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ESTIMATES OF THE HIGHER ORDER DERIVATIVES OF THE SOLUTIONS OF HYPOELLIPTIC EQUATIONS

Abstract. In this work we establish a connection between the behaviour of the higher order derivatives of the solutions of the hypoelliptic equation $P(D)u = f$ and the estimates of the derivatives of $f(x)$ in terms of multianisotropic Gevrey classes.

1. Introduction

The Gevrey classes play an important role in the theory of linear partial differential equations as intermediate spaces between the C^∞ and the analytic functions. In particular, whenever the properties of an operator differ in the C^∞ and in the analytic framework, it is natural to test its behaviour in the classes of the Gevrey functions and distributions. As a matter of facts, that weak solutions of the equation $P(D)u = f$ belong to C^∞, in particular to Gevrey classes, is important for the application of variational methods to the differential equation. A complete description of linear differential equations with constant coefficients having only C^∞ solutions for all infinitely differentiable right-hand sides has been given by L. Hörmander [8]. Equations of this type are called hypoelliptic.

It is well known (cf. [8], Chapter 11) that the regularity of the solutions of the hypoelliptic equation $P(D)u = f$ is determined by the behaviour of the function $d_P(\xi)$ as $\xi \to \infty$, where $d_P(\xi)$ is the distance from the point $\xi \in \mathbb{R}^n$ to the surface $\{\zeta : \zeta \in \mathbb{C}^n, P(\zeta) = 0\}$.

The behaviour of the function $d_P(\xi)$ at infinity is related to many properties of the solutions of an hypoelliptic equation $P(D)u = 0$, in particular it belongs to the Gevrey class $G^\lambda(\Omega)$, where $\lambda \in \mathbb{R}^n$ is determinated by the growth of the function $d_P(\xi)$ if $\xi \in \mathbb{R}^n$ and $|\xi|$ is sufficiently large (cf. [8], Theorem 11.4.1) (for the definition of the Gevrey classes $G^\lambda(\Omega)$, see for example [8] Def. 11.4.11).

V. Grushin [3, 4] proved that if $P(D)$ is an hypoelliptic operator with index of hypoelliplticity equal to λ, then all the solutions of the nonhomogeneous equation $P(D)u = f$ belong to $G^\lambda(\Omega)$ if $f \in G^\lambda(\Omega)$.

In [2] L. Cattabriga derived for an hypoelliptic operator $P(D)$ the algebraic conditions ensuring that the map $P(D) : G^\lambda(\mathbb{R}^n) \mapsto G^\lambda(\mathbb{R}^n)$ is an isomorphism. Such hypoelliptic operators are called G^λ-hypoelliptic operators. The G^λ-hypoelliptic operators have been studied by many authors: L.R.Volevich, B.Pini, L.Rodino, L.Zanghirati and others. Detailed references for G^λ-hypoelliptic operators can be found in the books

L. Rodino [12, 1].

G.Ghazaryan [9] introduced some functional characteristics, called weight of hypoellipticity, which coincides with the function $h(\xi) = |\xi|$ in the elliptic case and is specified in the general case. Moreover, more fine estimates of higher order derivatives of the solutions of an hypoelliptic equation $P(D)u = 0$ are obtained.

After introducing in [5] the concept of multianisotropic Gevrey classes, it became possible to improve the above mentioned results and formulate a general theorem, establishing the relationship between the growth of the derivatives of the solutions of the hypoelliptic equation $P(D)u = f$ and the growth of the function f. We shall prove:

THEOREM 1. *Let* $f \in G^B(\Omega)$. *Then any solution of the hypoelliptic equation* $P(D)u = f$ *belongs to* $G^{B\cap A_P}(\Omega)$.

For a convex set B, $G^B(\Omega)$ is the associated multianisotropic Gevrey class, and the set A_P is determined by the hypoelliptic operator $P(D)$.

2. Definitions and notations

Let $P(D) = \sum_\alpha \gamma_\alpha D^\alpha$ be a linear differential operator with constant coefficients, and let $P(\xi)$ be its characteristic polynomial . Here the sum goes over a finite set $(P) = \{\alpha : \alpha \in \mathbb{N}_0^n, \gamma_\alpha \neq 0\}$, where $\mathbb{N}_0^n = \{\alpha = (\alpha_1, \dots, \alpha_n) : \ \alpha_i \in \mathbb{N}_0, \ i = 1, \dots, n\}$ is the set of n-dimensional multi-indices.

We denote:

$$\mathbb{R}_0^n = \{\xi \in \mathbb{R}^n : \xi_1 ... \xi_n \neq 0\},$$
$$\mathbb{R}_+^n = \{\xi \in \mathbb{R}^n : \xi_j \geq 0, \ j = 1, ...n\}.$$

Let $A = \{\nu^k \in \mathbb{R}_+^n, k = 0, ..., m\}$.

DEFINITION 1. *The characteristic polyhedron (or Newton polyhedron) (C.P.)* $\mathcal{N}(A)$ *of the set A is defined to be the smallest convex polyhedron in* $\mathbb{R}_+^n$ *containing all the points* $A \cup \{0\}$. *The characteristic polyhedron (or Newton polyhedron) (C.P.)* $\mathcal{N} = \mathcal{N}(P)$ *of a polynomial* $P(\xi)$ *(or of a operator* $P(D)$*) is defined to be the smallest convex polyhedron in* $\mathbb{R}_+^n$ *containing all the points* $(P) \cup \{0\}$.

DEFINITION 2. *A polyhedron* $\mathcal{N}$ *is said to be completely regular (C.R.) if:*

a) $\mathcal{N}$ *has vertices at the origin and on all the coordinate axes of* $\mathbb{N}_0^n$ *different from the origin.*

b) all the coordinates of the exterior normals to the non-coordinate $(n-1)$ *- dimensional faces* $\mathcal{N}$ *are strictly positive.*

It is well known that if $P(D)$ is an hypoelliptic operator, then C.P. of $P(D)$ is a C.R.

Let $h(\xi) = \sum_{k=0}^{m} |\xi|^{\nu^k}$, where $\nu^0 = 0, \nu^k \in \mathbb{R}_+^n, |\xi|^{\nu^k} = |\xi_1|^{\nu_1^k} \cdot ... \cdot |\xi_n|^{\nu_n^k}$, and

$A_h = \{\nu^k\}_{k=0}^m$.

DEFINITION 3. *(cf. [9]) A function $h(\xi)$ is called weight of hypoellipticity of the polynomial $P(\xi)$ (or of the operator $P(D)$) if there exists a constant $C > 0$ such that:*

$$F_P(\xi) = \sum_{\alpha \neq 0} \left(\frac{|D^\alpha P(\xi)|}{|P(\xi)| + 1} \right)^{\frac{1}{|\alpha|}} \leq \frac{C}{h(\xi)}, \quad \forall \xi \in \mathbb{R}^n. \tag{1}$$

DEFINITION 4. *A weight of hypoellipticity of the operator $P(D)$ is called exact weight of hypoellipticity of the operator $P(D)$ if for any $\nu \in \mathbb{R}^n_+ \setminus \mathcal{N}(A_h)$:*

$$\sup_\xi |\xi^\nu| F_P(\xi) = +\infty.$$

By Lemma 11.1.4 of [8], for any weight of hypoellipticity of the operator $P(D)$, there exists a constant $C > 0$ such that:

$$h(\xi) \leq C \cdot (1 + d_P(\xi)), \ \ \forall \ \xi \in \mathbb{R}^n. \tag{2}$$

We denote by Λ^{n-1} the set of the exterior normals λ (relative to $\mathcal{N}(P)$) of the non-coordinate $(n-1)$ - dimensional faces $\mathcal{N}(P)$, for which $\min\limits_{1 \leq i \leq n} \lambda_i = 1$.

We set:

$$\begin{aligned} \mathcal{M}_P &= \{\nu : \nu \in \mathbb{R}^n_+, \sup_\xi |\xi^\nu| \cdot F_P(\xi) < \infty\}, \\ E(\mathcal{N}(P)) &= \{\nu \in \mathbb{R}^n_+, (\nu, \lambda) \leq 1, \forall \lambda \in \Lambda^{n-1}\}. \end{aligned}$$

It is well known (cf. [9], Lemma 3.5), that for any hypoelliptic operator $P(D)$ the set $\mathcal{M}_P$ is included in $E(\mathcal{N}(P))$.

LEMMA 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $P(D)$ be an hypoelliptic operator and $h(\xi)$ be a weight of hypoellipticity of the operator $P(D)$. Then there exists a constant $C > 0$ such that for any function $V \in C_0^\infty(\Omega)$, any $\varepsilon \in (0, 1)$ and any natural number l the following estimate is satisfied:*

$$\sum_{0 \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| P^{(\alpha)}(\xi)(\varepsilon h(\xi))^l F(V) \right|\right|^2_{L_2(\mathbb{R}^n)} \leq$$
$$C \sum_{\alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| P^{(\alpha)}(\xi)(\varepsilon h(\xi))^{l-1} F(V) \right|\right|^2_{L_2(\mathbb{R}^n)},$$

where $F(V)$ is the Fourier transform of the function $V(x)$.

This lemma can be proved similary to Lemma 11.1.4 of [8] using the estimates (2).

For a bounded set $\Omega \subset \mathbb{R}^n$ and for $\varepsilon > 0$, we denote $\Omega_\varepsilon = \{x : x \in \Omega, \ \rho(x, \partial\Omega) > \varepsilon\}$, where ρ is a distance in $\mathbb{R}^n$. Let $\delta \in (0, 1]$, r be a natural number, $B = \{x : x \in$

$\mathbb{R}^n, ||x|| < 1\}$ and $\varphi(x) \geq 0$ a function such that $\varphi \in C_0^\infty(B)$, $\int \varphi(x)dx = 1$. Denote by $\varphi_r^\delta(x) = \chi_{\Omega_{r\cdot\delta-\frac{\delta}{2}}} * \varphi^{\frac{\delta}{2}}(x)$, where for any $\varepsilon > 0$, $\varphi^\varepsilon(x) = \varepsilon^{-n} \cdot \varphi(\frac{x}{\varepsilon})$ and $\chi_{\Omega_\varepsilon}(x)$ is the characteristic function of the set Ω_ε.

LEMMA 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, l be a natural number. Then there exists a constant $C_l = C_l(\Omega) > 0$ such that:*

$$\sup_{x\in\Omega} |D^\alpha \varphi_j^\varepsilon| \leq C_l \cdot \varepsilon^{-|\alpha|}, \quad |\alpha| \leq l, \quad j = 1, 2, \ldots .$$

The proof easily follows from the following computations:

$$\begin{aligned} |D^\alpha \varphi_j^\varepsilon(x)| &= \left| \int \chi_{\Omega_{j\varepsilon-\frac{\varepsilon}{2}}}(y) \cdot D^\alpha \varphi^{\frac{\varepsilon}{2}}(x-y)dy \right| \\ &= \left| \left(\frac{\varepsilon}{2}\right)^{-|\alpha|} \int_{\Omega_{j\varepsilon-\frac{\varepsilon}{2}}} (D^\alpha \varphi)^{\frac{\varepsilon}{2}}(x-y)dy \right| \\ &\leq \left(\frac{\varepsilon}{2}\right)^{-|\alpha|} \int |D^\alpha \varphi(y)|dy| \leq C_l \varepsilon^{-|\alpha|}, \end{aligned}$$

where $C_l = \max_{|\alpha|\leq l} 2^{-|\alpha|} \int |D^\alpha \varphi(y)|dy$.

Let $\lambda^j \in \mathbb{R}_+^n \quad (j = 1, ..., k)$ be vectors with rational coordinates, for which $\min_{1\leq l\leq n} \lambda_l^j = 1$, and $d_j \quad (0 < d_j \leq 1,\ j = 1, ..., k)$ be rational numbers such that the set $A_P = \{\nu \in \mathbb{R}_+^n : (\nu, \lambda^i) \leq d_i, \quad i = 1, ..., k\} \subset \mathcal{M}_P$ is C.R.. We denote by A_P^0 the set of the vertices of the polyhedron A_P.

We let $h_{A_P}(\xi) = \sum_{\nu\in A_P^0} |\xi|^{\nu^k}$.

LEMMA 3. *Let $P(D)$ be an hypoelliptic operator (ordP=m), l be a natural number. Then there exists a constant $C > 0$ such that for any $\varepsilon \in (0, 1)$, $\beta \in lA_P \cap \mathbb{N}_0^n$ and any function $u \in C^\infty(\Omega)$ the following estimate is satisfied:*

(3)

$$\begin{aligned} \varepsilon^{2l} \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| D^\beta P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon j})} &\leq C \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon\cdot(j-1)})} \\ &+ C \sum_{i=1}^{l} \sum_{|\beta|\leq(i-1)m,\ \beta\in\mathbb{N}_0^n} \left|\left| (\varepsilon \cdot h_{A_P}(\xi))^{l-i} \cdot \varepsilon^{|\beta|} F(D^\beta \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(R^n)}, \\ & \qquad j = 1, 2, ... \end{aligned}$$

where $\omega \subset\subset \Omega$.

Proof. For some constant $C > 0$ and for any $\beta \in l \cdot A_P \cap \mathbb{N}_0^n$ we have $|\xi|^\beta \leq$

$Ch^l_{A_P}(\xi),\ \forall \xi \in \mathbb{R}^n$. Then by Parceval equality there is a constant $C_1 > 0$ such that:

$$\begin{aligned}
&\varepsilon^{2l} \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| D^\beta P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon j})} \\
&\qquad \le \varepsilon^{2l} \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| D^\beta P^{(\alpha)}(D)(u\varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&\qquad = \varepsilon^{2l} \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| \xi^\beta P^{(\alpha)}(\xi) F(u\varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&\qquad \le C_1 \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| (\varepsilon \cdot h_{A_P}(\xi))^l P^{(\alpha)}(\xi) F(u\varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)}.
\end{aligned}$$

By Lemma 1 there is a constant $C_2 > 0$ such that:

$$\begin{aligned}
&\varepsilon^{2l} \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| D^\beta P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon j})} \\
&\qquad \le C_2 \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| (\varepsilon \cdot h_{A_P}(\xi))^{l-1} P^{(\alpha)}(\xi) F(u\varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)} \qquad (4) \\
&\qquad + C_2 \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} P(\xi) F(u \cdot \varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)}.
\end{aligned}$$

By Newton-Leibniz formula, we can estimate the second term of the right hand side of (4) for a constant $C_3 > 0$:

$$\begin{aligned}
&\left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} P(\xi) F(u\varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&\le \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} F(\varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&+ C_3 \sum_{0\neq\alpha} \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} F(P^{(\alpha)}(D)u(D^\alpha \varphi_j^\varepsilon)) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&= \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} F(\varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&+ C_3 \sum_{0\neq\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} F(P^{(\alpha)}(D)u\varepsilon^{|\alpha|} D^\alpha \varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&\le \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} F(\varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\
&+ C_3 \sum_{\alpha\neq 0} \sum_{|\beta|\le m,\ 0\neq\beta\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} F(P^{(\alpha)}(D)u\varepsilon^{|\beta|} D^\beta \varphi_j^\varepsilon) \right|\right|^2_{L_2(\mathbb{R}^n)}.
\end{aligned}$$

By the estimate (4) there is a constant $C_4 > 0$ such that:

$$\varepsilon^{2l} \sum_{0 \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| D^\beta P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon j})}$$
$$\leq C_4 \sum_{0 \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \sum_{|\beta| \leq m} \left|\left| (\varepsilon h_{A_P}(\xi))^{l-1} P^{(\alpha)}(\xi) \varepsilon^{|\beta|} F(D^\beta \varphi_j^\varepsilon u) \right|\right|^2_{L_2(\mathbb{R}^n)}$$
$$+ C_4 \cdot \left|\left| (\varepsilon \cdot h_{A_P}(\xi))^{l-1} F(\varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)}.$$

Going on analogously as above, since $supp\varphi_j^\varepsilon \subset \omega_{\varepsilon \cdot (j-1)}$ at step $(l-1)$, then we obtain the estimate (3).

□

We denote by s the smallest natural number such that $s \cdot A_P^0 \subset \mathbb{N}_0^n$; and for any multi-index $\alpha \in s \cdot A_P^0$, there is $\beta \in \mathbb{N}_0^n$ such that $\alpha = 2 \cdot \beta$. We set $Q_P(\xi) = \sum_{\beta \in s A_P^0} \xi^\beta$, $q(\xi) = |Q(\xi)|^{\frac{1}{s}}$. Let $Q_P(D)$ be a differential operator, and $Q_P(\xi)$ its corresponding polynomial. In Lemma 3 we can take $q(\xi)$ in place of $h_{A_P}(\xi)$.

LEMMA 4. *Let $P(D)$ be an hypoelliptic operator (ordP=m). Then there is a constant $C > 0$ such that, for any $\varepsilon \in (0, 1)$, and any function $u \in C^\infty(\Omega)$ the following estimate is satisfied:*

$$\varepsilon^{2s} \cdot \sum_{0 \neq \alpha \in N_0^n} \varepsilon^{-2|\alpha|} \left|\left| Q_P(D) P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon j})}$$
$$\leq C \cdot \sum_{0 \neq \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| P^{(\alpha)}(D)u \right|\right|^2_{L_2(\omega_{\varepsilon(j-1)})} \tag{5}$$
$$+ C \cdot \sum_{i=1}^{s} \sum_{|\beta| \leq (i-1) \cdot m,\ \beta \in \mathbb{N}_0^n} \left|\left| (\varepsilon \cdot q(\xi))^{s-i} \cdot \varepsilon^{|\beta|} F(D^\beta \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)},$$
$$j = 1, 2, ...$$

where $\omega \subset\subset \Omega$.

The proof follows from Lemma 3 and the definition of the polynomial $Q_P(\xi)$.

LEMMA 5. *For any couple of multi-indeces β, α such that $\beta \in sA_P$ and $\alpha \in jA_P \setminus (j-1)A_P$, $\beta \geq \alpha$ $(j \in \mathbb{N}_0^1, j \leq s)$, we have $|\beta - \alpha| \leq s - j$.*

Proof. We prove it by contradiction. Let's suppose there are two multiindeces $\beta \in sA_P$ and $\alpha \in jA_P \setminus (j-1)A_P$, $\beta \geq \alpha$ such that $|\beta - \alpha| \geq s - j + 1$. Since $\alpha \notin (j-1)A_P$, then from the definition of the set A_P, it follows that there exists an index $i_0 : 1 \le i_0 \le k$ such that $(\alpha, \lambda^{i_0}) > d_{i_0}(j-1)$. As $\min\limits_{1\le j\le n} \lambda_j^{i_0} = 1,\ \ 0 < d_{i_0} \le 1$, then $(\beta, \lambda^{i_0}) = (\beta-\alpha, \lambda^{i_0}) + (\alpha, \lambda^{i_0}) \ge |\beta-\alpha| + (\alpha, \lambda^{i_0}) > s - j + 1 + (j-1)d_{i_0} \ge sd_{i_0}$, i.e. $\beta \notin sA_P \cap \mathbb{N}_0^n$.

□

LEMMA 6. *Let $P(D)$ be an hypoelliptic operator (ordP=m), j be a natural number. Then there is a constant $C > 0$ for which the following estimate is satisfied:*

$$\begin{aligned} &\varepsilon^{2s} \sum_{0\ne\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| Q_P(D) P^{(\alpha)}(D) u \right|\right|^2_{L_2(\omega_{\varepsilon j})} \\ &\le C \cdot \sum_{0\ne\alpha\in\mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| P^{(\alpha)}(D) u \right|\right|^2_{L_2(\omega_\varepsilon(j-1))} \\ &+ C \cdot \sum_{r=0}^{s} \sum_{\beta\in(rA_P\setminus(r-1)A_P)\cap\mathbb{N}_0^n} \varepsilon^{-2r} \left|\left| D^\beta P(D) u \right|\right|^2_{L_2(\omega_\varepsilon(j-1))}. \end{aligned} \tag{6}$$

Proof. By Lemma 3, it is sufficient to estimate the second term of the right hand side of (5). There is a constant $C_1 > 0$ for which it holds:

$$\begin{aligned} &\sum_{i=1}^{s} \sum_{|\gamma|\le(i-1)m} \left|\left| (\varepsilon q(\xi))^{s-i} \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\ &\le C_1 \sum_{|\gamma|\le(s-1)m} \left|\left| ((\varepsilon q(\xi))^s + 1) \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\ &= C_1 \cdot \sum_{|\gamma|\le(s-1)m} \left|\left| (\varepsilon^s \cdot Q_P(\xi) + 1) \cdot \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\ &\le C_1 \varepsilon^{2s} \sum_{|\gamma|\le(s-1)m} \left|\left| Q_P(\xi) \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\ &+ C_1 \cdot \sum_{|\gamma|\le(s-1)m} \left|\left| \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \end{aligned} \tag{7}$$

Applying Parceval equality and the Newton-Leibniz formula to the first term of the right hand side of (7), we obtain:

$$\begin{aligned} &\varepsilon^{2s} \sum_{|\gamma|\le(s-1)m} \left|\left| Q_P(\xi) \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)} \\ &= \varepsilon^{2s} \sum_{|\gamma|\le(s-1)m} \left|\left| Q_P(D)(\varepsilon^{|\gamma|} D^\gamma \varphi_j^\varepsilon) P(D) u \right|\right|^2_{L_2(\mathbb{R}^n)} \\ &\le C_1 \cdot \varepsilon^{2s} \sum_{|\gamma|\le(s-1)m} \sum_{\beta\in\mathbb{N}_0^n} \left|\left| Q_P^{(\beta)}(\varepsilon^{|\gamma|} D^\gamma \varphi_j^\varepsilon) D^\beta (P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)}. \end{aligned} \tag{8}$$

As from Lemma 5, for any $\alpha \in s \cdot A_P$, $\beta \in rA_P \backslash (r-1)A_P$, $\beta \geq \alpha$ $(s \geq r)$ $|\alpha - \beta| \leq s - r$, then by (8) there is a constant $C_2 > 0$ such that:

$$\varepsilon^{2s} \sum_{|\gamma| \leq (s-1)m} \left|\left| Q_P(\xi) \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)}$$

$$\leq C_1 \cdot \varepsilon^{2s} \sum_{|\gamma| \leq (s-1)m} \sum_{r=0}^{s} \sum_{\beta \in (rA_P \backslash (r-1)A_P) \cap \mathbb{N}_0^n} \left|\left| Q_P^{(\beta)} (\varepsilon^{|\gamma|} D^\gamma \varphi_j^\varepsilon) D^\beta (P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)}$$

$$\leq C_2 \cdot \sum_{|\gamma| \leq (s-1)m} \sum_{\alpha \in sA_P} \sum_{r=0}^{s} \sum_{\beta \in (rA_P \backslash (r-1)A_P) \cap \mathbb{N}_0^n} \varepsilon^{2r} \left|\left| \varepsilon^{|\alpha - \beta + \gamma|} (D^{\alpha - \beta + \gamma} \varphi_j^\varepsilon) D^\beta (P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)}$$

Then by Lemma 2 from (7), there exists a constant $C_3 > 0$ such that:

$$\sum_{i=1}^{s} \sum_{|\gamma| \leq (i-1)m} \left|\left| (\varepsilon \cdot q(\xi))^{s-i} \cdot \varepsilon^{|\gamma|} F(D^\gamma \varphi_j^\varepsilon P(D)u) \right|\right|^2_{L_2(\mathbb{R}^n)}$$

$$\leq C_3 \sum_{r=0}^{s} \sum_{\beta \in rA_P \cap \mathbb{N}_0^n} \varepsilon^{2r} \left|\left| D^\beta P(D)u \right|\right|^2_{L_2(\omega(j-1))}.$$

From this estimate we get the proof of the Lemma.

□

3. Estimates for higher order derivatives

For a convex set $A \subset \mathbb{R}^n_+$ we denote:

$$t \cdot A = \{\nu; \nu \in \mathbb{R}^n_+; \tfrac{\nu}{t} \in A\} \quad \text{for } t > 0,$$
$$0 \cdot A = 0,$$
$$t \cdot A = \emptyset \quad \text{for } t < 0.$$

DEFINITION 5. *(cf. [5]) Let $\Omega \subset \mathbb{R}^n$ be a open set. By $G^A(\Omega)$ we denote the set of the functions $f \in C^\infty(\Omega)$ such that for any compact subset $K \subset \Omega$ there exists a constant $C = C(K)$ for which:*

$$\sup_{x \in K} |D^\alpha f(x)| \leq C^{j+1} j^j, \ \ \alpha \in jA, \ \ j = 1, 2, ...$$

The class $G^A(\Omega)$ is called multianisotropic Gevrey class.

In [5] it was proved that if $A = \{\nu : \nu \in \mathbb{R}^n_+; (\nu, \lambda) \leq 1\}$ for some $\lambda \in \mathbb{R}^n_+ \cap \mathbb{R}^n_0$, $\min_{1 \leq i \leq n} \lambda_i = 1$, then $G^A(\Omega) = G^\lambda(\Omega)$. If $\lambda = (1, ..., 1)$, then the class $G^A(\Omega)$ is the class of the analytic functions with real variables.

LEMMA 7. *Let $\mathcal{N}$ be a C.R. polyhedron, l a natural number, $\Omega' \subset \Omega \subset \mathbb{R}^n$ an open set with diameter less than 2. If $f \in G^{\mathcal{N}}(\Omega')$, then there is a constant $C = C(l, f) > 0$ such that, for any $j > l$ $(j \in \mathbb{N}_0^1)$, and any multiindex $\alpha \in j \cdot \mathcal{N}$ and $\delta \in (0,1)$ the following estimate is satisfied:*

$$\delta^j \cdot \sup_{x \in \Omega'_{(j-l)\delta}} |D^\alpha f(x)| \le C^{j+1}. \tag{9}$$

Proof. Since if $(j-l)\delta \ge 1$, then $\Omega'_{(j-l)\delta} = \emptyset$, therefore it is sufficient to prove the estimate (9) in the case $\delta(j-l) < 1$. Then

$$\sup_{x \in \Omega'_{(j-l)\delta}} |D^\alpha f(x)| \le C^{j+1} \cdot j^j = C^{j+1} \cdot (j-l)^j \cdot \left(\frac{j}{j-l}\right)^j$$
$$\le C_1^{j+1} \cdot (j-l)^j \le C_1^{j+1} \cdot \left(\frac{1}{\delta}\right)^j.$$

Now the proof easily follows.

□

THEOREM 2. *Let $u(x)$ be a solution of the hypoelliptic equation $P(D)u = f$, where $f \in G^{A_P}(\Omega)$. Then there is a constant $K = K(u, \omega) > 0$ $(\omega \subset\subset \Omega)$ such that:*

$$\varepsilon^{2js+2m} \cdot \sum_{0 \ne \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| Q_P^j(D) P^{(\alpha)}(D) u \right|\right|^2_{L_2(\omega_{\varepsilon j})} \le K^{2(j+1)}, \quad j = 1, 2 \ldots, \tag{10}$$

where m denotes the order of $P(D)$.

Proof. Since any solution $u(x)$ of the hypoelliptic equation $P(D)u = f$ belongs to $C^\infty(\Omega)$ if $f \in C^\infty(\Omega)$, then there is a constant $K > 0$ such that the inequality (10) is true for $j = 0$. We proceed by induction. Let's suppose that the estimate (10) is true for any $j \le l (l \ge 0)$. Then we prove it for $j = l+1$. Since $V(x) = Q_P^l(D)u(x)$ is a solution of the equation $P(D)V = Q_P^l(D) f$, then by Lemma 6 we get:

(11)

$$\varepsilon^{2s(l+1)+2m} \cdot \sum_{0 \ne \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| Q_P^{l+1}(D) P^{(\alpha)}(D) u \right|\right|^2_{L_2(\omega_{(l+1)\varepsilon})}$$
$$= \varepsilon^{2s(l+1)+2m} \cdot \sum_{0 \ne \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| Q_P(D) P^{(\alpha)}(D) Q_P^l(D) u \right|\right|^2_{L_2(\omega_{(l+1)\varepsilon})}$$
$$\le C \cdot \varepsilon^{2sl+2m} \cdot \sum_{0 \ne \alpha \in \mathbb{N}_0^n} \varepsilon^{-2|\alpha|} \left|\left| P^{(\alpha)}(D) Q_P^l(D) u \right|\right|^2_{L_2(\omega_{l\varepsilon})}$$
$$+ C \cdot \varepsilon^{2sl+2m} \cdot \sum_{r=0}^{s} \sum_{\beta \in rA \cap \mathbb{N}_0^n} \varepsilon^{2r} \left|\left| D^\beta Q_P^l(D) f \right|\right|^2_{L_2(\omega_{l\varepsilon})}.$$

And by induction we have:

$$\varepsilon^{2sl+2m} \cdot \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|} \left\| P^{(\alpha)}(D) Q_P^l(D) u \right\|^2_{L_2(\omega_{l\varepsilon})} \leq K^{2(l+1)}. \tag{12}$$

For the second term of the right hand side of the estimate (11), by Lemma 7, and as $f \in G^{A_P}(\Omega)$, then there is a constant $q = q(f, \omega) > 0$ such that:

$$\varepsilon^{2sl+2m} \cdot \sum_{r=0}^{s} \sum_{\beta \in rA_P \cap N_0^n} \varepsilon^{2r} \left\| D^\beta Q_P^l(D) f \right\|^2_{L_2(\omega_{l\varepsilon})} \leq q^{2(l+2)}. \tag{13}$$

And by (12)-(13), we obtain from (11):

$$\begin{aligned} \varepsilon^{2s(l+1)+2m} \cdot \sum_{\alpha\neq 0} \varepsilon^{-2|\alpha|} \left\| Q_P^{l+1}(D) P^{(\alpha)}(D) u \right\|^2_{L_2(\omega_{(l+1)\varepsilon})} &\leq C \cdot (K^{2(l+1)} + q^{2(l+2)}) \\ &\leq K^{2(l+2)}, \end{aligned}$$

if K is sufficiently large.

□

THEOREM 3. *Let $u(x)$ be a solution of the hypoelliptic equation $P(D)u = f$, where $f \in G^{A_P}(\Omega)$. Then for any $\omega \subset\subset \Omega$ there is a constant $K_1 = K_1(u, \omega) > 0$ such that:*

$$\left\| Q_P^j(D) u \right\|^2_{L_2(\omega)} \leq K_1^{2(j+1)} \cdot j^{2sj}, \qquad j = 1, 2, \ldots.$$

Proof. Since $\rho = \rho(\omega, \partial\Omega) > 0$, then for any $\delta \in (0, \rho)$ there is $\Omega' \subset\subset \Omega$ such that $\omega \subset \Omega'_\delta$. Then for any natural number j, taking $\varepsilon = \frac{\delta}{j}$ from Theorem 2, we have:

$$\left(\frac{\delta}{j}\right)^{2sj} \left\| Q_P^j(D) u \right\|^2_{L_2(\omega)} \leq \left(\frac{\delta}{j}\right)^{2sj} \left\| Q_P^j(D) u \right\|^2_{L_2(\Omega'_\delta)} \leq K^{2(j+1)}.$$

It follows:

$$\left\| Q_P^j(D) u \right\|^2_{L_2(\omega)} \leq K^{2(j+1)} \cdot \left(\frac{j}{\delta}\right)^{2sj} = K_1^{2(j+1)} \cdot j^{2sj}; \qquad j = 1, 2, \ldots \quad .$$

□

PROPOSITION 1. *For any multiindex $\alpha \notin (s-1)A_P$ we have $D^\alpha Q_P(\xi) \equiv const$.*

Proof. Since for any multiindex α:

$$D^\alpha Q_P(\xi) = \sum_{\beta \in sA_P^0, \beta \geq \alpha} \frac{\beta!}{(\beta-\alpha)!} \xi^{\beta-\alpha},$$

then it is sufficient to consider the case $\alpha \in sA_P$. Let $\beta_0 \in sA_P^0 \cap \mathbb{N}_0^n$ be such that $\alpha \leq \beta_0$, $\alpha \neq \beta_0$, then $|\beta_0 - \alpha| \geq 1$. By the difinition of the set A_P, there is a natural number i_0, $(1 \leq i_0 \leq k)$ such that $(\alpha, \lambda^{i_0}) > d_{i_0}(s-1)$ and $\min\limits_{1\leq j\leq n} \lambda_j^{i_0} = 1$. So we obtain $(\beta_0, \lambda^{i_0}) = (\beta_0-\alpha, \lambda^{i_0}) + (\alpha, \lambda^{i_0}) > |\beta_0-\alpha| + d_{i_0}(s-1) \geq 1 + d_{i_0}(s-1) \geq sd_{i_0}$. This leads to a contradiction, therefore such $\beta_0 \in sA_P^0 \cap \mathbb{N}_0^n$ can't exist. The Proposition is proved.

□

LEMMA 8. *For any $\varepsilon > 0$ and any function $\varphi \in C_0^\infty(\mathbb{R}^n)$, there is a constant $C > 0$ for which the following estimate is satisfied:*

$$\varepsilon^{-(s-j)}||D^\alpha\varphi||_{L_2(\mathbb{R}^n)} \leq C(||Q_P(D)\varphi||_{L_2(\mathbb{R}^n)} + \varepsilon^{-(s)}||\varphi||_{L_2(\mathbb{R}^n)}, 0 \leq j \leq s, \ \forall\alpha \in jA_P.$$

Proof. By the definition of the polynomial $Q_P(\xi)$, for any $\alpha \in jA_P$, $(0 \leq j \leq s)$ there is a constant $C_1 > 0$ such that $|\xi^{2\alpha}| \leq C_1|Q_P(\xi)|^{\frac{2j}{s}}$, $\forall\xi \in \mathbb{R}^n$.

Multiply the latter by $\varepsilon^{-2(s-j)}$, for $\varepsilon > 0$, then by Hӧlder's inequality there is a constant $C_2 > 0$ such that:

$$\varepsilon^{-2(s-j)}|\xi^{2\alpha}| \leq C_1\varepsilon^{-2(s-j)}|Q_P(\xi)|^{\frac{2j}{s}} \leq C_2(Q_P^2(\xi) + \varepsilon^{-2s}). \tag{14}$$

Applying Parceval equality, then for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ the following is satisfied:

$$\varepsilon^{-(s-j)}||D^\alpha\varphi||_{L_2(\mathbb{R}^n)} \leq C(||Q_P(D)\varphi||_{L_2(\mathbb{R}^n)} + \varepsilon^{-(s)}||\varphi||_{L_2(\mathbb{R}^n)}).$$

□

LEMMA 9. *Let $\Omega \subset \mathbb{R}^n$ be an open set, j be a natural number, $0 \leq j \leq s$, $\delta \in (0, 1)$. Then for any function $\varphi_\delta \in C_0^\infty(\Omega)$, $0 \leq \varphi_\delta \leq 1$, $\varphi_\delta = 1$ on Ω_δ, there is a constant $C_s > 0$ such that:*

$$\sup_\Omega |Q_P^{(\alpha)}(D)\varphi_\delta| \leq C_s\delta^{-(s-j)}, \ \ \forall\alpha \in jA_P \setminus (j-1)A_P.$$

The proof easily follows from Lemma 4.1 in [7] and Proposition 1.

For any number μ, any open set $\Omega \subset \mathbb{R}^n$ and any function $f \in L_2^{loc}(\Omega)$ we write (cf. [7]):

$$N_{\Omega,\mu}(f) = N_\mu(f) = \sup_{\delta>0} \delta^\mu||f||_{L_2(\Omega_\delta)}.$$

THEOREM 4. *For any multiindex $\beta \in jA_P$ $(j \in \mathbb{N}_0^1, \ 0 \leq j \leq s)$ there is a constant $C > 0$ such that:*

$$N_j(D^\beta u) \leq C(N_s(Q_P(D)u) + N_0(u)), \qquad \forall u \in C^\infty(\Omega).$$

Proof. Let $\varphi_\delta \in C_0^\infty(\Omega)$ be a function satisfying the condition of Lemma 9, then by Newton-Leibniz formula, we have:

$$Q_P(D)(\varphi_\delta u) = \sum_\alpha \frac{Q^{(\alpha)}_{P(D)}\varphi_\delta}{\alpha!} D^\alpha u \tag{15}$$
$$= \sum_{\alpha \notin (s-1)A_P} \frac{Q_P^{(\alpha)}(D)\varphi_\delta}{\alpha!} D^\alpha u + \sum_{\alpha \in (s-1)A_P} \frac{Q_P^{(\alpha)}(D)\varphi_\delta}{\alpha!} D^\alpha u.$$

By Proposition 1 the first term of (15) is equal to $\varphi_\delta Q_P(D)u$. Therefore:

$$Q_P(D)(\varphi_\delta u) = \varphi_\delta Q_P(D)u + \sum_{j=0}^{s-1} \sum_{\alpha \in (jA_P \setminus (j-1)A_P)} \frac{Q_P^{(\alpha)}(D)\varphi_\delta \alpha!}{D}^{\alpha} u.$$

So there is a constant $C > 0$ such that:

$$||Q_P(D)(\varphi_\delta u)||_{L_2(\mathbb{R}^n)} \le C\{||\varphi_\delta Q_P(D)u||_{L_2(\mathbb{R}^n)} + \sum_{j=0}^{s-1} \sum_{\alpha \in (jA_P \setminus (j-1)A_P)} ||Q_P^{(\alpha)}(D)\varphi_\delta D^\alpha u||_{L_2(\mathbb{R}^n)}\}.$$

By the definition of N_μ, for a suitable constant $C_1 > 0$ we have:

$$||Q_P(D)(\varphi_\delta u)||_{L_2(\mathbb{R}^n)} \le C_1 \delta^{-s} \left\{ N_s(Q_P(D)u) + \sum_{j=0}^{s-1} \sum_{\alpha \in (jA_P \setminus (j-1)A_P)} N_j(D^\alpha u) \right\} \tag{16}$$
$$\le C_1 \delta^{-s} \left\{ N_s(Q_P(D)u) + \sum_{j=0}^{s-1} \sum_{\alpha \in jA_P} N_j(D^\alpha u) \right\}.$$

Since $||\varphi_\delta u||_{L_2(\mathbb{R}^n)} \le N_0(u)$, then by Lemma 8 from (16) it follows that for a constant $C_2 > 0$ we have:

$$\varepsilon^{-(s-j)} ||D^\beta(\varphi_\delta u)||_{L_2(\mathbb{R}^n)} \le C_2 \left\{ \delta^{-s} \left\{ N_s(Q_P(D)u) + \sum_{j=0}^{s-1} \sum_{\alpha \in jA_P} N_j(D^\alpha u) \right\} + \varepsilon^{-s} N_0(u) \right\}, \tag{17}$$

$\forall \beta \in jA_P$. Taking $\varepsilon = \frac{\delta}{\sigma}$, $\delta < \frac{dim\Omega}{2}$ and multiplying by (17) δ^s, we get:

$$\sigma^{(s-j)} N_j(D^\beta u) \le C_2 \left\{ N_s(Q_P(D)u) + \sum_{j=0}^{s-1} \sum_{\alpha \in jA_P} N_j(D^\alpha u) \right\}, \quad \forall \beta \in jA_P. \tag{18}$$

Taking the sum of (18) for all j, $j = 0, ..., s$:

$$\sum_{j=0}^{s} \sigma^{(s-j)} \sum_{\beta \in jA_P} N_j(D^\beta u)$$
$$\leq (s+1) \cdot C_2 \left\{ N_s(Q(D)u) + \sum_{j=0}^{s-1} \sum_{\alpha \in jA_P} N_j(D^\alpha u) + \sigma^s N_0(u) \right\}.$$

For sufficiently large σ, we can find a constant $C_3 = C_3(\sigma) > 0$ such that:

$$\sum_{j=0}^{s} \sigma^{(s-j)} \sum_{\beta \in jA_P} N_j(D^\beta u) \leq C_3\{N_s(Q_P(D)u) + N_0(u)\}.$$

The proof of the Lemma follows.

□

THEOREM 5. *Any solution of an hypoelliptic equation $P(D)u = f$ belongs to $G^{A_P}(\Omega)$, if $f \in G^{A_P}(\Omega)$.*

Proof. Let $\omega \subset\subset \Omega$. By Theorem 4, for any $v \in C^\infty(\Omega)$, we have:

$$||D^\beta v||_{L_2(\omega_{s+t})} \leq C_2(||Q_P(D)v||_{L_2(\omega_s)} + t^{-s}||v||_{L_2(\omega_s)}),$$

where $t > 0$. Taking $t = \frac{\delta}{l}$, $s = (1 - \frac{1}{l})\delta$ we get:

$$(19) \qquad ||D^\beta v||_{L_2(\omega_\delta)} \leq C_2 \left(||Q_P(D)v||_{L_2(\omega_{(1-\frac{1}{l})\delta})} + \left(\frac{\delta}{l}\right)^{-s} ||v||_{L_2(\omega_{(1-\frac{1}{l})\delta})} \right).$$

By Theorem 1.1 of [6], for the polyhedron sA_P there is a natural number $j_0 \geq s$ such that any multi-index $\alpha \in jA_P$, $j \geq j_0$, can be represented in the form $\alpha = \beta + \gamma$, where $\beta \in sA_P \cap \mathbb{N}_0^n$, $\gamma \in (j-s)A_P \cap \mathbb{N}_0^n$. For simplicity let $j_0 = s$. Therefore, every multiindex α can be represented as $\alpha = \sum_{k=1}^{l} \alpha^{(k)}$, where $l = [\frac{j}{s}]$ if $[\frac{j}{s}]$ is integer, and $l = [\frac{j}{s}] + 1$ otherwise, $\alpha^{(k)} \in sA_P \cap \mathbb{N}_0^n$, $k = 1, .., l$. Now let $\beta = \alpha^1$, then by (19) we get:

$$||D^{\alpha^1}(D^{\alpha-\alpha^1})u||_{L_2(\omega_\delta)}$$
$$\leq C_2 \left(||Q_P(D)(D^{\alpha-\alpha^1})u||_{L_2(\omega_{(1-\frac{1}{l})\delta}} + \left(\frac{\delta}{l}\right)^{-s} ||D^{\alpha-\alpha^1}u||_{L_2(\omega_{(1-\frac{1}{l})\delta})} \right)$$
$$(20) \qquad \leq C_2 \left(||D^{\alpha^2} D^{\alpha-\alpha^1-\alpha^2} Q_P(D)u||_{L_2(\omega_{(1-\frac{1}{l})\delta}} \right.$$
$$\left. + \left(\frac{\delta}{l}\right)^{-s} ||D^{\alpha^2}(D^{\alpha-\alpha^1-\alpha^2})u||_{L_2(\omega_{(1-\frac{1}{l})\delta})} \right).$$

Taking the function $v = D^{\alpha-\alpha^1-\alpha^2} Q_P(D)u$ in the first term of the right hand side of (20) and taking $v = D^{\alpha-\alpha^1-\alpha^2}u$ in the second term of the right hand side of (20), applying to (20) the estimate (19) and working anologously to step $(l-1)$, we obtain:

$$\begin{aligned} ||D^\alpha u||_{L_2(\omega_\delta)} &\leq C_2^l \sum_{i=0}^{l} C_l^i \left(\frac{l}{\delta}\right)^{si} ||Q_P^{(l-i)}(D)u||_{L_2(\omega_s)} \\ &\leq C_2^l \sum_{i=0}^{l} C_l^i \left(\frac{l}{\delta}\right)^{si} K^{l-i+1}(l-i)^{s(l-i)} \\ &\leq C_3^l \sum_{i=0}^{l} C_l^i \left(\frac{1}{\delta}\right)^{si} (l)^{sl} \leq (C_4(\delta))^{j+1} j^j, \end{aligned}$$

i.e. $u \in G^{A_P}(\Omega)$.

□

Let $\mu \in \mathbb{R}_+^n$, $i = 1, ..., n$, $\min\limits_{1\leq i\leq n} \mu_i = 1$ and $0 < \rho_i \leq 1$, $i = 1, ...n$. We denote by $B = \{\nu \in \mathbb{R}_+^n, (\nu, \mu_i) \leq \rho_i, i = 1, ..., n\}$.

THEOREM 6. *Let $f \in G^B(\Omega)$. Then any solution of the hypoelliptic equation $P(D)u = f$ belongs to $G^{B\cap A_P}(\Omega)$.*

The theorem was proved analogously to Theorem 2.4 with some modifications. We now present two examples clarifying the previous results.

EXAMPLE 1. Let $n = 2$, $P(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = (\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y})(\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x})$. Using the notations $D_1 = \frac{1}{i}\frac{\partial}{\partial x}$, $D_2 = \frac{1}{i}\frac{\partial}{\partial y}$, we have:

$$P(D) = (-D_1^2 - iD_2)(-D_2^2 - iD_1) = iD_1^3 + iD_2^3 + D_1^2 D_2^2 - D_1 D_2,$$

and its characteristic polynomial is:

$$P(\xi) = (-\xi_1^2 - i\xi_2)(-\xi_2^2 - i\xi_1) = i\xi_1^3 + i\xi_2^3 + \xi_1^2\xi_2^2 - \xi_1\xi_2.$$

It is easy to see that $P(\xi)$ is a multi-quasi-elliptic polynomial, and therefore the set M_P is a C.R. polyhedron. Simple computations show that $M_P = \{\nu \in R_+^2, 2\nu_1 + \nu_1 \leq 1; \nu_1 + 2\nu_1 \leq 1\}$. The exact weight hypoellipticity of the operator $P(D)$ is

$$h(\xi) = |\xi_1|^{\frac{1}{2}}| + |\xi_2|^{\frac{1}{2}}| + |\xi_1|^{\frac{1}{3}}|\xi_2|^{\frac{1}{3}}.$$

By Hörmander Theorem (cf. 8, Theorem 11.4.1), all the solutions of the equation $P(D)u = 0$ belong to the Gevrey class $G^{2,2}(\Omega)$ and this result is sharp remaining in the frame of the anisotropic Gevrey classes. However, from the hypoellipticity and the form of the operator $P(D)$, it follows that any solution can be represented in the form:

$$u(x, y) = u_1(x, y) + u_2(x, y),$$

where $u_1(x, y) \in G^{1,2}(\Omega)$,$u_2(x, y) \in G^{2,1}(\Omega)$. Using this fact, we can estimate $D_1^{\alpha_1} D_2^{\alpha_2} u$, where $\alpha_1 = \alpha_2 = j$, $j = 1, 2, ...$ as follows: for any compact subset $K \subset \Omega$ there exist two constants $C_1 = C_1(K, u_1) > 0$ and $C_2 = C_2(K, u_2) > 0$ such that:

$$\sup_{x\in K} |D_1^j D_2^j u(x, y)| \le \sup_{x\in K} |D_1^j D_2^j u_1(x, y)| + \sup_{x\in K} |D_1^j D_2^j u_2(x, y)|$$
$$\le C_1^{2j+1} j^{1j} j^{2j} + C_2^{2j+1} j^{2j} j^{1j} \le C_3^{2j+1} j^3,$$

where $C_3 = \max(C_1, C_2)$. Therefore, the classical Gevrey classes don't describe completely the behaviour of the solutions of the hypoelliptic equation $P(D)u = 0$. Using the multianisotropic classes Gevrey and noticing that $(j, j) \in 3jM_P$, we have:

$$\sup_{x\in K} |D_1^j D_2^j u(x, y)| \le C^{2j+1} j^{3j}.$$

Let for example $f \in G^B(\Omega)$, where:

$$B = \left\{ \nu \in R_+^2, 3\nu_1 + \frac{3}{2}\nu_2 \le 1 \right\}.$$

Then $A = M_P \cap B = \{\nu \in R_+^2, 3\nu_1 + \frac{3}{2}\nu_2 \le 1; \nu_1 + 2\nu_2 \le 1\}$.

From Theorem 2.5 we have that all the solutions of the equation $P(D)u = f$ belong to $G^A(\Omega)$.

EXAMPLE 2. Let $n = 2$ and $P(D)$ be the operator with symbol:

$$P(\xi) = \xi_1^6(\xi_1 - \xi_2)^6 + \xi_1^8\xi_2^2 + \xi_1^8 + 1.$$

The polynomial $P(\xi)$ is not multi-quasi-elliptic. Simple computations show that:

$$M_P = \left\{ \nu \in R_+^2, 2\nu_1 + 3\nu_2 \le 1; \nu_1 + \nu_2 \le \frac{2}{3} \right\}.$$

Let $P(D)u = f$, where $f \in G^B(\Omega)$, B for instance has the form:

$$B = \{\nu \in R_+^2, 2\nu_1 + \frac{3}{2}\nu_2 \le 1\}.$$

Since $B \cap M_P = M_P$, then from Theorem 2.5 it follows that $u \in G^{M_P}\Omega$.

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