) when $w_{\boldsymbol{\alpha}} = 1$ and $\widetilde{\omega}_S^2\big(f, \frac{1}{\sqrt{n}}\big)_p$.)

The sharp Marchaud inequality on the simplex with the K-functional given in (12.27) was proved in [Da-Di,07, Theorem 5.1] and is given by

$$K_{2\beta}\big(f, \widetilde{P}_{S,\boldsymbol{\alpha}}(D)^\beta, t^{2\beta}\big)_p \le Ct^{2\beta}\Big\{\int_t^C \frac{K_{2\gamma}\big(f, P_{S,\boldsymbol{\alpha}}(D)^\gamma, u^{2\gamma}\big)_p^q}{u^{2\beta q+1}}\,du\Big\}^{1/q} \tag{12.29}$$

for $1 < p < \infty$, $\beta < \gamma$ and $q = \min(p, 2)$ where $\widetilde{P}_{S,\boldsymbol{\alpha}}(D)$ is given in (12.22) and the K-functionals are defined following (4.10).

The Nikol'skii inequality on $I_d = [-1, 1] \times \cdots \times [-1, 1]$ (see [Di-Ti,07, 6.9]) is given by

$$\|P_n\|_{L_{w_{\boldsymbol{\alpha},\boldsymbol{\beta}},q}(I_d)} \le Cn^{\gamma(\frac{1}{p}-\frac{1}{q})}\|P_n\|_{L_{w_{\boldsymbol{\alpha},\boldsymbol{\beta}},p}(I_d)} \tag{12.30}$$

for $0 < p < q \le \infty$, $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) = \prod_{i=1}^d (1 - x_i)^{\alpha_i}(1 + x_i)^{\beta_i}$ with $\alpha_i, \beta_i > -1$ and $\gamma = \sum_{i=1}^d \max\big(2 + 2\max(\alpha_i, \beta_i), 1\big)$.

The results (12.19) - (12.29) can easily be extended to replace S with a cube. In fact, following remarks in [Di,95,I, Section 5], these results can be extended to Cartesian products of simplices. The multivariate Jackson result for weighted doubling or for Freud-type weights was not studied.

13 Ul'yanov-type result

For trigonometric polynomials Ul'yanov established relations between moduli of smoothness in $L_p(T)$ and moduli of smoothness in $L_q(T)$, $p < q$. (For the most general form of these types of relations on T^d see [Di-Ti,05, Section 2].) A Ul'yanov-type inequality shows quantitatively how measures of smoothness of f in L_p influence the measure of smoothness or the norm of f in L_q when $q > p$. Here we present first the analogous relations for $\omega_\varphi^r(f, t)_p$ proved in [Di-Ti,05].

For $f \in L_p[-1, 1]$, $0 < p < q \le \infty$, $\omega_\varphi^r(f, t)_p$ and $E_n(f)_p$ given by (1.1) and (2.7) respectively, we have (see [Di-Ti,05, Section3])

$$\|f\|_{L_q[-1,1]} \le C\Big[\Big\{\int_0^1 \big(u^{-\theta}\omega_\varphi^r(f, u)_p\big)^{q_1}\,\frac{du}{u}\Big\}^{1/q_1} + \|f\|_{L_p[-1,1]}\Big], \tag{13.1}$$

$$\omega_\varphi^r(f, t)_q \le C\Big(\int_0^t \big(u^{-\theta}\omega_\varphi^r(f, u)_p\big)^{q_1}\,\frac{du}{u}\Big)^{1/q_1}, \tag{13.2}$$

$$\|f\|_{L_q[-1,1]} \le C\Big[\Big\{\sum_{k=1}^\infty k^{q_1\theta-1}E_k(f)_p^{q_1}\Big\}^{1/q_1} + \|f\|_{L_p[-1,1]}\Big], \tag{13.3}$$

and

$$E_n(f)_q \le C\Big\{\sum_{k=n}^\infty k^{q_1\theta-1}E_k(f)_p^q\Big\}^{1/q_1} \tag{13.4}$$

where

$$q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases} \quad \text{and} \quad \theta = 2\Big(\frac{1}{p} - \frac{1}{q}\Big).$$

While (13.1) and (13.2) are valid for $r = 1, 2, \dots$ they are useful only for r big enough ($r > 2(\frac{1}{p} - \frac{1}{q})$ for $p \ge 1$ and $r + \frac{1}{p} - 1 > 2(\frac{1}{p} - \frac{1}{q})$ for $0 < p < 1$).

For a simple polytope $S \subset \mathbb{R}^d$ (see Section 12) the inequalities (13.1) - (13.4) were generalized in [Di-Ti,05, Section 8]. It was proved that for $0 < p < q \le \infty$, $\widetilde{\omega}^r_S(f,t)_{L_p(S)}$ and $E_n(f)_{L_p(S)}$ given by (12.6) and (12.9) respectively, one has

$$\|f\|_{L_q(S)} \le C\Big[\Big\{\int_0^1 \big(u^{-\theta}\widetilde{\omega}^r_S(f,u)_{L_p(S)}\big)^{q_1}\,\frac{du}{u}\Big\}^{1/q_1} + \|f\|_{L_p(S)}\Big], \tag{13.5}$$

$$\widetilde{\omega}^{\,r}_S(f,t)_{L_q(S)} \le C\Big(\int_0^t \big(u^{-\theta}\,\widetilde{\omega}^{\,r}_S(f,u)_{L_p(S)}\big)^{q_1}\,\frac{du}{u}\Big)^{1/q_1}, \tag{13.6}$$

$$\|f\|_{L_q(S)} \le C\Big[\Big\{\sum_{k=1}^{\infty} k^{q_1\theta-1}E_k(f)^{q_1}_{L_p(S)}\Big\}^{1/q_1} + \|f\|_{L_p(S)}\Big], \tag{13.7}$$

and

$$E_n(f)_{L_q(S)} \le C\Big\{\sum_{k=n}^{\infty} k^{q_1\theta-1}E_k(f)^q_p\Big\}^{1/q_1} \tag{13.8}$$

where

$$q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases} \quad \text{and} \quad \theta = 2d\big(\frac{1}{p} - \frac{1}{q}\big).$$

In fact, (13.5) - (13.8) contain (13.1) - (13.4) (when $d = 1$) and (13.1) - (13.4) were presented here explicitly for those interested mainly in the one-dimensional case and in the moduli defined by $\omega^r_\varphi(f,t)_p$, which is less intricate than $\widetilde{\omega}^r_S(f,t)_{L_p(S)}$.

For the weighted $L_p(\mathbb{R})$ with the Freud weight $W_\alpha(x) = \exp(-|x|^\alpha)$ a set of Ul'yanov-type inequalities was given in [Di-Ti,05, Section 9]. For $0 < p < q \le \infty$, $W_\alpha = \exp(-|x|^\alpha)$ ($\alpha > 1$), $E_n(f)_{W_\alpha,p}$ and $\omega^r(f, W_\alpha, t)_p$ given by (11.9) and (11.5) respectively, we have

$$\|W_\alpha f\|_{L_q(\mathbb{R})} \le C\Big[\Big\{\sum_{k=1}^{\infty} k^{q_1\theta-1}E_k(f)^{q_1}_{W_\alpha,p}\Big\}^{1/q_1} + \|W_\alpha f\|_{L_p(\mathbb{R})}\Big], \tag{13.9}$$

$$E_n(f)_{W_\alpha,q} \le C\Big\{\sum_{k=n}^{\infty} k^{q_1\theta-1}E_k(f)^{q_1}_{W_\alpha,p}\Big\}^{1/q_1}, \tag{13.10}$$

$$\|W_\alpha f\|_{L_q(\mathbb{R})} \le C\Big[\Big\{\int_0^1 \big(u^{-\eta}\omega^r(f,W_\alpha,t)_p\big)^{q_1}\,\frac{du}{u}\Big\}^{1/q_1} + \|W_\alpha f\|_{L_p(\mathbb{R})}\Big], \tag{13.11}$$

and

$$\omega^r(f,W_\alpha,t)_q \le C\Big\{\int_0^t \big(u^{-\eta}\omega^r(f,W_\alpha,t)_p\big)^{q_1}\,\frac{du}{u}\Big\}^{1/q_1} \tag{13.12}$$

where

$$q_1 = \begin{cases} q, & q < \infty \\ 1, & q = \infty \end{cases}, \quad \theta = \frac{\alpha-1}{\alpha}\,\big(\frac{1}{p} - \frac{1}{q}\big) \quad \text{and} \quad \eta = \frac{1}{p} - \frac{1}{q}\,.$$

It turns out that the Nikol'ski-type inequalities and realization results using best approximants following (5.5), (11.11) or (12.18) are crucial for the proof of the above-mentioned Ul'yanov-type inequalities.

We note that an inequality like (13.1) can be stated using Besov spaces terminology. The Besov space $B^{\theta}_{p,q}(\varphi, r)$ is given by the norm or quasi-norm

$$\|f\|_{B^{\theta}_{p,q}(\varphi,r)} = \Big(\int_0^1 \big(u^{-\theta}\omega^r_{\varphi}(f,u)_p\big)^q \, \frac{du}{u}\Big)^{1/q} + \|f\|_{L_p[-1,1]}. \tag{13.13}$$

The inequality (13.1) means that for $\theta = 2\big(\frac{1}{p} - \frac{1}{q}\big)$ and $0 < p < q < \infty$, $B^{\theta}_{p,q}(\varphi, r)$ is continuously embedded in $L_q[-1, 1]$, which can be written as

$$B^{\theta}_{p,q}(\varphi, r) \hookrightarrow L_q[-1, 1]. \tag{13.14}$$

In [Di-Ti,05] examples are given to show that the power $q_1 = q$ is optimal when $q < \infty$.

14 $\omega^r_{\varphi^\lambda}(f, t)_\infty$, $0 \le \lambda \le 1$, filling the gap

For $C[-1, 1]$ and $\omega^r(f, t)_{C[-1,1]}$ the classical "pointwise estimate" theory established by Dzyadic, Timan, Brudnyi and others (see [Ti,A]) yields a complete (pointwise) description of polynomial approximation on $C[-1, 1]$. Estimates using $\omega^r_{\varphi}(f, t)_{C[-1,1]}$ yield a complete (norm) description of polynomial approximation on $C[-1, 1]$. While the estimates using $\omega^r_{\varphi}(f, t)_p$ are applicable to all p, $0 < p \le \infty$, one has two different ways to characterize polynomial approximation when $p = \infty$ (for which the relevant theory is on $C[-1, 1]$). In an effort to unify these two theories (for $C[-1, 1]$), one can use the moduli $\omega_{\varphi^\lambda}(f, t)_{C[-1,1]}$ given for $0 \le \lambda \le 1$ and $\varphi(x) = \sqrt{1 - x^2}$ by

$$\omega^r_{\varphi^\lambda}(f, t)_{C[-1,1]} = \sup_{|h| \le t} \|\Delta^r_{h\varphi^\lambda} f\|_{C[-1,1]} \tag{14.1}$$

where

$$\Delta^r_{h\varphi^\lambda} f(x) = \begin{cases} \sum\limits_{k=0}^{r} (-1)^k \binom{r}{k} f\big(x + (\frac{r}{2} - k)h\varphi^\lambda(x)\big) & \text{when} \quad x \pm \frac{r}{2}\, h\varphi^\lambda(x) \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \tag{14.2}$$

(see [Di-Ji]).

Clearly, $\omega^r_{\varphi^\lambda}(f, t)_{C[-1,1]}$ is $\omega^r(f, t)_{C[-1,1]}$ when $\lambda = 0$ and it is $\omega^r_{\varphi}(f, t)_{C[-1,1]}$ when $\lambda = 1$.

The direct estimate using $\omega^r_{\varphi^\lambda}(f, t)_{C[-1,1]}$ proved in [Di-Ji, Theorem 2.1] states that for $f \in C[-1, 1]$ there exists a sequence of polynomials P_n that satisfies

$$|f(x) - P_n(x)| \le C(r, \lambda)\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)_{C[-1,1]} \tag{14.3}$$

where $\delta_n(x) = n^{-1} + \sqrt{1 - x^2}$ and $C(r, \lambda)$ is independent of f and n. The inequality (14.3) fills the gap between (2.6) for $C[-1, 1]$ (when $\lambda = 1$) and the classical estimate (when $\lambda = 0$). The converse result with $\omega^r_{\varphi^\lambda}(f, t)_{C[-1,1]}$ was given in [Di-Ji, Theorem 5.1] as follows. For $f \in C[-1, 1]$ and $\omega(t)$ an increasing function satisfying for some s

$$\omega(\mu t) \le C(\mu^s + 1)\omega(t), \tag{14.4}$$

the existence of a sequence of polynomials P_n satisfying

$$|f(x) - P_n(x)| \le M\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \tag{14.5}$$

implies

$$\omega^r_{\varphi^\lambda}(f,t) \le Mt^r \sum_{0<n\le 1/t} n^{r-1}\omega(n^{-1}). \tag{14.6}$$

Other estimates were given as well, and results using $\omega^r_{\varphi^\lambda}(f,t)_{C[-1,1]}$ were followed in many papers which are not referenced here.

The results mentioned in this section, particularly (14.3) and (14.6), answer a natural question, and filling the gap was a necessary endeavor. However, I feel that one is better off dealing with either $\lambda = 1$ (and the norm estimate) or with $\lambda = 0$ (and the pointwise estimate).

15 Shape-preserving polynomial approximation

Sometimes it is desirable that the polynomial approximating a function on a given interval have the same shape there as the function itself. For example, one may want to approximate a nondecreasing or convex function on $[-1,1]$ by a nondecreasing or convex polynomial on $[-1,1]$. This aspect of polynomial approximation has attracted much attention and dozens of papers have been published, mostly in the last twenty years, covering its many variations. It is clear to me that in this survey, I will not be able to do justice to the topic, which may require a separate survey. I refer the reader to a survey by Leviatan (see [Le]) and two subsequent papers by Kopotun, Leviatan and Shevchuk (see [Ko-Le-Sh,05] and [Ko-Le-Sh,06]) where many of the related results are described. (The words "final frontier" and "conclusion" in the last two articles do not mean that the whole subject of shape-preserving polynomial approximation is to be abandoned by these authors.) One can probably consider this section as an introduction to the subject, rather than a survey of the main results.

For a long time it was known that if $f(x)$ satisfies $\Delta^k_h f(x) \ge 0$ on $\left[\frac{kh}{2}, 1-\frac{kh}{2}\right]$ for some k ($k = 0,1,2,\dots$), then its Bernstein polynomials, $B_n(f,x)$ given in (8.1) satisfy $\Delta^k_h B_n(f,x) \ge 0$ (or $(\frac{d}{dx})^k B_n(f,x) \ge 0$) for that k. We recall that $\Delta^k_h f(x) > 0$ represents a condition on the shape of f; for example, when $k = 0$, then f is positive, when $k = 1$, f is nondecreasing, and when $k = 2$, then f is convex etc. (Recall $\Delta^k_h f(x)$ is given by (2.3).)

It is known that

$$|B_n(f,x) - f(x)| \le C\omega^2\big(f, \sqrt{\frac{x(1-x)}{n}}\,\big)_{C[0,1]}$$

(the pointwise estimate) and

$$\|B_n f - f\|_{C[0,1]} \le C\omega^2_\varphi(f, 1/\sqrt{n})_{C[0,1]} \quad \text{with} \quad \varphi^2 = x(1-x)$$

(the norm estimate).

The approximation by a general polynomial gives rise to faster convergence, that is $n^{-1/2}$ is replaced by n^{-1} and a higher degree of smoothness may be considered. The problem of shape-preserving polynomial approximation is the relation between the shape that is preserved and the rate of approximation achievable under this constraint.

One defines the best constrained polynomial approximation of f satisfying $\Delta^k_h f(x) \ge 0$ on $[-1,1]$ by

$$E^{(k)}_n(f)_p = E^{(k)}_n(f)_{L_p[-1,1]} = \inf\Big(\|f - P_n\|_{L_p[-1,1]} : \Delta^k_h P_n(x) \ge 0 \text{ in } [-1,1],\ P_n \in \Pi_n\Big), \tag{15.1}$$

where $\Delta_h^r f(x) \ge 0$ in $[-1,1]$ means that for all x and h

$$\Delta_h^r f(x) = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} f\big(x + (\frac{r}{2} - \ell)h\big) \ge 0 \quad \text{where} \quad x \pm \frac{rh}{2} \in [-1,1]. \tag{15.2}$$

Shvedov proved (see [Sh, Theorem 3]) that for any constant $A > 0$ and $1 \le p \le \infty$ there exists a function $f \in C^{(k)}[-1,1]$ such that $\Delta_h^k f(x) \ge 0$ in $[-1,1]$ and

$$E_n^{(k)}(f)_{L_p[-1,1]} \ge A\omega^{k+2}(f, 1/n)_{L_p[-1,1]} \text{ for } n \ge k+2 \tag{15.3}$$

where

$$\omega^r(f,h)_{L_p[-1,1]} = \sup \|\Delta_h^r f(\cdot)\|_{L_p[-1+\frac{rh}{2}, 1-\frac{rh}{2}]}. \tag{15.4}$$

The inequality (15.3) shows that not all Jackson-type results can be followed. As $\omega_\varphi^r(f,t)_p \le C\omega^r(f,t)_p$, Shvedov's negative result applies to $\omega_\varphi^r(f,t)_p$ as well, though at the time of publication of Shvedov's article, estimates of polynomial approximation by $\omega_\varphi^r(f,1/n)_p$ were not known. The knowledge that (as expected) not all Jackson-type estimates for polynomial approximation can be followed for shape-preserving polynomial approximation made the pursuit of the remaining possible estimates more interesting. Recently, (see [Bo-Pr, Theorems 1 and 2]) it was shown (in addition to (15.3)) that for the function

$$f(x) = x_+^{k-1} = \begin{cases} x^{k-1}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

which clearly satisfies $\Delta_h^k f(x) \ge 0$ (everywhere) one has

$$E_n^{(k)}(x_+^{k-1})_{L_p[-1,1]} \ge \frac{C(k,p)}{n^2} \quad \text{for} \quad k > 3, \tag{15.5}$$

and, as

$$\omega_\varphi^3(x_+^{k-1}, t)_p \approx \omega^3(x_+^{k-1}, t)_p \approx t^3 \quad \text{for} \quad k > 3, \tag{15.6}$$

it follows that $E_n^{(k)}(x_+^{k-1})_p \ge C\omega^3(x_+^{k-1}, 1/n)_p$ for $n \ge n_0(C)$, $k > 3$, and $p \le \infty$. The same method (see [Bo-Pr, Remark 5]) shows that

$$E_n^{(3)}(x_+^2)_p \ge C\omega^3(x_+^2, 1/n)_p \text{ for } n \ge n_0(C) \text{ and } p < \infty. \tag{15.7}$$

For monotonic functions on $[-1,1]$ satisfying $f \in L_p[-1,1]$, the Jackson theorem for $0 < p \le \infty$ is given in [De-Le-Yu] by

$$E_n^{(1)}(f)_p \le C(p)\omega_\varphi^2(f, 1/n)_p\,. \tag{15.8}$$

For convex functions on $[-1,1]$, i.e. when $\Delta_h^2 f(x) \ge 0$, it was shown that

$$E_n^{(2)}(f)_p \le C\omega_\varphi^3(f, 1/n)_p, \text{ for } 0 < p \le \infty. \tag{15.9}$$

In fact, it was known earlier that $E_n^{(2)}(f)_p \le C\omega_\varphi^2(f,1/n)_p$, and it was clear that a gap existed between that result and (15.3). This gap was closed for $p = \infty$ by Kopotun (see [Ko,94]), and following much of his method, for $0 < p < \infty$ in [De-Hu-Le]. Kopotun (see [Ko,94, p. 156]) also gave the analogue for the pointwise Jackson inequality. That is, he showed that there exists a sequence of convex polynomials $P_n \in \Pi_n$ such that

$$|f(x) - P_n(x)| \le C\omega^3\Big(f,\, \frac{1}{n^2} + \frac{1}{n}\,\sqrt{1-x^2}\Big)_{C[-1,1]}. \tag{15.10}$$

Recently Bondarenko (see [Bo]) showed that when $\Delta_h^3 f(x) \ge 0$ in $[-1,1]$, one has

$$E_n^{(3)}(f)_\infty \le C\omega_\varphi^3(f, 1/n)_\infty. \tag{15.11}$$

In addition, many other related questions were answered, for instance, simultaneous approximation of a function and its derivatives under a shape-preserving constraint or the analogous pointwise estimate under such constraints. As there were over fifty articles on the subject of this section, I could not describe all the results or even just quote them. (At the beginning of this section, I already referred to other sources, i.e. [Le], [Ko-Le-Sh,05] and [Ko-Le-Sh,06].) Perhaps I will mention what might be some unanswered questions:

(I) Is

$$E_n^{(k)}(f)_p \le C(p,k)\omega_\varphi^2(f, 1/n)_p \tag{15.12}$$

valid for all k, $0 < p \le \infty$ and $n \ge n_0(k,p)$? (This is known for $k = 1, 2, 3$ see (15.8), (15.9) and (15.11).)

(II) Can one obtain the estimate

$$E_n^{(3)}(f)_\infty \le C\omega_\varphi^4(f, 1/n)_\infty \text{ for } n \ge n_0? \tag{15.13}$$

16 Average moduli of smoothness (Ivanov's moduli)

In the text by Sendov and Popov (see [Se-Po]) an alternative to the moduli of smoothness on $[a,b]$, T or $\mathbb{R}$ is given and is called averaged moduli of smoothness. These moduli are defined there (see [Se-Po, p. 7]) by

$$\tau_k(f,t)_{L_p[a,b]} = \|\omega_k(f,\cdot\,;t)\|_{L_p[a,b]} \tag{16.1}$$

for a bounded measurable function f where

$$\omega_k(f,x,\delta) = \sup_{|h|\le\delta}\Big\{|\vec{\Delta}_h^k f(\zeta)| : \zeta, \zeta + kh \in \Big[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\Big] \cap [a,b]\Big\} \tag{16.2}$$

and

$$\vec{\Delta}_h^k f(x) = \begin{cases} \sum\limits_{\ell=0}^{k} (-1)^{k-\ell}\binom{k}{\ell} f(x+\ell h), & x, x+kh \in [a,b] \\ 0, & \text{otherwise.} \end{cases} \tag{16.3}$$

We note that $\tau_k(f,t)_{L_p[a,b]}$ given above is not necessarily finite for all $f \in L_p[a,b]$.

In a series of articles (see [Iv]) K. Ivanov introduced averaged moduli to deal with algebraic polynomial approximation. The moduli introduced for $1 \le p,\, q \le \infty$ are given by (see [Iv, p. 187])

$$\tau_k\big(f;\psi(t,\cdot)\big)_{q,p} = \|\omega_k\big(f;\psi(t,\cdot)\big)_q\|_p \tag{16.4}$$

where

$$\omega_k\big(f,x;\psi(t,x)\big)_q = \Big[\frac{1}{2\psi(t,x)}\int_{-\psi(t,x)}^{\psi(t,x)} |\vec{\Delta}_u^k f(x)|^q du\Big]^{1/q},\ q<\infty \tag{16.5}$$

and

$$\omega_k\big(f,x;\psi(t,x)\big)_\infty = \sup\big(|\vec{\Delta}_h^k f(x)|; |h|\le\psi(t,x)\big). \tag{16.6}$$

The restriction $1\le p,q$ is not necessary, and some results in case $0<p=q<1$ were discussed in [Ta,90], [Ta,91], [Ta,95] and [Di-Hr-Iv]. We note that (16.4) is finite for any $f\in L_q$ for a fixed t and p. Moreover, for $f\in L_p$, $0<p\le\infty$, $\tau_k\big(f;\psi(t,\cdot)\big)_{p,q}$ is finite whenever $q\le p$.

For $\psi(t,x)=t^2+t\sqrt{1-x^2}$ and $[a,b]=[-1,1]$ Ivanov proved [Iv, Corollary 5.2] that

$$\tau_r\big(f,\psi(t,\cdot)\big)_{p,p}\approx K_{r,\varphi}(f,t^r)_p \quad\text{for}\quad 1\le p\le\infty. \tag{16.7}$$

One also has

$$\tau_r\big(f,\psi(t,\cdot)\big)_{p,p}\approx \omega_\varphi^r(f,t)_p \quad\text{for}\quad 0<p\le\infty, \tag{16.8}$$

which, for $0<p<1$ was proved by Tachev (see [Ta,95]) and also follows from [Di-Hr-Iv, Section 7]. Ivanov also treated the weighted τ moduli with weights w and ψ satisfying some mild conditions (see [Iv, (3.9), (3.10) and (3.11)]).

The moduli $\tau_r\big(f;\psi(t,\cdot)\big)_{q,p}$ given by (16.4) are a somewhat more cumbersome method to describe smoothness than $\omega_\varphi^r(f,t)_p$, and their computation is more difficult. However, they have some advantages. For instance, the versatility of having separate q and p may prove useful. Also in many proofs one resorts to local averages for obtaining results, and in that direction the averaged moduli may also be helpful. Many of the results of this paper follow for the averaged moduli because of (16.8), and some were proved by Ivanov directly for the τ moduli independent of (16.7) and (16.8).

In conclusion, one should keep in mind the concept given in (16.4) and tools developed by K. Ivanov for possible use in polynomial approximation and other problems.

17 Algebraic addition (Felten's moduli)

In the definition of $\omega_\varphi^r(f,t)_p$ it was clear from the start that $x\pm h\varphi(x)$ may not be in the interval $[-1,1]$ and that $\Delta_{h\varphi}^r f\ne\Delta_{h\varphi}(\Delta_{h\varphi}^{r-1}f)$ (where $\Delta_{h\varphi}^r$ is defined by (1.2)). With his goal to alleviate those two inconveniences (or difficulties) M. Felten (see [Fe,97,I] and [Fe,97,II]) defined the elegant addition

$$a\oplus b = a\sqrt{1-b^2}+b\sqrt{1-a^2}\quad\text{for}\quad a,b\in[-1,1]. \tag{17.1}$$

Felten introduced the difference

$${}_*\Delta_h f(x)=f(x\oplus h)-f(x),\quad {}_*\Delta_h^r f(x)={}_*\Delta_h\big({}_*\Delta_h^{r-1}f(x)\big). \tag{17.2}$$

He then dealt with the space $L_{p,\varphi^{-1}}[-1,1]$ given by the norm

$$\|f\|_{p,\varphi^{-1}}=\Big\{\int_{-1}^1|f(x)|^p\,\frac{dx}{\varphi(x)}\Big\}^{1/p},\quad \|f\|_{\infty,\varphi^{-1}}\equiv\|f\|_\infty=\sup_{-1\le x\le1}|f(x)| \tag{17.3}$$

where $\varphi(x)^2=1-x^2$. (For $p=\infty$ he considered $f\in C[-1,1]$.)

He defined the moduli of smoothness

$$W_\varphi^r(f,t)_p = \sup_{0<h\le t} \| {}_*\Delta_h^r f\|_{p,\varphi^{-1}}. \tag{17.4}$$

Felten proved that $x \oplus h \in [-1,1]$ if $x, h \in [-1,1]$ and used (17.2) for iteration, thus overcoming the inconveniences mentioned above.

For $W_\varphi^r(f,t)_p$ Felten proved that

$$E_n(f)_{p,\varphi^{-1}} = O(n^{-\alpha}) \Longleftrightarrow W_\varphi^r(f,t)_p = O(n^{-\alpha}) \tag{17.5}$$

for $0<\alpha<r$ and $1\le p\le\infty$ where

$$E_n(f)_{p,\varphi^{-1}} = \inf_{P\in\Pi_n} \|f-P\|_{p,\varphi^{-1}}\,. \tag{17.6}$$

Furthermore, he showed that

$$W_\varphi^r(f,t)_p \approx \inf\left(\|f-g\|_{p,\varphi^{-1}} + t^r\|D^r g\|_{p,\varphi^{-1}}\right) \tag{17.7}$$

where $Dg=\varphi g'$ and $D^r g = D(D^r g)$ and the infimum is taken on the class of functions for which $\|D^r g\|_{p,\varphi^{-1}}$ is bounded and $g^{(r-1)}$ is locally absolutely continuous. (One might as well assume that $g^{(r)}$ is continuous in $(-1,1)$ with no ill effect on (17.7) .)

If not for the fact that the weight in (17.3), (17.6) and (17.7) has to be $(1-x^2)^{-1/2}$, this could have been a very important development. Unfortunately though, that weight seems to be crucial (it does not work for the weight 1) and that was perhaps the justified reason why this direction was not pursued. Still I feel this was an interesting effort.

18 Generalized translations

For functions on T the translations $T_t f(x) = f(x+t)$ are multiplier operators given by

$$T_t f^\wedge(n) = e^{int}\widehat{f}(n) \quad \text{where} \quad \widehat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx. \tag{18.1}$$

As trigonometric polynomial approximation is the model for investigation of algebraic polynomial approximation in so many directions, translations using multiplier operators have been examined for this purpose over the last forty years (see [Lo-Pe]). Much work was done by Butzer and mathematicians working with him and under his direction. A survey of those works including some new results was published in 1992 (see [Bu-Ja-St]). The Jacobi translation τ_t is given in [Bu-Ja-St] by

$$(\tau_t f)^\wedge(k) = \psi_k(t)\widehat{f}(k) \tag{18.2}$$

where $\psi_k(t)=\varphi_k(t)/\varphi_k(1)$ and $\varphi_k(t)$ is given by (10.5) and $\widehat{f}(k)=a_k$ of (10.6) for $\alpha,\beta>-1$ as described by (10.1), (10.5) and (10.6).

The moduli of smoothness were given (see [Bu-Ja-St, (3.7), p.171]) by

$$\omega_s^J(f,t)_X = \sup_{\substack{1-t\le h_j<1\\ j=1,\dots,s}} \|\Delta_{h_1}^J \dots \Delta_{h_s}^J f\|_X \tag{18.3}$$

where X is an underlying Banach space (mainly $L_{p,w}[-1,1]$, $1 \le p \le \infty$, with $w = w_{\alpha,\beta}$ of (10.1)) and $\Delta_h^J f$ is given by

$$\Delta_h^J f(x) = \tau_h f(x) - f(x). \tag{18.4}$$

Properties of $\omega_s^J(f,t)_X$ and their relation to best weighted algebraic approximation are described in [Bu-Ja-St]. In that investigation a class of sequences $\{\phi(n)\}$ ($\phi(n) \to 0$) is given for which $E_n(f)$, $\omega_s^J(f,1/n)_X$ and the appropriate K-functionals behave like $\phi(n)$ (see for instance [Bu-Ja-St, Theorem 4.1, p. 183]). In fact, the natural gap between $E_n(f)_X$ and other measures of smoothness is left bigger than necessary and $E_n(f)_X$ and $\omega_s^J(f,t)_X$ are related via some selected sequences and not by direct and weak converse inequalities.

The advantage of using generalized translation is that one has the multiplier operators (18.2) which yield commutativity and other nice properties. The disadvantages are that the computation of $\tau_t f(x)$ and $\omega_s^J(f,t)_X$ for a given f is prohibitive. In fact, rather than learning about the behaviour of $E_n(f)_X$ using that of $\omega_s^J(f,t)_X$, it is actually the behaviour of $\omega_s^J(f,t)_X$ that we learn about by using $E_n(f)_X$. The rate of convergence of $E_n(f)_X$ now has to be investigated using other moduli of smoothness.

For $f \in L_p$, $0 < p < 1$, $\tau_t f$ and $\omega_s^J(f,t)_p$ cannot be defined.

M.K. Potapov continued to explore relations between the rate of approximation by algebraic polynomials and generalized translations, and has published (together with some coauthors) over twenty articles on the subject in the last twenty years. Potapov described in detail generalized translation as an integral operator for various situations. This description is far too long and involved to give here. Another feature of Potapov's investigation is that a relation is given between algebraic polynomial approximation in $L_{p,w_{\alpha,\beta}}[-1,1]$ and a translation induced by the weight $w_{\mu,\nu}$ and the differential operator $w_{\mu,\nu}^{-1}\frac{d}{dx}(1-x^2)w_{\mu,\nu}\frac{d}{dx}$. (Relations are given between the pairs (μ,ν) and (α,β) for which the results are valid.) Potapov and his coauthors in the situation they investigated achieved a direct (Jackson-type) and a weak converse result, which is an improvement on proving that for a sequence $\varphi(n)$, $\varphi(n) \to 0$, that satisfies certain conditions,

$$E_n(f)_{L_p(w_{\alpha,\beta})} \approx \varphi(n) \Longleftrightarrow \widetilde{\omega}^r(f,1/n)_X \approx \varphi(n).$$

Relations with appropriate K-functionals were given but not in the form of the usual equivalence (see [Po,01,I, Theorem 3]).

Clearly, moduli that are defined by integrals or multipliers cannot be defined for $L_p(w_{\alpha,\beta})$ when $0 < p < 1$. Also, computation of the behaviour of these moduli is essentially impossible if one does not use relations with $E_n(f)_{L_p(w_{\alpha,\beta})}$ and learn about $E_n(f)_{L_p(w_{\alpha,\beta})}$ by using other moduli.

I refer the reader who is interested in the approach of Potapov and his coauthors to some of his more recent articles, such as [Po-Ka], [Po,01,I], [Po,01,II] and [Po,05].

19 Lipschitz-type and Besov-type spaces

The Besov-type space that is induced by $\omega_\varphi^r(f,t)_p$ is given by the norm or quasinorm

$$\|f\|_{B^s_{p,q}(\varphi,r)} = \left(\int_0^1 u^{-sq}\omega_\varphi^r(f,u)_p^q \frac{du}{u}\right)^{1/q} + \|f\|_{L_p[-1,1]} \tag{19.1}$$

for $0 < s$, $0 < p \le \infty$ and $q < \infty$, and by

$$\|f\|_{B^s_{p,\infty}(\varphi,r)} = \sup_u \frac{\omega_\varphi^r(f,u)_p}{u^s} + \|f\|_{L_p[-1,1]} \tag{19.2}$$

for $0 < p \le \infty$ and $q = \infty$. The norm or quasi-norm $\|f\|_{B^s_{p,\infty}(\varphi,r)}$ represents a Lipschitz-type space.

It was shown in [Di-To,87, Corollary 7.2.5] for $1 \le p \le \infty$ and in [Di-Ji-Le, Theorem 1.1] for $0 < p < 1$ that for $0 < s < r$

$$\omega^r_\varphi(f,u)_p = O(u^s) \iff E_n(f)_p = O(n^{-s}). \tag{19.3}$$

Therefore, for $s < r$

$$\|f\|_{B^s_{p,\infty}(\varphi,r)} \approx \sup_n n^{-s} E_n(f)_p + \|f\|_{L_p[-1,1]}. \tag{19.4}$$

It was proved for $s < r$ and $1 \le p \le \infty$ (see [Di-To,88, Theorem 2.1] that

$$\|f\|_{B^s_{p,q}(\varphi,r)} \approx \Big\{ \sum_{n=1}^{\infty} n^{sq-1} E_n(f)_p^q \Big\}^{1/q} + \|f\|_{L_p[-1,1]}. \tag{19.5}$$

This equivalence is valid for $0 < p < 1$ as well, which now follows easily the proof in [Di-To,88] and the realization results (see Section 5). Similar results are valid for the moduli using Freud weights $\omega^r(f,W,t)_p$ given in (11.5) and for moduli $\widetilde{\omega}^r_S(f,t)_p$ given by (12.6). For $1 \le p \le \infty$ one can obtain analogues for the K-functionals $K_{2\alpha}\big(f, (-\frac{d}{dx}(1-x^2)\frac{d}{dx})^\alpha, t^{2\alpha}\big)_p$ (see (4.10)), in which case the result is for $s < 2\alpha$.

We observe that the measure of smoothness that is indicated by belonging to a given Besov space is not as sharp as the measures discussed in earlier sections, and that the results described in this section are mere corollaries of results described in earlier sections. We also note that while for $1 \le p \le \infty$ one has to avoid in (19.1) $s = r$ (justifiably), for $0 < p < 1$ we have to avoid $r \le s \le r - 1 + \frac{1}{p}$.

20 Other methods

To investigate K-functionals with step weights such as $\varphi = \sqrt{1-x^2}$ for example, one can use appropriate transformation of the variable and study K-functionals without step weight or weights, which is simpler. Recently, Draganov and Ivanov discussed this method extensively in a long article (see [Dr-Iv]), and it seems that this direction of investigation will continue in their forthcoming papers. The parts relevant to best algebraic polynomial approximation are Corollary 5.3 (p. 139) and 8.1 (p. 145) of [Dr-Iv]. In [Dr-Iv, p. 146] transformations related to Bernstein and Szasz-Mirakian operators on $C[0,1]$ and $C[0,\infty)$ respectively are discussed. For $L_p[0,1]$ the Kantorovich and Durrmeyer operators are discussed in [Dr-Iv, p. 147]. Of course, the K-functional of the transformed function is simpler, but often the difficulty is hidden in the fact that we no longer deal with the original function but with its transformation. Nevertheless, there are cases in which this method yields a real advantage.

In a long paper on the subject the talented M. Dubiner (see [Dub]) described smoothness by the rate of local polynomial approximation. Using local polynomial approximation to obtain global polynomial approximation is not new and was used extensively by many authors. Dubiner's innovation, however, is that he used the local polynomial approximation as the basis for his investigation rather than an intermediate step.

The advantage of the method is that it allegedly yields treatment for multivariate domains that is not accessible by other methods. The disadvantages are that for cases which were not investigated earlier by other methods one cannot estimate the behaviour of such measures of smoothness. Another important deficiency is the lack of converse results.

Moreover, I have difficulty in closing what I perceive to be many gaps in the proofs and in understanding some of the concepts. (While some friends assured me that everything is okay in the article by Dubiner, those that tried to explain were stuck like myself.)

Apart from the above and the lack of inverse theorems, it would be nice if direct and weak converse inequalities for polynomial approximation on a compact subdomain of $\mathbb{R}^d$ could be given. Without weak converse, this type of result is classical see (12.13). With weak converse, it is not known for domains as simple as the unit ball in $\mathbb{R}^d$ when $d > 1$.

Operstein (see [Op]) studied the analogue of the classical theorems on the rate of pointwise polynomial approximation to $L_p[-1,1]$. Operstein proved for $f \in L_p[-1,1]$, $\rho_n = 2^{-n}(1-x^2)^{1/2} + 2^{-2n}$ and $\omega(t)$ satisfying $\omega(t_1+t_2) \le \omega(t_1)+\omega(t_2)$ that there exists a sequence of polynomials $P_n \in \Pi_{2^n+r-1}$ satisfying

$$\Big\|\Big\{ \frac{\|f-P_n\|_{L_p[-1,1]}}{\omega(\rho_n)} \Big\}_{n=1}^{\infty}\Big\|_{\ell_p} \le C\Big\|\Big\{ \frac{\omega^r(f,2^{-n})_p}{\omega(2^{-n})} \Big\}_{n=1}^{\infty}\Big\|_{\ell_p} \tag{20.1}$$

where

$$\|\{a_n\}\|_{\ell_p} = \Big\{ \sum_{n=1}^{\infty} |a_n|^p \Big\}^{1/p}. \tag{20.2}$$

Furthermore, Operstein showed that if for some sequence $\{P_n\}$, $P_n \in \Pi_{2^n}$ one has

$$\Big\|\Big\{ \frac{\|f-P_n\|_{L_p[-1,1]}}{\omega(\rho_n)} \Big\}_{n=1}^{\infty}\Big\|_{\ell_p} \le 1, \quad 1<p<\infty, \tag{20.3}$$

then

$$\omega^r(f,t)_p \le Ct^r \Big\{ \int_t^1 \Big(\frac{\omega(u)}{u^r}\Big)^q \frac{du}{u} \Big\}^{1/q}, \quad \frac{1}{p}+\frac{1}{q}=1. \tag{20.4}$$

One can view (20.1) and (20.4) as the analogues of the pointwise direct and converse result.

21 Epilogue

I have endeavored to mention all directions and progress made regarding the rate of polynomial approximation in the last twenty years. Even though my list of references is quite long, there are over a hundred possible references that might also have been included.

The issue of best constants was not considered. Still, I hope that this survey will be helpful to students and researchers interested in quantitative estimates of polynomial approximation.

I would like to thank F. Dai, A. Prymak and S. Tikhonov for reading a draft of this manuscript and eliminating many misprints.

22 Appendix

The Jackson-type inequality

$$E_n^*(f)_B = \inf(\|f-T_n\|_B : T_n \in \boldsymbol{T}_n) \le C\omega^r(f,1/n)_B \tag*{(2.1)$'$}$$

for a Banach space B on T satisfying (2.4) and (2.5) is essentially known, but as I could not locate a reference for the exact form (2.1)$'$, I am adding a proof here. One first observes that

$$\|\sigma_n f\|_B \le \|f\|_B \quad \text{where} \quad \sigma_n f = \frac{1}{2\pi n}\int_{-\pi}^{\pi} \Big(\frac{\sin\frac{n(t-x)}{2}}{\sin\frac{t-x}{2}}\Big)^2 f(t)dt.$$

Therefore, $E_{2n}(f)_B \le \|f-2\sigma_{2n}f+\sigma_n f\|_B \le 4E_n(f)_B$. We define $F = f*g = \int_{-\pi}^{\pi} f(x+t)g(t)dt$ with g of norm 1 in B^*, the dual of B. For appropriately chosen g (and still $\|g\|_{B^*}=1$)

$$\|f-2\sigma_{2n}f+\sigma_n f\|_B - \varepsilon \le |F(0)-2\sigma_{2n}F(0)+\sigma_n F(0)|.$$

We now have

$$\begin{aligned} E_{2n}(f)_B - \varepsilon &\le \|f-2\sigma_{2n}f+\sigma_n f\|_B - \varepsilon \\ &\le |F(0)-2\sigma_{2n}F(0)+\sigma_n F(0)| \\ &\le \|F-2\sigma_{2n}F+\sigma_n F\|_{C(T)} \le 4E_n(F)_{C(T)} \\ &\le 4C\omega^r(F,1/n)_{C(T)} \le 4C\omega^r(f,1/n)_B, \end{aligned}$$

and as ε is arbitrary (independent of n) and $2^r\omega^r(f,1/2n)_B \ge \omega^r(f,1/n)_B$, (2.1)$'$ is proved. □

References

[Be-Sc-Xu] Berens, H., Schmid, H., and Xu, Y. [1992] Bernstein-Durrmeyer polynomials on a simplex, *J. Approx. Theory* **68**, 247–261.

[Bo] Bondarenko, A.V. [2002] Jackson-type inequality in 3-convex approximation, *East J. Approx.* **8**, 291–302.

[Bo-Pr] Bondarenko, A., and Prymak, A. [2004] Negative results in form preserving approximation of higher order, *Mat. Zametki* **76**(6), 812-826. English translation in *Math. Notes* **76**(6) (2004), 758–769.

[Bu-Ja-St] Butzer, P.L., Jansche, S., and Stens, R.L. [1992] Functional analytic methods in the solution of the fundamental theorems on best weighted algebraic approximation, in *Approximation Theory*, (Memphis, TN, 1991), G. Anastassiou, ed., Lecture Note in Pure and Applied Mathematics, Marcel Dekker, 151–205.

[Ch-Di,93] Chen, W., and Ditzian, Z. [1993] A note on Bernstein-Durrmeyer operators in $L_2(S)$, *J. Approx. Theory* **72**, 234–236.

[Ch-Di,94] Chen, W., and Ditzian, Z. [1994] Strong converse inequality for Kantarovich polynomials, *Constr. Approx.* **10**, 95–106.

[Ch-Di,97] Chen, W., and Ditzian, Z. [1997] Best approximation and K-functionals, *Acta Math. Hungar.* **75**(3), 165–208.

[Ch-Di-Iv] Chen, W., Ditzian, Z., and Ivanov, K. [1993] Strong converse inequality for the Bernstein-Durrmeyer operator, *J. Approx. Theory* **75**, 25–43.

[Da,03] Dai, F. [2003] Some equivalence theorems with K-functionals, *J. Approx. Theory* **121**, 143–157.

[Da,06] Dai, F. [2006] Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights, *J. Funct. Anal.* **235**(1), 137–170.

[Da-Di,05] Dai, F., and Ditzian, Z. [2005] Littlewood-Paley theory and a sharp Marchaud inequality, *Acta Sci. Math. (Szeged)* **71**, 65–90.

[Da-Di,07] Dai, F., and Ditzian, Z. [2007] Cesàro summability and Marchaud inequality, *Constr. Approx.* **25**(1), 73–88.

[Da-Di-Ti] Dai, F., Ditzian, Z., and Tikhonov, S. [2008] Sharp Jackson inequalities, *J. Approx. Theory*, (to appear).

[De,81] Derriennic, M.M. [1981] Sur l'approximation de functions intégrable sur $[0,1]$ par le polynômes de Bernstein modifies, *J. Approx. Theory* **31**, 325–343.

[De,85] Derriennic, M.M. [1985] On multivariate approximation by Bernstein-type polynomials, *J. Approx. Theory* **45** , 155–166.

[De-Hu-Le] DeVore, R.A., Hu, Y.K., and Leviatan, D. [1996] Convex polynomial and spline approximation in L_p, $0 < p < \infty$, *Constr. Approx.* **12**, 409–422.

[De-Le-Yu] DeVore, R.A., Leviatan, D., and Yu, X.M. [1992] Polynomial approximation in L_p, $(0 < p < 1)$, *Constr. Approx.* **8**, 187–201.

[De-Lo] DeVore, R.A., and Lorentz, G.G. [1993] *Constructive Approximation*, Springer Verlag, Berlin.

[De-Sh] DeVore, R., and Sharpley, R. [1993] Besov spaces on domains in R^d, *Trans. Amer. Math. Soc.* **335**, 843–864.

[Di,88] Ditzian, Z. [1988] On the Marchaud-type inequality, *Proc. Amer. Math. Soc.* **103**, 198–202.

[Di,92] Ditzian, Z. [1992] Multivariate Bernstein and Markov inequalities, *J. Approx. Theory* **70**(3), 273–283.

[Di,95] Ditzian, Z. [1995] Multidimensional Jacobi-type Bernstein-Durrmeyer operators, *Acta Sci. Math. (Szeged)* **60**, 225–243.

[Di,95,I] Ditzian, Z. [1995] On best polynomial approximation in $L^2_W(S)$, *J. Math. Anal. Appl.* **194**, 548–559.

[Di,96] Ditzian, Z. [1996] Polynomial approximation in $L_p(S)$ for $p > 0$, *Constr. Approx.* **12**, 241–269.

[Di,98] Ditzian, Z. [1998] Fractional derivatives and best approximation, *Acta Math. Hungar.* **81**, 323–348.

[Di-Hr-Iv] Ditzian, Z., Hristov, V.H., and Ivanov, K.G. [1995] Moduli of smoothness and K-functionals in L_p, $0 < p < 1$, *Constr. Approx.* **11**, 67–83.

[Di-Iv] Ditzian, Z., and Ivanov, K. [1993] Strong converse inequalities, *J. Anal. Math.* **61**, 61–111.

[Di-Ji] Ditzian, Z., and Jiang, D. [1992] Approximation of functions by polynomials in $C[-1,1]$, *Canad. J. Math.* **44**, 924–940.

[Di-Ji-Le] Ditzian, Z., Jiang, D., and Leviatan, D. [1994] Inverse theorems for best polynomial approximation in L_p, $0 < p < 1$, *Proc. Amer. Math. Soc.* **120**, 151–155.

[Di-Lu] Ditzian, Z., and Lubinsky, D.S. [1997] Jackson and smoothness theorems for Freud weights in L_p, $(0 < p \le \infty)$, *Constr. Approx.* **13**, 99–152.

[Di-Ti,05] Ditzian, Z., and Tikhonov, S. [2005] Ul'yanov and Nikol'skii-type inequalities, *J. Approx. Theory* **133**, 100–133.

[Di-Ti,07] Ditzian, Z., and Tikhonov, S. [2007] Moduli of smoothness of functions and their derivatives, *Studia Math.* **180**(2), 143–160.

[Di-To,87] Ditzian, Z., and Totik, V. [1987] *Moduli of Smoothness*, Springer Verlag.

[Di-To,88] Ditzian, Z., and Totik, V. [1988] Remarks on Besov spaces and best polynomial approximations, *Proc. Amer. Math. Soc.* **104**, 1059–1066.

[Di-To,90] Ditzian, Z., and Totik, V. [1990] K-functionals and weighted moduli of smoothness, *J. Approx. Theory* **63**, 3–29.

[Di-Zh] Ditzian, Z., and Zhou, X.L. [1993] Optimal approximation class for multivariate Bernstein operators, *Pacific J. Math.* **158**, 93–120.

[Dr-Iv] Draganov, B.R., and Ivanov, K.G. [2005] A new characterization of weighted Peetre K-functionals, *Constr. Approx.* **21**, 113–148.

[Dub] Dubiner, M. [1995] The theory of multidimensional polynomial approximation, *J. Anal. Math.* **67**, 39–116.

[Du] Durrmeyer, J.L. [1967] Une formule d'inversion de la transformée de Laplace, Applications a la theorie des moments, Thèse de 3e cycle Faculté des Sciences de l'Université de Paris.

[Du-Xu] Dunkl, C.F., and Xu, Y. [2001] *Orthogonal Polynomials of Several Variables*, Cambridge University Press.

[Er] Erdelyi, T. [1999] Notes on inequalities with doubling weights, *J. Approx. Theory* **100**, 60–72.

[Fe,97,I] Felten, M. [1997] Characterization of best algebraic approximation by an algebraic modulus of smoothness, *J. Approx. Theory* **89**, 1–25.

[Fe,97,II] Felten, M. [1997] A modulus of smoothness based on algebraic addition, *Aequationes Math.* **54**, 56–73.

[Fr] Freud, G. [1971] *Orthogonal Polynomials*, Pergamon Press.

[Go-Zh] Gonska, H.H., and Zhou, X.L. [1995] The strong converse inequality for Bernstein-Kantorovich operators, concrete analysis, *Comput. Math. Appl.* **30**, 103–128.

[Hr-Iv] Hristov, V.H., and Ivanov, K. [1990] Realization of K-functionals on subsets and constrained approximation, *Math. Balkanica (N.S.)* **4**, 236–257.

[Iv] Ivanov, K.G. [1989] A characterization of weighted Peetre K-functionals, *J. Approx. Theory* **56**, 185–211.

[Kn-Zh,94] Knoop, H.B., and Zhou, X.L. [1994] The lower estimate for linear positive operators, II, *Results Math.* **25**, 315–330.

[Kn-Zh,95] Knoop, H.B., and Zhou, X.L. [1995] The lower estimate for linear positive operators, I, Constr. Approx. **11**, 53–66.

[Ko,94] Kopotun, K. [1994] Pointwise and uniform estimates for convex approximation of functions by algebraic polynomials, *Constr. Approx.* **10**, 153–174.

[Ko,98] Kopotun, K. [1998] Approximation of k-monotone functions, *J. Approx. Theory* **94**, 481–493.

[Ko-Le-Sh,05] Kopotun, K., Leviatan, D., and Shevchuk, I.A, [2005] Convex polynomial approximation in the uniform norm: conclusion, *Canad. J. Math.* **57**(6), 1224–1248.

[Ko-Le-Sh,06] Kopotun, K., Leviatan, D., and Shevchuk, I.A. [2006] Coconvex approximation in the uniform norm: the final frontier, *Acta Math. Hungar.* **110**(1-2), 117–151.

[Le] Leviatan, D. [2000] Shape preserving approximation by polynomials, *J. Comp. Applied Math.* **121**, 73–94.

[Le-Lu] Levin, E., and Lubinsky, D.S. [2001] *Orthogonal polynomials for exponential weights*, in *CMS Books in Mathematics*, Springer Verlag.

[Lo-Pe] Löfstrom, E.J., and Peetre, J. [1969] Approximation theorems connected with generalized translations, *Math. Ann.* **181**, 255–268.

[Lu] Lubinsky, D.S. [2007] A survey of weighted polynomial approximation with exponential weights, *Surv. Approximation Theory* **3**, 1–105.

[Lu-To] Lubinsky, D., and Totik, V. [1994] Best weighted polynomial approximation via Jacobi expansions, *SIAM J. Math. Anal.* **25**, 555–570.

[Ma-To,98] Mastroianni, G., and Totik, V. [1998] Jackson type inequalities for doubling and A_P weights, in *Proceedings of the Third International Conference on Functional Analysis and Approximation Theory*, Vol I (Aequafredda di Maratea 1996) Rend. Circ. Mat. Palermo (2), suppl. 52 Vol **1**, 83–99.

[Ma-To,99] Mastroianni, G., and Totik, V. [1999] Jackson type inequalities for doubling weights II, *East J. Approx.* **5**, 101–116.

[Ma-To,00] Mastroianni, G., and Totik, V. [2000] Weighted polynomial inequalities with doubling and A_∞ weights, *Constr. Approx.* **16**, 37–71.

[Ma-To,01] Mastroianni, G., and Totik, V. [2001] Best approximation and moduli of smoothness for doubling weights, *J. Approx. Theory* **110**, 180–199.

[Mh] Mhaskar, H.N. [1996] *Introduction to the theory of weighted polynomial approximation*, World Scientific.

[Ni] Nikol'skii, S.M. [1946] On the best approximation by polynomials of functions which satisfy Lipschitz condition, *Izv. Akad. Nauk SSSR* **10**, 295–318 (Russian).

[Op] Operstein, V.A. [1995] A characterization of smoothness in terms of approximation by algebraic polynomials in L_p, *J. Approx. Theory* **81**, 13–22.

[Po,01,I] Potapov, M.K. [2001] On the properties and applications in approximation theory of a family of generalized shift operators (Russian), *Mat. Zametki* **69** 412–426, English translation in *Math. Notes* **69** (2001), 373–386.

[Po,01,II] Potapov, M.K. [2001] Direct and inverse theorems in approximation theory for the m-th generalized modulus of smoothness, *Trudi Mat. Inst. Steklova* **232**, 289–297, translation in *Proc. Steklov Inst. Math.* **232**, 281-289.

[Po,05] Potapov, M.K. [2005] On the interrelations between generalized moduli of smoothness of differentiable functions and their best approximation by algebraic polynomials (Russian), *Vestnik Moskov Univ. Ser. 1 Math. Mekh.* **64** 6, 10–17, translation in *Moscow Univ. Math. Bull.* **60** (2005), 6, 9–16 (2006).

[Po-Ka] Potapov, M.K., and Kazimirov, G.N. [1998] On the approximation of functions that have a given order of the k^{th} generalized modulus of smoothness (Russian), *Math. Zametki* **63**, 425–436, translation in *Math. Notes* **63**, 374–383.

[Sa] Sanguesa, C. [2002] Lower estimates for centered Bernstein-type operators, *Constr. Approx.* **18**, 145–159.

[Se-Po] Sendov, B., and Popov, V. [1988] *The Averaged Moduli of Smoothness*, Wiley-Interscience, John Wiley & Sons.

[Sh] Shvedov, A.S. [1981] Orders of coapproximation of functions by algebraic polynomials, *Mat. Zanetki* **29**, 117–130, English translation in *Math. Notes* **29** (1981), 63–70.

[Ta,90] Tachev, G.T. [1990] A direct theorem for the best algebraic approximation in $L_p[-1,1]$, $(0 < p < 1)$, *Math. Balkanica* **4**, 381–390.

[Ta,91] Tachev, G.T. [1991] A converse theorem for the best algebraic approximation in $L_p[-1,1]$, $(0 < p < 1)$, *Serdica* **17**, 161–166.

[Ta,95] Tachev, G.T. [1995] A note on two moduli of smoothness, *J. Approx. Theory* **81**, 136–140.

[Ti,A] Timan, A.F. [1963] *Theory of Approximation of Functions of Real Variable*, Pergamon Press.

[Ti,M,58] Timan, M.F. [1958] Inverse theorems of the constructive theory of functions in the space L_p, *Mat. Sb. N.S.* **46**(88), 125–132 (Russian).

[Ti,M,66] Timan, M.F. [1966] On Jackson's theorem in L_p spaces, *Ukr. Mat. Zh.* **18**, 134–137 (Russian).

[To,88] Totik, V. [1988] Sharp converse theorem of L_p polynomial approximation, *Constr. Approx.* **4**, 419–433.

[To,94] Totik, V. [1994] Approximation by Bernstein polynomials, *Amer. J. Math.* **116**, 995–1018.

[Tr-Be] Trigub, R.M., and Belinsky, E.S. [2004] *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers.

[Wa] Wang, H. [1993] *Approximation by polynomials in Orlicz spaces*, Master dissertation (Chinese), Beijing Normal University, Beijing.

[Zh] Zhou, X.L. [1994] Approximation by multivariate Bernstein operators, *Results Math.* **25**, 166–191.

[Zh-Ca-Xu] Zhang, P., Cao, F., and Xu, Z. [2003] Multivariate modulus of smoothness in Orlicz spaces and its application, *Adv. Math.* **32**, No. 6, 695–705.

[Zy] Zygmund, A. [1950] A remark on the integral modulus of continuity, *Univ. Nac. Tucuman Rev. Ser.* **A7**, 259–269.

Department of Mathematical
and Statistical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1
zditzian@math.ualberta.ca