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A SIMPLE PROOF THAT DETERMINACY IMPLIES LEBESGUE MEASURABILITY

Abstract. It is well-known that the Axiom of Determinacy implies that all sets of reals are Lebesgue measurable and that weaker determinacy hypotheses consistent with the Axiom of Choice have corresponding measurability consequences. Here a new and simple proof of these facts, a proof arising from the author's work on imperfect information determinacy, is presented.

Mycielski–Świerczkowski [1] gave the first proof that the Axiom of Determinacy implies that all sets of reals are Lebesgue measurable. For a very different proof due to Leo Harrington, see the proof of the first assertion of Theorem 36.20 of Kechris [2]. These proofs are "local": e.g., they show that projective determinacy implies that all projective sets are Lebesgue measurable, and—with an extra trick—they show that Σ^1_n determinacy implies that all Σ^1_{n+1} sets are Lebesgue measurable.

Our reduction in [3] of imperfect information determinacy to ordinary determinacy led to a proof that determinacy implies measurability, a proof unlike either of the two proofs mentioned above. Vervoort [4] earlier showed that imperfect information determinacy implies Lebesgue measurabilty. By combining his idea with methods of [3] and then eliminating the imperfect information games, one gets a proof of Lebesgue measurability that is very natural and is at least as simple as the proofs mentioned above.

The basic idea of our proof was given in [3]. (See Theorems 8 and 9 of that paper.) But it seems worthwhile to present the proof in detail, and that is the reason for this note.

We write ω for the set $\{0, 1, 2, \ldots\}$ of all natural numbers. 2^ω is the set of all functions from ω to $\{0, 1\}$, i.e., the set of all infinite sequences of natural numbers. $2^{<\omega}$ is the set of all finite sequences of natural numbers.

For background information about determinacy, see [2].

For our purposes, let us think of Lebesgue measure as the usual "coin-flipping" measure μ on 2^ω. Let μ_* and μ^* be respectively the corresponding inner and outer measures.

Let $A \subseteq 2^\omega$. For each $v \in (0, 1]$, we define a game G_v. Play in the game G_v is as follows:

$$\begin{array}{lccccccc} \text{I} & h_0 & & h_1 & & h_2 & & \ldots \\ \text{II} & & e_0 & & e_1 & & e_2 & & \ldots \end{array}$$

With each postion in G_v of length $2i$, we associate inductively $v_i \in (0, 1]$. Set $v_0 = v$. For each i, h_i must be a function from $\{0, 1\}$ to $[0, 1]$ such that

$$\frac{1}{2}h_i(0) + \frac{1}{2}h_i(1) \geq v_i.$$

For each i, e_i must be a member of $\{0, 1\}$ such that $h_i(e_i) \neq 0$. Set $v_{i+1} = h_i(e_i)$.

For each position p^* in G_v, let $\pi(p^*)$ be the sequence of all the moves made by II in arriving at p^*. For any play x^* of G_v, let $\pi(x^*) = \bigcup_i \pi(x^* \restriction i)$, i.e., let $\pi(x^*)$ be element of 2^ω consisting of the moves made by II in the play x^*. A play x^* is a win for I if and only if

$$\pi(x^*) \in A.$$

The idea behind G_v is as follows. Player I is trying to show that $\mu_*(A) \geq v$. He begins by asserting lower bounds for $\mu_*(\{x \mid \langle 0\rangle^\frown x \in A\})$ and $\mu_*(\{x \mid \langle 1\rangle^\frown x \in A\})$ which are large enough to imply that $\mu_*(A) \geq v$. These alleged lower bounds are the two values of h_0. Player II then challenges one of the two by choosing e_0. II is not allowed to challenge a vacuous assertion that a lower bound is 0. I continues by asserting lower bounds for $\mu_*(\{x \mid \langle e_0, 0\rangle^\frown x \in A\})$ and $\mu_*(\{x \mid \langle e_0, 1\rangle^\frown x \in A\})$; etc.

LEMMA 1. *If I has a winning strategy for G_v, then $\mu_*(A) \geq v$.*

Proof. Let σ be a winning strategy for I for G_v.

By induction on the length $\ell\text{h}(p)$ of p, we define the notion of an *acceptable* $p \in 2^{<\omega}$, and we associate with each acceptable p a position $\psi(p)$ in G_v that is consistent with σ and is such that $\ell\text{h}(\psi(p)) = 2\ell\text{h}(p)$ and $\pi(\psi(p)) = p$. When p is acceptable and extends q, then q will be acceptable and $\psi(p)$ will extend $\psi(q)$.

The initial position $\emptyset$ is acceptable, and of course $\psi(\emptyset) = \emptyset$.

Assume that p is acceptable and that $\psi(p)$ is defined. Let h^p be the h_i given by σ at $\psi(p)$. For $e \in \{0, 1\}$, we declare $p^\frown\langle e\rangle$ to be acceptable just in case $h^p(e) \neq 0$, in which case we set

$$\psi(p^\frown\langle e\rangle) = \psi(p)^\frown\langle h^p, e\rangle.$$

For acceptable p, let v^p be the $v_{\ell\text{h}(p)}$ of $\psi(p)$. Define $f : 2^{<\omega} \to [0, 1]$ by

$$f(p) = \begin{cases} v^p & \text{if } p \text{ is acceptable;} \\ 0 & \text{otherwise.} \end{cases}$$

For p acceptable,

$$\begin{aligned} \frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle) &= \frac{1}{2}h^p(0) + \frac{1}{2}h^p(1) \\ &\geq v^p \\ &= f(p). \end{aligned}$$

Note that the inequality $\frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle) \geq f(p)$ trivially holds also for unacceptable p.

Say that $x \in 2^\omega$ is *acceptable* just in case all $p \subseteq x$ are acceptable. For acceptable x, let $\psi(x)$ be the play of G_v extending all the $\psi(x \restriction n)$. If x is acceptable then $x \in A$, since $\psi(x)$ is consistent with the winning strategy σ.

The set C of all acceptable x is a closed subset of A. Using the fact that $\frac{1}{2} f(p^\frown\langle 0\rangle) + \frac{1}{2} f(p^\frown\langle 1\rangle) \geq f(p)$, it is easy to see by induction that, for all n,

$$\Sigma_{\ell\mathrm{h}(p)=n} 2^{-n} f(p) \geq v.$$

Since $f(p) \leq 1$ for all p and $f(p) = 0$ for unacceptable p, it follows that, for all n,

$$\mu(\{x \mid x \restriction n \text{ is acceptable}\}) \geq v.$$

But this means that $\mu(C) \geq v$ and so that $\mu_*(A) \geq v$.

□

LEMMA 2. *If II has a winning strategy for G_v, then $\mu^*(A) \leq v$.*

Proof. Let τ be a winning strategy for II for G_v.

Let $\delta > 0$. We shall prove that $\mu^*(A) \leq v + \delta$.

By induction on $\ell\mathrm{h}(p)$, we define a new notion of an *acceptable* $p \in 2^{<\omega}$, and we associate with each acceptable p a position $\psi(p)$ in G_v that is consistent with τ and is such that $\ell\mathrm{h}(\psi(p)) = 2\ell\mathrm{h}(p)$ and $\pi(\psi(p)) = p$. When p is acceptable and extends q, then q will be acceptable and $\psi(p)$ will extend $\psi(q)$.

The initial position $\emptyset$ is acceptable, and $\psi(\emptyset) = \emptyset$.

Assume that p is acceptable and that $\psi(p)$ is defined. Let v^p be the $v_{\ell\mathrm{h}(p)}$ of $\psi(p)$. For $e \in \{0, 1\}$, set

$$u^p(e) = \inf\{h(e) \mid h \text{ is legal at } \psi(p) \ \wedge\ \tau(\psi(p)^\frown\langle h\rangle) = e\},$$

where we adopt the convention that $\inf \emptyset = 1$.

We first show that

$$\frac{1}{2}u^p(0) + \frac{1}{2}u^p(1) \leq v^p.$$

If this is false, then there is an $\varepsilon > 0$ such that the move h given by $h(e) = u^p(e) - \varepsilon$ is legal at $\psi(p)$, an impossibility by the definition of u^p.

We define $p^\frown\langle e\rangle$ to be acceptable just in case $u^p(e) \neq 1$. For e such that $p^\frown\langle e\rangle$ is acceptable, let $h^{p,e}$ be legal at $\psi(p)$ and such that

$$h^{p,e}(e) \leq u^p(e) + 2^{-(\ell\mathrm{h}(p)+1)}\delta \ \text{ and } \ \tau(\psi(p)^\frown\langle h^{p,e}\rangle) = e.$$

Then set

$$\psi(p^\frown\langle e\rangle) = \psi(p)^\frown\langle h^{p,e}, e\rangle.$$

For e such that $p^\frown\langle e\rangle$ is not acceptable, let $h^{p,e} = 1$. Note that the inequality $h^{p,e}(e) \leq u^p(e) + 2^{-(\ell\mathrm{h}(p)+1)}\delta$ holds for these e also.

Define $f : 2^{<\omega} \to (0, 1]$ by

$$f(p) = \begin{cases} v^p & \text{if } p \text{ is acceptable;} \\ 1 & \text{otherwise.} \end{cases}$$

For acceptable p with $\ell\text{h}(p) = n$,

$$\begin{aligned} \frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle) &= \frac{1}{2}h^{p,0}(0) + \frac{1}{2}h^{p,1}(1) \\ &\leq \frac{1}{2}(u^p(0) + 2^{-(n+1)}\delta) + \frac{1}{2}(u^p(1) + 2^{-(n+1)}\delta) \\ &\leq v^p + 2^{-(n+1)}\delta \\ &= f(p) + 2^{-(n+1)}\delta. \end{aligned}$$

Note that the inequality $\frac{1}{2}f(p^\frown\langle 0\rangle) + \frac{1}{2}f(p^\frown\langle 1\rangle) \leq f(p) + 2^{-(n+1)}\delta$ holds trivially for unacceptable p with $\ell\text{h}(p) = n$.

As before, say that $x \in 2^\omega$ is *acceptable* just in case all $p \subseteq x$ are acceptable. For acceptable x, let $\psi(x)$ be the play of G_v extending all the $\psi(x \restriction n)$. If x is acceptable then $x \notin A$, since $\psi(x)$ is consistent with the winning strategy τ.

The set C of all acceptable x is a closed subset of the complement of A. It is easy to see by induction that, for all n,

$$\Sigma_{\ell\text{h}(p)=n} 2^{-n} f(p) \leq v + (2^n - 1)\delta/2^n.$$

Since $f(p) \geq 0$ for all p and $f(p) = 1$ for unacceptable p, it follows that, for all n,

$$\mu(\{x \mid x \restriction n \text{ is unacceptable}\}) \leq v + (2^n - 1)\delta/2^n.$$

But this means that $\mu(C) \geq 1 - v - \delta$ and so that $\mu^*(A) \leq v + \delta$.

□

Note that the proofs of both our lemmas go through unchanged if we modify G_v so that the values of the h_i are required to be rational.

If all G_v are determined, then the given set A is Lebesgue measurable and

$$\mu(A) = \sup\{v \mid \text{I has a winning strategy for } G_v\}.$$

Thus our arguments show that AD implies that all sets are Lebesgue measurable. Moreover the complexity of the G_v is essentially the same as that of A, so our proof is "local" in the sense mentioned earlier.

References

[1] MYCIELSKI J. AND ŚWIERCZKOWSKI S., *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. **54** (1964), 67–71.

[2] KECHRIS A.S., *Classical descriptive set theory*, Springer-Verlag, New York 1994.

[3] MARTIN D.A., *The determinacy of Blackwell games*, Jour. Symb. Logic **63** (1998), 1565–1581.

[4] VERVOORT M.R., *Blackwell games*, in: "Statistics, probability, and game theory: papers in honor of David Blackwell", (Eds. Ferguson T., Shapley L. and McQueen J.), IMS Lecture Notes-Monograph Series **30**, Institute of Mathematical Statistics 1996, 369–390.

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