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GENERALIZED GEVREY CLASSES AND MULTI-QUASI-HYPERBOLIC OPERATORS

Abstract. In this paper we consider generalized Gevrey classes defined in terms of Newton polyhedra. In such functional frame we prove a theorem of solvability of the Cauchy Problem for a class of partial differential operators, called multi-quasi-hyperbolic. We then give a result of regularity of the solution with respect to the space variables and finally analyze the regularity with respect to the time variable.

Introduction

It is well known from the Cauchy-Kovalevsky theorem that the Cauchy Problem for partial differential equations with constant coefficients or analytic coefficients, and analytic data admits a unique, analytic solution.
But there are problems that are not C^∞ well-posed, i.e. starting with C^∞ data, there is not a C^∞ solution. In these cases it is natural to consider the behaviour of the operator in the Gevrey classes G^s, $1 < s < \infty$ (for definition and properties see for example Rodino [11]). Solvability of the Cauchy Problem in Gevrey spaces has been obtained for a class of partial differential operators with constant coefficients, the so called s-hyperbolic operators.
More precisely, we recall the following definition and the corresponding result, for the proof see Cattabriga [3], Hörmander [9], Rodino [11].

DEFINITION 1. *Let's consider partial differential operators in $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x$, non-characteristic with respect of the t-hyperplanes, i.e. operators that can be written in the form:*

$$P(D_t, D_x) = D_t^m + \sum_{j=1}^{m-1} a_j(D_x) D_t^j \tag{1}$$

with $order(a_j(D_x)) \leq m - j$.
We say that $P(D)$ is s-hyperbolic (with respect to the t variable), $1 < s < \infty$, if its symbol satisfies for a suitable $C > 0$ the condition:
if $\lambda^m + \sum_{j=0}^{m-1} a_j(\xi)\lambda^j = 0$ *for* $(\lambda, \xi) \in \mathbb{C}_t \times \mathbb{R}^n_x$, *then* $\Im\lambda \geq -C(1 + |\xi|^{\frac{1}{s}})$.
In the case $\Im\lambda \geq -C$ we say that $P(D)$ is hyperbolic.

THEOREM 1. *Let $P(D)$ be a differential operator in $\mathbb{R}_t \times \mathbb{R}^n_x$ of the form* (1) *and let P be s-hyperbolic with respect to t, with $1 < s < \infty$. Let $1 < r < s$ and assume $f_k(x) \in G^r_0(\mathbb{R}^n_x)$ for $k = 0, 1, \dots, m-1$. Then there exists a Gevrey function $u \in G^r(\mathbb{R}^{n+1})$ satisfying the Cauchy Problem:*

$$\begin{cases} P(D)u = D^m_t u + \sum_{j=0}^{m-1} a_j(D_x) D^j_t u = 0 \\ D^k_t u(0,x) = f_k(x) \quad \forall x \in \mathbb{R}^n, \forall k = 0, 1, \dots, m-1. \end{cases} \tag{2}$$

In the case $P(D)$ is hyperbolic , we have the corresponding result of existence in the C^∞ class.

The previous Theorem 1 can be extended to operators with variable coefficients, for example we refer to the important contribution of Bronstein [2].
Here, remaining in the frame of constant coefficients, we want to extend the previous theorem in order to assure the solvability of the Cauchy Problem for a larger class of data.
To this end, we define generalized Gevrey classes $G^{s\mathcal{P}}$, $1 < s < \infty$, based on a complete polyhedron $\mathcal{P}$, following Zanghirati [13], Corli [6], and give equivalent definitions of these classes (for details see Section 1).
Let us observe that $G^s \subset G^{s\mathcal{P}}$. The classes $G^{s\mathcal{P}}$ allow to express a precise result of regularity for the so called multi-quasi-elliptic equations, defined in terms of the norm $|\xi|_{\mathcal{P}}$ associated to $\mathcal{P}$, see Cattabriga [4], Hakobyan-Margaryan [8], Boggiatto-Buzano-Rodino [1] and the subsequent Section 1.
We then introduce a class of differential operators with constant coefficients, modelled on a complete polyhedron $\mathcal{P}$, that is natural to name as multi-quasi-hyperbolic operators.

DEFINITION 2. *Let $1 < s < \infty$ and let $\mathcal{P}$ be a complete polyhedron. We say that a differential operator with constant coefficients in $\mathbb{R}^n_x \times \mathbb{R}_t$ of the form* (1) *is multi-quasi-hyperbolic of order s with respect to $\mathcal{P}$ if there exists a constant $C > 0$ such that for $(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n$ the condition:*

$$P(\lambda, \xi) = \lambda^m + \sum_{j=0}^{m-1} a_j(\xi)\lambda^j = 0$$

implies:

$$\Im\lambda \geq -C|\xi|^{\frac{1}{s}}_{\mathcal{P}}$$

where $|\xi|_{\mathcal{P}}$ is the weight associated to $\mathcal{P}$ as in Section 1.

The algebraic properties of the symbol of multi-quasi-hyperbolic operators will be studied in Section 2, where we shall also give some examples.
Since $|\xi|_{\mathcal{P}} \leq const(1 + |\xi|)$, the previous definition implies s-hyperbolicity; therefore we may apply to $P(D)$ the previous Theorem 1 and conclude the well-posedness of (2) in G^r with $r < s$. However, for multi-quasi-hyperbolic operators of order s we have

the well-posedness of the Cauchy Problem in the larger classes $G^{r\mathcal{P}}$, $r < s$. More precisely, in Section 3 we will prove the following theorem:

THEOREM 2. *Let $P(D)$ be a differential operator in $\mathbb{R}_t \times \mathbb{R}^n_x$ as in* (1) *and let P be multi-quasi-hyperbolic of order s with respect to a complete polyhedron $\mathcal{P}$ in $\mathbb{R}^n$, with $1 < s < \infty$. Let $1 < r < s$ and assume $f_k \in G_0^{r\mathcal{P}}(\mathbb{R}^n_x)$ for $k = 0, 1, \ldots, m-1$. Then there exists $u(t,\cdot) \in G^{r\mathcal{P}}(\mathbb{R}^n_x)$ for $t \in \mathbb{R}$ satisfying the Cauchy Problem* (2)*.*

This gives a regularity of the solution u with respect to the space variables. To test the regularity with respect to the time variable, we need to define a new polyhedron $\mathcal{P}'$ that extends the polyhedron $\mathcal{P}$ to $\mathbb{R}^{n+1}$. We shall then be able to conclude $u \in G^{r\mathcal{P}'}(\mathbb{R}^{n+1})$, see to Section 4 for details.

1. Complete polyhedra and generalized Gevrey classes

A convex polyhedron $\mathcal{P}$ in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$. There is univocally determined by $\mathcal{P}$ a finite set $\mathcal{V}(\mathcal{P})$ of linearly independent points, called the set of vertices of $\mathcal{P}$, as the smallest set whose convex hull is $\mathcal{P}$.
Moreover, if $\mathcal{P}$ has non-empty interior, there exists a finite set:

$\mathcal{N}(\mathcal{P}) = \mathcal{N}_0(\mathcal{P}) \bigcup \mathcal{N}_1(\mathcal{P})$
such that :
$|\nu| = 1, \forall \nu \in \mathcal{N}_0(\mathcal{P})$ and
$\mathcal{P} = \{z \in \mathbb{R}^n | \nu \cdot z \geq 0, \forall \nu \in \mathcal{N}_0(\mathcal{P}) \wedge \nu \cdot z \leq 1, \forall \nu \in \mathcal{N}_1(\mathcal{P})\}.$

The boundary of $\mathcal{P}$ is made of faces $\mathcal{F}_\nu$ of equation:

$$\begin{array}{ll} \nu \cdot z = 0 & \text{if } \nu \in \mathcal{N}_0(\mathcal{P}) \\ \nu \cdot z = 1 & \text{if } \nu \in \mathcal{N}_1(\mathcal{P}). \end{array}$$

We now introduce a class of polyhedra that will be very useful in the following.

DEFINITION 3. *A complete polyhedron is a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n_+$ such that:*

1. *$\mathcal{V}(\mathcal{P}) \subset \mathbb{N}^n$ (i.e. all vertices have integer coordinates);*
2. *the origin $(0, 0, \ldots, 0)$ belongs to $\mathcal{P}$;*
3. *$dim(\mathcal{P}) = n$;*
4. *$\mathcal{N}_0(\mathcal{P}) = \{e_1, e_2, \ldots, e_n\}$, with $e_j = (0, 0, \ldots, 0, 1_{j-th}, 0, \ldots, 0) \in \mathbb{R}^n$ for $j = 1, \ldots, n$;*
5. *$\mathcal{N}_1(\mathcal{P}) \subset \mathbb{R}^n_+$.*

We note that 5. means that the set: $Q(x) = \{y \in \mathbb{R}^n | 0 \leq y \leq x\} \subset \mathcal{P}$ if $x \in \mathcal{P}$ and if s belongs to a face of $\mathcal{P}$ and $r > s$ then $r \notin \mathcal{P}$.
We can consider also polyhedra with rational vertices instead of integer vertices, as in Zanghirati (see [13]); the properties below remain valid.

PROPOSITION 1. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ with natural (or rational) vertices $s^l = (s_1^l, \dots, s_n^l)$, $l = 1, \dots, n(\mathcal{P})$, where $n(\mathcal{P})$ is the number of the vertices of $\mathcal{P}$, then:*

1. *for every $j = 1, 2, \dots, n$, there is a vertex s^{l_j} of $\mathcal{P}$ such that:*
$$(0, \dots, 0, s_j^{l_j}, 0, \dots, 0) = s_j^{l_j} e_j, \quad s_j^{l_j} = \max_{s \in \mathcal{P}} s_j =: m_j(\mathcal{P}).$$
2. *there is a finite non-empty set $\mathcal{N}_1(\mathcal{P}) \subset \mathbb{Q}_+^n \backslash \{0\}$ such that:*
$$\mathcal{P} = \bigcap_{\nu \in \mathcal{N}_1(\mathcal{P})} \{s \in \mathbb{R}_+^n : \nu \cdot s \leq 1\};$$
3. *for every $j = 1, \dots, n$ there is at least one $\nu \in \mathcal{N}_1(\mathcal{P})$ such that:*
$$m_j = m_j(\mathcal{P}) = \nu_j^{-1};$$
4. *if $s \in \mathcal{P}$, then:*
$$|\xi^s| \leq \sum_{l=1}^{n(\mathcal{P})} |\xi^{s^l}| \quad \left(\xi^s = \prod_{j=1}^{n} \xi_j^{s_j} \right).$$

The proof is trivial and we need only to point out that 4. is a consequence of the following lemma, for whose proof we refer to Boggiatto-Buzano-Rodino [1], Lemma 1.1.

LEMMA 1. *Given a subset $A \subset (\mathbb{R}_0^+)^n$ and a linear convex combination $\beta = \sum_{\alpha \in A} c_\alpha \alpha$, then for any $x \in (\mathbb{R}_0^+)^n$ the following inequality is satisfied:*

$$x^\beta \leq \sum_{\alpha \in A} c_\alpha x^\alpha \tag{3}$$

We now give some notations related to a complete polyhedron $\mathcal{P}$.
Let's denote by $\mathcal{L}(\mathcal{P})$ the cardinality of the smallest set $\mathcal{N}_1(\mathcal{P})$ that satisfies 2. of Proposition 1.
We denote:
$\mathcal{F}_\nu(\mathcal{P}) = \{s \in \mathcal{P} : \nu \cdot s = 1\}$, $\forall \nu \in \mathcal{N}_1(\mathcal{P})$ a face of $\mathcal{P}$;
$\mathcal{F} = \bigcup_{\nu \in \mathcal{N}_1(\mathcal{P})} \mathcal{F}_\nu(\mathcal{P})$ the boundary of $\mathcal{P}$;
$\mathcal{V}(\mathcal{P})$ the set of vertices of $\mathcal{P}$;
$\delta\mathcal{P} = \{s \in \mathbb{R}_+^n : \delta^{-1} s \in \mathcal{P}\}$, $\delta > 0$;
$k(s, \mathcal{P}) = \inf\{t > 0 : t^{-1} s \in \mathcal{P}\} = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu \cdot s$, $s \in \mathbb{R}_+^n$.
Now let $\mathcal{P}$ be a complete polyhedron, we say:
$\mu_j(\mathcal{P}) = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu_j^{-1}$;
$\mu = \mu(\mathcal{P}) = \max_{j=1,\dots,n} \mu_j$ the formal order of $\mathcal{P}$;
$\mu^{(0)}(\mathcal{P}) = \min_{\gamma \in \mathcal{V}(\mathcal{P}) \backslash \{0\}} |\gamma|$ the minimum order of $\mathcal{P}$;

$\mu^{(1)}(\mathcal{P}) = \max_{\gamma\in\mathcal{V}(\mathcal{P})} |\gamma|$ the maximum order of $\mathcal{P}$;
$q(\mathcal{P}) = \left(\frac{\mu(\mathcal{P})}{\mu_1(\mathcal{P})}, \dots, \frac{\mu(\mathcal{P})}{\mu_n(\mathcal{P})}\right)$;
$|\xi|_{\mathcal{P}} = (\sum_{s\in\mathcal{V}(\mathcal{P})} \xi^{2s})^{\frac{1}{2\mu}}, \forall\xi \in \mathbb{R}^n$ the weight of ξ associated to the polyhedron $\mathcal{P}$.
Considering a polynomial with complex coefficients, we can regard it as the symbol of a differential operator, and associate a polyhedron to it, as in the following.

DEFINITION 4. *Let $P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha$, $c_\alpha \in \mathbb{C}$ be a differential operator with complex coefficients in $\mathbb{R}^n$ and $P(\xi) = \sum_{|\alpha|\le m} c_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^n$ its characteristic polynomial. The Newton polyhedron or characteristic polyhedron associated to $P(D)$ is the convex hull of the set:*

$$\{0\} \bigcup \{\alpha \in \mathbb{Z}^n_+ : c_\alpha \neq 0\}.$$

There follow some examples of Newton polyhedra related to differential operators:

1. If $P(\xi)$ is an elliptic operator of order m, then its Newton polyhedron is complete and is the polyhedron of vertices $\{0, me_j,\ j = 1, \dots, n\}$ and so: $\mathcal{P} = \{\xi \in \mathbb{R}^n : \xi \ge 0,\ \sum_{i=1}^n \xi_i \le m\}$.
   The set $\mathcal{N}_1(\mathcal{P})$ is reduced to a point:
   $\nu = m^{-1}\sum_{j=1}^m e_j = (m^{-1}, \dots, m^{-1})$.
   $m_j(\mathcal{P}) = \mu_j(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \mu(\mathcal{P}) = m, \quad j = 1, 2, \dots, n$;
   $q(\mathcal{P}) = (1, 1, \dots, 1)$;
   $k(s,\mathcal{P}) = m^{-1}|s| = m^{-1}\sum_{j=1}^n s_j, \quad s \in \mathbb{R}^n_+$.

2. If $P(\xi)$ is a quasi-elliptic polynomial of order m (see for example Hörmander [9], Rodino [11], Zanghirati [12]), its characteristic polyhedron $\mathcal{P}$ is complete and has vertices $\{0, m_j e_j,\ j = 1, \dots, n\}$ where $m_j = m_j(\mathcal{P})$ are fixed integers.
   The set $\mathcal{N}_1(\mathcal{P})$ is again reduced to a point:
   $\nu = \sum_{j=1}^n m_j^{-1} e_j$.
   $\mathcal{P} = \{\xi \in \mathbb{R}^n : \xi \ge 0, \sum_{j=1}^n m_j^{-1}\xi_j \le 1\}$;
   $\mu_j(\mathcal{P}) = m_j, \quad j = 1, \dots, n$;
   $\mu^{(0)}(\mathcal{P}) = \min_{j=1,\dots,n} m_j$;
   $\mu(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \max_{j=1,\dots,n} m_j = m$;
   $q(\mathcal{P}) = (\frac{m}{m_1}, \dots, \frac{m}{m_n})$;
   $k(s,\mathcal{P}) = \mu(\mathcal{P})^{-1} q \cdot s, \quad s \in \mathbb{R}^n_+$.
   In this case the unique face of $\mathcal{P}$ is defined by the equation:

$$\frac{1}{m_1}x_1 + \ldots + \frac{1}{m_n}x_n = 1.$$

We note in general that s belongs to the boundary of $k(s,\mathcal{P})\mathcal{P}$ and $k(s,\mathcal{P})$ is univocally determined for complete polyhedra.
$k(s,\mathcal{P})$ satisfies the following inequality that will be very useful in the following:

$$\frac{|s|}{\mu^{(1)}} \le k(s,\mathcal{P}) \le |\frac{|s|}{\mu^{(0)}} \le |s|, \quad \forall s \in \mathbb{R}^n_+ \,. \tag{4}$$

We remember (see [1]) that the polyhedron of a hypoelliptic polynomial is complete, but the converse is not true in general.
We now introduce a class of generalized Gevrey functions associated to a complete polyhedron, as in Corli[6], Zanghirati [13].
They can be regarded as a particular case of inhomogeneous Gevrey classes with weight $\lambda(\xi) = |\xi|_{\mathcal{P}}$, in the sense of the definition of Liess-Rodino [11], and can be expressed also by means of the derivatives of u.
Following Corli [6] we give the following definition:

DEFINITION 5. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$. Let Ω be an open set in $\mathbb{R}^n$ and $s \in \mathbb{R},\ s > 1$. We denote by $G^{s\mathcal{P}}(\Omega)$ the set of all $u \in C^\infty(\Omega)$ such that:*

$$\begin{aligned} &\forall K \subset\subset \Omega, \quad \exists C > 0 : \\ &|D^\alpha u(x)| \leq C^{|\alpha|+1}(\mu k(\alpha, \mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall \alpha \in \mathbb{Z}^n_+,\ \forall x \in K. \end{aligned} \tag{5}$$

We also define:

$$G_0^{s\mathcal{P}}(\Omega) = G^{s\mathcal{P}}(\Omega) \cap C_0^\infty(\Omega).$$

The space $G^{s\mathcal{P}}(\Omega)$ can be endowed with a natural topology. Namely, we denote by $C^\infty(\mathcal{P}, s, K, C)$ the space of funcions $u \in C^\infty(\Omega)$ such that:

$$\begin{aligned} &\text{supp} u \subset K \\ &\|u\|_{K,C} = \sup_{\alpha \in \mathbb{Z}^n_+} \sup_{x \in K} C^{-|\alpha|}(\mu k(\alpha, \mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})}|D^\alpha u(x)| < \infty \end{aligned} \tag{6}$$

With such a norm, $C^\infty(\mathcal{P}, s, K, C)$ is a Banach space. Then:

$$G^{s\mathcal{P}}(\Omega) = \bigcap_{K \subset\subset \Omega} \bigcup_{C>0} C^\infty(\mathcal{P}, s, K, C)$$

endowed with the topology of projective limit of inductive limit.

REMARK 1. If $\mathcal{P}$ is the Newton polyhedron of an elliptic operator, then $G^{s\mathcal{P}}(\Omega)$ coincides with $G^s(\Omega)$, the set of the standard s-Gevrey functions in Ω.

REMARK 2. If $\mathcal{P}$ is the Newton polyhedron of a quasi-elliptic operator, then:

$$G^{s\mathcal{P}}(\Omega) = G^{sq}(\Omega), \quad \text{where } q = \left(\frac{m}{m_1}, \ldots, \frac{m}{m_n}\right)$$

the set of the anisotropic Gevrey functions, for definition see Hörmander [9], Rodino [11], Zanghirati [12].

REMARK 3. We have the following inclusion:

$$G^{s\frac{\mu}{\mu(1)}} \subset G^{s\mathcal{P}} \subset G^{s\frac{\mu}{\mu(0)}}, \quad \forall s > 1,\ \forall \mathcal{P}$$

as follows immediately from Definition 5 and the inequality (4).

We give now equivalent definitions of generalized Gevrey classes.
The arguments are similar to those in Corli [6], Zanghirati [13], but simpler, since for our purposes we need to consider only classes for $s > 1$; to be definite, we prefer to give here self-contained proofs.
Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ and let K be a compact set in $\mathbb{R}^n$.

DEFINITION 6. *If $\nu \in \mathcal{N}_1(\mathcal{P})$, let:*

$$C(\nu) = \{\alpha \in \mathbb{Z}^n_+ : k(\alpha, \mathcal{P}) = \alpha \cdot \nu\}.$$

$C(\nu)$ is a cone of $\mathbb{Z}^n_+$ and $C(\nu) \bigcap \mathcal{F} = \mathcal{F}_\nu$. This means that $k(\alpha, \mathcal{P})^{-1}\alpha \in \mathcal{F}_\nu$.

LEMMA 2. *Let $s > 1$, there is a function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that:*

$$\begin{aligned} &\chi(x) = 1, \quad x \in K, \\ &|D^\alpha \chi| \le C(CN^{s\mu})^{\alpha\cdot\nu}, \ \text{if } \alpha \cdot \nu \le N, \ \forall N = 1, 2, \dots, \ \forall \nu \in \mathcal{N}_1(\mathcal{P}). \end{aligned} \tag{7}$$

Proof. Every $u \in G_0^s(\mathbb{R}^n)$ satisfies the conditions 7. In fact, every $u \in G_0^s(\mathbb{R}^n)$ satisfies:

$$|D^\alpha u(x)| \le CC^{|\alpha|}|\alpha|^{s|\alpha|} \le CC^{|\alpha|}N^{s|\alpha|} \quad \text{if } |\alpha| \le N.$$

In fact, as $0 < \nu_j \le 1, \forall j = 1, \dots, n, \forall \nu \in \mathcal{N}_1(\mathcal{P})$ and $\alpha \cdot \nu \le |\alpha|$, we get:

$$\begin{aligned} &|\alpha| \le \alpha \cdot \nu \max(\nu_j)^{-1} = \alpha \cdot \nu\mu \\ &|\alpha| \le N \Rightarrow \alpha \cdot \nu \le N. \end{aligned}$$

So, taking $R = C^\mu \mu^{s\mu}$, we obtain:

$$|D^\alpha u(x)| \le C(RN^{s\mu})^{\alpha\cdot\nu}, \ \forall \nu \in \mathcal{N}_1(\mathcal{P}), \ \forall \alpha : \alpha \cdot \nu \le N.$$

Then we can proceed as in the C^∞ case to construct $\chi \in G_0^s(\mathbb{R}^n)$ such that $\chi \equiv 1$ in K.

□

LEMMA 3. *With the previous notations, if $u \in G^{s\mathcal{P}}(\mathbb{R}^n)$, then taking χ as in Lemma 2, we obtain the estimate:*

$$|\widehat{\chi u}(\xi)| \le C\left(\frac{CN^s}{|\xi|_{\mathcal{P}} + N^s}\right)^{\mu N} \qquad N = 1, 2, \dots \tag{8}$$

Proof. By Leibniz formula we can write:

$$|D^\alpha(\chi u)| \le \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta}\chi||D^\beta u|$$

Let's choose any $\beta \le \alpha$, then $\beta \in C(\nu)$ for some $\nu \in \mathcal{N}_1(\mathcal{P})$ (not necessarily unique) and for that ν we get:

$$\sup_{x \in \text{supp}\chi} |D^\beta \chi(x)| \le C(CN^{s\mu})^{\beta\cdot\nu}$$

and by Lemma 2:

$$\sup_{x\in \mathrm{supp}\chi} |D^{\alpha-\beta}\chi(x)| \le C(CN^{s\mu})^{(\alpha-\beta)\cdot\nu}$$
$$\text{if } \alpha\cdot\nu \le N,\ \ N = 1, 2, \dots$$

So we get:

$$\begin{aligned}\sup_{x\in \mathrm{supp}\chi} |D^{\alpha-\beta}\chi(x)||D^{\beta}u(x)| &\le C(CN^{s\mu})^{(\alpha-\beta)\cdot\nu} C_1^{|\alpha-\beta|+1}(\mu k(\beta,\mathcal{P}))^{s\mu k(\beta,\mathcal{P})}\\ &\le C(CN^{s\mu})^{(\alpha-\beta)\cdot\nu} C_1(C_2\mu k(\beta,\mathcal{P}))^{s\mu k(\beta,\mathcal{P})}\end{aligned}$$

as $|\beta| \le \mu k(\beta,\mathcal{P})$, taking $C_2 = C_1^{\frac{1}{s}}$.
But we have supposed that $k(\beta,\mathcal{P}) = \beta\cdot\nu$ and $\beta \le \alpha$, moreover $\alpha\cdot\nu \le N$ implies $\beta\cdot\nu \le N$. We now proceed to estimate:

$$\begin{aligned}\sup_{x\in \mathrm{supp}\chi} |D^{\alpha-\beta}\chi(x)||D^{\beta}u(x)| &\le C'(CN^{s\mu})^{(\alpha-\beta)\cdot\nu}(C_1N^{s\mu})^{\beta\cdot\nu}\\ &\le C'(C''N^{s\mu})^{\alpha\cdot\nu}\end{aligned}$$

$\forall\alpha$, if $\alpha\cdot\nu \le N,\ \forall\nu \in \mathcal{N}_1(\mathcal{P}),\ N = 1, 2, \dots$. Taking $C'' = \max\{C, C_1\}$, using the linearity of scalar product and observing that:

$$(\alpha-\beta)\cdot\nu + \beta\cdot\nu = \alpha\cdot\nu,$$
$$k(\alpha,\mathcal{P}) = \max\{\alpha\cdot\nu,\ \nu\in\mathcal{N}_1(\mathcal{P})\}$$

we get the inequality:

$$|D^{\alpha}(\chi u)| \le C'(C''N^{s})^{\mu k(\alpha,\mathcal{P})}.$$

On the other hand we have:

$$\begin{aligned}|\widehat{D^{\alpha}(\chi u)}| &= |\int e^{-ix\cdot\xi} D^{\alpha}(\chi u)|\\ &\le \int_{\mathrm{supp}\chi} |D^{\alpha}(\chi u)| \le C \sup_{\mathrm{supp}\chi} |D^{\alpha}(\chi u)|\end{aligned}$$

as χ has compact support. Using the properties of the Fourier transform we conclude:

$$|\widehat{D^{\alpha}(\chi u)}| = |\xi^{\alpha}\widehat{\chi u}| \le C \sup_{\mathrm{supp}\chi} |D^{\alpha}(\chi u)| \le C(CN^{s})^{\mu k(\alpha,\mathcal{P})}.$$

Let now $\alpha = vN$, for any $v \in \mathcal{V}(\mathcal{P})$, the set of vertices of $\mathcal{P}$, summing up the previous inequalities for $\alpha = 0,\ \alpha = vN, \forall v \in \mathcal{V}(\mathcal{P})$, we obtain:

$$|\widehat{\chi u}(\xi)|N^{s\mu N} + \sum_{v\in\mathcal{V}(\mathcal{P})} |\widehat{\chi u}(\xi)\xi^{vN}| \le C(CN^{s})^{\mu N}.$$

Using the following inequality:

$$\sum_{v\in\mathcal{V}(\mathcal{P})} |\xi^{vN}| n(\mathcal{P})^{N\mu-1} \le |\xi|_{\mathcal{P}}^{N\mu} \le 2^{n(\mathcal{P})(\mu N-1)} \sum_{v\in\mathcal{V}(\mathcal{P})} |\xi^{vN}|$$

where $n(\mathcal{P})$ denotes the number of vertices of $\mathcal{P}$ different from the origin.
So we can conclude that:

$$\begin{aligned} |\widehat{\chi u}(\xi)| &\le \frac{C(CN^s)^{\mu N}}{N^{s\mu N} + \sum_{s\in\mathcal{V}(\mathcal{P})} |\xi^{sN}|} \\ &\le \frac{C(CN^s)^{\mu N}}{N^{s\mu N} + \frac{|\xi|_{\mathcal{P}}^{N\mu}}{2^{n(\mathcal{P})(\mu N-1)}}} \le C' \left(\frac{C'N^s}{|\xi|_{\mathcal{P}} + N^s} \right)^{\mu N}, \quad N = 1, 2, \ldots . \end{aligned}$$

$\square$

THEOREM 3. *Let Ω be an open set in $\mathbb{R}^n$, $x_0 \in \Omega$, $u \in \mathcal{D}'(\Omega)$, then u is of class $G^{s\mathcal{P}}$ in a neighborhood of x_0 if and only if there is a neighborhood U of x_0 and $v \in \mathcal{E}'(\Omega)$ or $v \in \mathcal{S}'(\mathbb{R}^n)$ such that:*

1. *$v = u$ in U*

2. *$\hat{v}$ satisfies:*

$$(9) \qquad |\hat{v}(\xi)| \le C \left(\frac{CN^s}{|\xi|_{\mathcal{P}}} \right)^{\mu N} = C \left(\frac{C'N}{|\xi|_{\mathcal{P}}^{\frac{1}{s}}} \right)^{s\mu N}, \quad N = 1, 2, \ldots .$$

REMARK 4. The previous Theorem 3 admits the more general formulation:
Let $K \subset\subset \Omega$, $u \in \mathcal{D}'(\Omega)$, then u is of class $G^{s\mathcal{P}}$ in a neighborhood U of K if and only if there is $v \in \mathcal{E}'(\Omega)$ or $v \in \mathcal{S}'(\mathbb{R}^n)$, $v = u$ in U such that $\hat{v}$ satisfies the estimate (9). The proof is analogous to that of Theorem 3.

Proof. Proof of necessity: Let $u \in G^{s\mathcal{P}}$ in the set $\{x : |x - x^0| \le 3r\}$, $0 < r \le 1$, χ as in Lemma 3, with $K = \{x : |x - x_0| \le r\}$ and $\operatorname{supp}\chi \subset \{x : |x - x_0| \le 2r\}$. Then the function $v = \chi u$ satisfies conditions 1.,2. of the theorem.
Proof of sufficiency:
Let $v \in \mathcal{E}'(\Omega)$ satisfy the conditions 1.,2.. Then there are two constants $M_0, C > 0$ such that:

$$|\hat{v}(\xi)| \le C(1 + |\xi|)^{M_0}.$$

So:

$$|\hat{v}(\xi)| \le C|\xi|_{\mathcal{P}}^{M}, \qquad M = \mu M_0$$

Let's fix $\alpha \in \mathbb{Z}_+^n$, the integral $\int |\xi^\alpha \hat{v}(\xi)| d\xi$ converges by condition 2..
By 1., if $x \in U$, then:

$$D^\alpha u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \xi^\alpha \hat{v}(\xi) d\xi .$$

Now we use the property:

$$|\xi^\alpha| \leq |\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})}. \tag{10}$$

In fact, given $\alpha \in \mathbb{Z}_+^n$, then $\frac{\alpha}{k(\alpha,\mathcal{P})} \in \mathcal{F}$ and so, by the definition of convex hull, told $s^{l_1}, \ldots, s^{l_r}$ the vertices of the face where $\frac{\alpha}{k(\alpha,\mathcal{P})}$ lies, we have:

$$\alpha = k(\alpha,\mathcal{P}) \sum_{i=1}^{r} \lambda_i s^{l_i}, \quad \sum_{i=1}^{r} \lambda_i = 1, \quad \lambda_i \geq 0 \,,$$

and hence by Lemma 1:

$$\begin{aligned} |\xi^\alpha| = \prod_{j=1}^{n} |\xi_j^{\alpha_j}| &\leq \sum_{i=1}^{r} \lambda_i \left(\prod_{j=1}^{n} |\xi_j|^{s_j^{l_j}} \right)^{k(\alpha,\mathcal{P})} \\ &\leq \left(\sum_{s^l \in \mathcal{V}(\mathcal{P})} \prod_{j=1}^{n} |\xi_j|^{2s_j^{l_j}} \right)^{\frac{1}{2}k(\alpha,\mathcal{P})} \leq |\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})}. \end{aligned} \tag{11}$$

Now, splitting the integral into the two regions:

$$|\xi|_{\mathcal{P}} < N^s, \quad |\xi|_{\mathcal{P}} > N^s$$

we get:

$$\begin{aligned} |D^\alpha u(x)| \leq (2\pi)^{-n}(1+N^s)^{M+s\mu k(\alpha,\mathcal{P})} \int_{|\xi|_{\mathcal{P}}<N^s} d\xi \\ + C(CN^s)^{\mu N} \int_{|\xi|_{\mathcal{P}}>N^s} |\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})-\mu N} d\xi \,. \end{aligned}$$

The first integral is limited for all N and the second converges for large N, namely we set $N = k(\alpha,\mathcal{P}) + R$ for R depending only on $\mathcal{P}$ and M. Then:

$$|D^\alpha u(x)| \leq C'(C'\mu k(\alpha, P) + R)^{s\mu(k(\alpha,\mathcal{P})+R)}$$

implies:

$$|D^\alpha u(x)| \leq C^{|\alpha|+1}(\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}.$$

□

We now give a characterization of generalized Gevrey functions by means of exponential estimates for the Fourier transform, that is possible if $s > 1$ and will be of main interest in the proof of Theorem 2.

THEOREM 4. *1. Let $u \in G_0^{s\mathcal{P}}(\mathbb{R}^n)$, then there exist two constants $C > 0$, $\epsilon > 0$ such that:*

$$|\hat{u}(\xi)| \leq C \exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}}); \tag{12}$$

2. if the Fourier transform of $u \in \mathcal{E}'(\mathbb{R}^n)$, or $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies (12)*, then $u \in G^{s\mathcal{P}}(\mathbb{R}^n)$.*

In the proof of Theorem 4 we shall use the following lemma:

LEMMA 4. *The estimates:*

$$|\hat{u}(\xi)| \leq C \left(\frac{CN}{|\xi|_{\mathcal{P}}^{\frac{1}{s}}} \right)^N, \quad N = 1, 2, \ldots \tag{13}$$

are equivalent to the following:

$$|\hat{u}(\xi)| \leq C^{N+1} N! |\xi|_{\mathcal{P}}^{-\frac{N}{s}}, \quad N = 1, 2, \ldots \tag{14}$$

for suitable different constants $C > 0$ independent of N.

The proof of the lemma is trivial and based on the inequalities: $N! \leq N^N$, $\forall N = 1, 2, \ldots$ and $N^N \leq e^N N!$. Now we prove Theorem 4.

Proof. Let's suppose that $u \in G_0^{s\mathcal{P}}(\mathbb{R}^n)$, then taking supp$u \subset K$ in Remark 4, we obtain:

$$|\hat{u}(\xi)|^{\frac{1}{s\mu}} \leq C \left(\frac{CN}{|\xi|_{\mathcal{P}}^{\frac{1}{s}}} \right)^N, \quad N = 1, 2, \ldots$$

Then by the previous lemma, u satisfies for a suitable constant $C' > 0$:

$$|\hat{u}(\xi)|^{\frac{1}{s\mu}} \leq (C')^{N+1} N! |\xi|_{\mathcal{P}}^{-\frac{N}{s}}$$

and fixing $\epsilon = \frac{1}{2C'}$ we get:

$$|\hat{u}(\xi)|^{\frac{1}{s\mu}} \epsilon^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \leq \frac{1}{2\epsilon} \frac{1}{2^N}.$$

Summing up for $N = 1, 2, \ldots$:

$$|\hat{u}(\xi)|^{\frac{1}{s\mu}} \sum_{N=0}^{\infty} \epsilon^N \frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \leq \frac{1}{2\epsilon} \sum_{N=0}^{\infty} \frac{1}{2^N}$$

and hence:

$$|\hat{u}(\xi)|^{\frac{1}{s\mu}} exp(\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}}) \leq \frac{1}{\epsilon}\ .$$

So we obtain for a suitable constant $C > 0$:

$$|\hat{u}(\xi)| \leq Cexp(-\epsilon s\mu|\xi|_{\mathcal{P}}^{\frac{1}{s}}) = Cexp(-\epsilon'|\xi|_{\mathcal{P}}^{\frac{1}{s}})$$

letting $\epsilon' = \epsilon s\mu$.

If $u \in \mathcal{E}'(\mathbb{R}^n)$ satisfies:

$$|\hat{u}(\xi)| \leq C\exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}}) = C\left(\exp\left(-\frac{\epsilon}{\mu s}|\xi|_{\mathcal{P}}^{\frac{1}{s}}\right)\right)^{\mu s}$$

then:

$$|\hat{u}(\xi)|^{\frac{1}{\mu s}}\exp\left(\frac{\epsilon}{\mu s}|\xi|_{\mathcal{P}}^{\frac{1}{s}}\right) \leq C^{\frac{1}{\mu s}}\ .$$

Hence, by expanding the exponential into Taylor series we have:

$$\sum_{N=0}^{\infty}|\hat{u}(\xi)|^{\frac{1}{\mu s}}(\epsilon')^N\frac{|\xi|_{\mathcal{P}}^{\frac{1}{s}}}{N!} \leq C'$$

with $C' = C\frac{1}{\mu s},\ \epsilon' = \frac{\epsilon}{\mu s}$.
This implies:

$$|\hat{u}(\xi)|^{\frac{1}{\mu s}}(\epsilon')^N\frac{|\xi|_{\mathcal{P}}^{\frac{N}{s}}}{N!} \leq C'$$

and hence for a new constant $C > 0$:

$$|\hat{u}(\xi)| \leq C'\left(\frac{CN}{|\xi|_{\mathcal{P}}^{\frac{1}{s}}}\right)^{s\mu N}$$

that means that $u \in G^{s\mathcal{P}}(\mathbb{R}^n)$ as the conditions of Theorem 4 are satisfied in a neighborhood of any $x_0 \in \mathbb{R}^n$.

□

2. Multi-quasi-hyperbolic operators

For any complete polyhedron $\mathcal{P}$ we define the corresponding class of multi-quasi-hyperbolic operators, according to Definition 2. For short, we denote multi-quasi-hyperbolic operators of order s with respect to $\mathcal{P}$ by $(s,\mathcal{P})$-hyperbolic.
Obviously, if $P(D)$ is multi-quasi-hyperbolic of order $s > 1$ with respect to $\mathcal{P}$, $P(D)$ is also multi-quasi-hyperbolic of order $r,\ \forall r,\ 1 < r < s$ with respect to $\mathcal{P}$.
We now prove some properties for this class of operators.

PROPOSITION 2. *If $P(D)$ is $(s,\mathcal{P})$-hyperbolic for $1 < s < \infty$, then for any $(\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n$ such that $P(\lambda,\xi) = 0$, there is $C > 0$ such that:*

$$|\Im\lambda| \leq C|\xi|_{\mathcal{P}}^{\frac{1}{s}}. \tag{15}$$

Proof. The coefficient of λ^{m-1} in $P(\lambda,\xi)$ is a linear function of ξ. If the zeros of $P(\lambda,\xi)$ are denoted by λ_j, it follows that $\sum_{j=1}^m \lambda_j$ is a linear function of ξ. Then also $\sum_{j=1}^m \Im\lambda_j$ is a linear combination of ξ, and if $P(D)$ is $(s,\mathcal{P})$-hyperbolic, then:

$$\sum_{j=0}^m \Im\lambda_j \geq -mC|\xi|_{\mathcal{P}}^{\frac{1}{s}}$$

implies $\sum_{j=0}^m \Im\lambda_j = C_0$ for a suitable constant C_0. So we obtain for all λ_k root of $P(\lambda,\xi)$:

$$\Im\lambda_k = C_0 - \sum_{j\neq k}\Im\lambda_j \leq C_0 + C(m-1)|\xi|_{\mathcal{P}}^{\frac{1}{s}} \leq C'|\xi|_{\mathcal{P}}^{\frac{1}{s}}.$$

That completes the inequality:

$$|\Im\lambda_k| \leq C|\xi|_{\mathcal{P}}^{\frac{1}{s}}$$

for all roots λ_k of $P(\lambda,\xi)$.

□

PROPOSITION 3. *If $P(D)$ is $(s,\mathcal{P})$-hyperbolic for $1 < s < \infty$, then the principal part $P_m(D)$ of $P(D)$ is hyperbolic, i.e. the homogeneous polynomial $P_m(\lambda,\xi)$ satisfies:*

$$P_m(\lambda,\xi) = 0 \quad (\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n \Rightarrow \Im\lambda = 0. \tag{16}$$

Proof. Taking $\sigma > 0$, $\lambda\in\mathbb{C}$, $\xi\in\mathbb{R}^n$, we get:

$$P_m(\lambda,\xi) = \lim_{\sigma\to\infty} P(\sigma\lambda,\sigma\xi)\cdot\sigma^{-m}$$

From Proposition 2 the zeros of $P(\sigma\lambda,\sigma\xi)$ must satisfy:

$$|\Im\lambda_k| \leq C\frac{|\sigma\xi|_{\mathcal{P}}^{\frac{1}{s}}}{\sigma}$$

So for $\sigma\to\infty$, $\Im\lambda = 0$ for all the roots $\lambda\in\mathbb{C}$ of $P_m(\lambda,\xi)$, that is $P_m(D)$ is hyperbolic.
□

PROPOSITION 4. *For a differential operator $P_m(D)$ associated to an homogeneous polynomial $P_m(\lambda,\xi)$, the notion of hyperbolicity and $(s,\mathcal{P})$-hyperbolicity coincide.*

The proof follows easily from Proposition 3.

PROPOSITION 5. *Let $P(D)$ be a differential operator of the form:*

$$P(D) = P_m(D) + \sum_{j=0}^{m-1} a_j(D_x)D_t^j \tag{17}$$

with homogeneous principal part:

$$P_m(D) = D_t^m + \sum_{j=0}^{m-1} b_j(D_x)D_t^j, \tag{18}$$

with:

$$\begin{aligned} order(b_j(D_x)) &= m - j \\ order(a_j(D_x)) &\leq m - j - 1 \end{aligned}$$

and assume:

$$\begin{aligned} &P_m(\lambda, \xi) = 0 \quad \textit{for } \lambda \in \mathbb{C},\ \xi \in \mathbb{R}^n \textit{ implies } \Im\lambda = 0 \\ &|a_j(\xi)| \leq C|\xi|_{\mathcal{P}}^{m-1}(1 + |\xi|)^{-j} \quad \textit{for } \ j = 1, \dots, m-1 \end{aligned} \tag{19}$$

(for a $C > 0$).
Then $P(D)$ is $(\frac{m}{m-1}, \mathcal{P})$ hyperbolic.

Proof. By definition, the terms $a_j(D_x),\ b_j(D_x)$ satisfy for a suitable $C > 0$:

$$\begin{aligned} |b_j(\xi)| &\leq C|\xi|^{m-j} \\ |a_j(\xi)| &\leq C|\xi|^{m-j-1}. \end{aligned} \tag{20}$$

In the region $\{\epsilon|\lambda| > |\xi|\}$ (for $\epsilon > 0$ sufficiently small), the following inequality is satisfied:

$$|P(\lambda, \xi) - \lambda^m| \leq C \sum_{j=0}^{m-1} |\xi|^{m-j}|\lambda|^j < \frac{\lambda^m}{2}$$

that implies:

$$|P(\lambda, \xi)| > \frac{\lambda^m}{2}$$

and consequently $P(\lambda, \xi)$ can't have roots in this region and the roots must so satisfy for $\epsilon > 0$:

$$|\lambda| \leq \epsilon^{-1}|\xi|. \tag{21}$$

On the other hand, for (λ, ξ) such that $P(\lambda, \xi) = 0$:

$$P_m(\lambda, \xi) = -(P - P_m)(\lambda, \xi) = -\sum_{j=0}^{m-1} a_j(\xi)\lambda^j.$$

In view of the estimates (21) and (19), we obtain:

$$\begin{aligned}|P_m(\lambda,\xi)| &\le \sum_{j=0}^{m-1} |a_j(\xi)||\lambda|^j \le C\sum_{j=0}^{m-1} |\xi|_{\mathcal{P}}^{m-1}(1+|\xi|)^{-j}|\lambda|^j \\ &\le C'\sum_{j=0}^{m-1} |\xi|_{\mathcal{P}}^{m-1}(1+|\xi|)^{-j}|\xi|^j \le C''|\xi|_{\mathcal{P}}^{m-1}\end{aligned}$$

In view of the hyperbolicity of P_m we can write:

$$P_m(\lambda,\xi) = \prod_{j=0}^{m}(\lambda-\lambda_j), \quad \lambda_j \in \mathbb{R}.$$

Hence:

$$\begin{aligned}&|\Im\lambda|^m \le |P_m(\lambda,\xi)| \le C''|\xi|_{\mathcal{P}}^{m-1}\\ &|\Im\lambda| \le C'''|\xi|_{\mathcal{P}}^{\frac{m-1}{m}}\end{aligned}$$

i.e. $P(D)$ is $(\frac{m}{m-1},\mathcal{P})$ hyperbolic.

□

PROPOSITION 6. *Any differential operator $P(D) = D_t^m + \sum_{j=0}^{m-1} a_j(D_x)D_t^j$ satisfying the condition:*

(22) $$|a_j(\xi)| \le C|\xi|_{\mathcal{P}}^{m-j-1} \quad j = 0,1,\ldots,m-1, \text{ for } C>0$$

is $(\frac{m}{m-1},\mathcal{P})$ hyperbolic.

We note that the principal part is only D_t^m and is obviously hyperbolic; Proposition 6 states that in this particular case we may replace (19) with the weaker assumption (22).

Proof. By the estimates (22) we have:

$$|P(\lambda,\xi)-\lambda^m| \le C\sum_{j=0}^{m-1}|\xi|_{\mathcal{P}}^{m-j-1}|\lambda|^j < \frac{|\lambda|^m}{2}$$

in the region $\{(\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n : |\xi|_{\mathcal{P}} < \epsilon|\lambda|\}$ for a sufficiently small $\epsilon > 0$. Consequently $|P(\lambda,\xi)| > \frac{|\lambda|^m}{2}$ and $P(\lambda,\xi)$ can't have roots in this region and so they must satisfy:

(23) $$|\lambda| \le \epsilon^{-1}|\xi|_{\mathcal{P}}.$$

For (λ,ξ) such that $P(\lambda,\xi)=0$, we write:

$$\lambda^m = -(P(\lambda,\xi)-\lambda^m) = -\sum_{j=0}^{m-1} a_j(\xi)\lambda^j\ .$$

In view of the estimate (23) for λ and (22) for $a_j(\xi)$:

$$|\lambda|^m \leq C' \sum_{j=0}^{m-1} |\xi|_{\mathcal{P}}^{m-j-1} |\lambda|^j \leq C'' |\xi|_{\mathcal{P}}^{m-1}$$

Hence:

$$|\Im\lambda|^m \leq C'' |\xi|_{\mathcal{P}}^{m-1}$$
$$|\Im\lambda| \leq C''' |\xi|_{\mathcal{P}}^{\frac{m-1}{m}}$$

i.e. $P(D)$ is $(\frac{m}{m-1}, \mathcal{P})$ hyperbolic.

□

REMARK 5. A more general version of Proposition 6 is easily obtained by assuming as in Proposition 5 that $P(D)$ has hyperbolic homogeneous principal part $P_m(D) = \sum_{j=0}^{m-1} b_j(D_x) D_t^j$ with:

$$|b_j(\xi)| \leq C|\xi|_{\mathcal{P}}^{m-j}, \quad j = 0, 1, \ldots, m-1 \tag{24}$$

and keeping condition (22) for the lower order terms.
Observe however that (24) implies $b_j(\xi) \equiv 0$, but in the quasi-homogeneous case.

There follow some examples of multi-quasi-hyperbolic operators, that follow from the previous propositions.

1. If $P(D)$ is a differential operator in $\mathbb{R}^n$ with symbol $P(\xi)$ and Newton polyhedron $\mathcal{P}$ of formal order μ, then the differential operator in $\mathbb{R}^{n+1}$:

$$Q(D) = D_t^m + P(D_x),$$

   with $m > \mu$, is multi-quasi-hyperbolic of order $\frac{m}{\mu}$ with respect to $\mathcal{P}$.
   In fact, the roots of the symbol of $Q(D)$ satisfy:

$$|\Im\lambda| \leq C|\xi|_{\mathcal{P}}^{\frac{\mu}{m}}.$$

2. A particular case of Proposition 5 is the following:
   if $\mathcal{P}$ is the polyhedron in $\mathbb{R}^2$ of vertices $(0,0)$, $(0,2)$, $(1,0)$, then $\mu = 2$ and the following operator:

$$P(D_x, D_t) = P_3(D_x, D_t) + C_1 D_{x_2}^2 + C_2 D_{x_1} + C_3 D_{x_2} + C_4 D_t + C_5$$

   where $P_3(D_x, D_t)$ is an hyperbolic homogeneous operator of order 3 and $C_1, \ldots, C_5 \in \mathbb{C}$, is multi-quasi-hyperbolic of order $\frac{3}{2}$ with respect to $\mathcal{P}$.

3. Another particular case of Proposition 5 is the following:
if $\mathcal{P}$ is the polyhedron in $\mathbb{R}^2$ of vertices $(0,0),\ (0,3),\ (1,2),\ (2,0)$, then the formal order $\mu = 4$ and the operator of order 4:

$$\begin{aligned} P(D_t, D_x) = P_4(D_t, D_x) + c_1 D_{x_2}^2 + c_2 D_{x_1} D_{x_2} + c_3 D_{x_2} D_t \\ + c_4 D_{x_1} + c_5 D_{x_2} + c_6 D_t + c_7 \end{aligned}$$

where $P_4(D_x, D_t)$ is an hyperbolic homogeneous operator of order 4 and $C_1, ..., C_7 \in \mathbb{C}$, is multi-quasi-hyperbolic of order $\frac{4}{3}$ with respect to $\mathcal{P}$.

4. Let $P(D)$ be a differential operator in $\mathbb{R}^n$ with symbol $P(\xi)$, then we consider the differential operator in $\mathbb{R}^{n+1}$:

$$Q(D) = (D_t^2 + \triangle_x)^m - P(D_x)$$

with $order\, P(D) < 2m$.
The roots of the symbol of $Q(D)$ satisfy:

$$(\lambda^2 - |\xi|^2)^m - P(\xi) = 0$$

and then, denoting by $P(\xi)^{\frac{1}{m}}$ the generic $m - th$ root of $P(\xi)$:

$$\Im\lambda = |\xi^2 + P(\xi)^{\frac{1}{m}}|^{\frac{1}{2}} sen\theta$$

where for $\theta > 0$:

$$tg2\theta = \frac{\Im(\xi^2 + P(\xi)^{\frac{1}{m}})}{\Re(\xi^2 + P(\xi)^{\frac{1}{m}})} \leq \frac{|P(\xi)|^{\frac{1}{m}}}{|\xi|^2}.$$

We consider the first term of the Taylor expansion to estimate $sen\theta$:

$$sen\theta \leq C \frac{|P(\xi)|^{\frac{1}{m}}}{2|\xi|^2}$$
$$|\Im\lambda| \leq C \frac{P(\xi)^{\frac{1}{m}}}{|\xi|}$$

Let $\mathcal{P}'$ be a given complete polyhedron. If for some $\rho < 1$ we have:

$$\text{(25)} \qquad \begin{aligned} |P(\xi)|^{\frac{1}{m}} \leq C |\xi|_{\mathcal{P}'}^{\rho} |\xi| \qquad \text{i.e.} \\ |P(\xi)| \leq C |\xi|_{\mathcal{P}'}^{\rho m} |\xi|^m \end{aligned}$$

then $Q(D)$ is multi-quasi-hyperbolic of order $\frac{1}{\rho}$ with respect to $\mathcal{P}'$. If we consider in particular the Newton polyhedron associated to $P(D_x)$ with formal order $\mu < 2m$, then $Q(D)$ is multi-quasi-hyperbolic of order $\frac{m}{\mu}$, but we can consider also a larger class of polyhedra satisfying condition (25), and in any case stronger with respect to what we may deduce from Proposition 5.

3. Proof of Theorem 2

Now we prove Theorem 2.

Proof. We try to satisfy the Cauchy Problem:

$$\begin{cases} P(D)u = D_t^m u + \sum_{k=0}^{m-1} a_k(D_x) D_t^k u = 0 \\ D_t^k u(0,x) = f_k(x) \quad \forall x \in \mathbb{R}^n, \forall k = 0,1,\dots,m-1 \end{cases}$$

by a function $u(t,x)$ such that $u(t,x) \in \mathcal{S}(\mathbb{R}^n_x)$ for any fixed $t \in \mathbb{R}$.
We apply partial Fourier transform with respect to x, considering t as a parameter, so the Cauchy Problem admits the following equivalent formulation:

$$\begin{cases} P(D_t,\xi)\hat{u} = D_t^m \hat{u} + \sum_{k=0}^{m-1} a_k(\xi) D_t^j \hat{u} = 0 \\ D_t^k \hat{u}(0,\xi) = \hat{f}_k(\xi) \quad \forall \xi \in \mathbb{R}^n, \ k = 0,1,\dots,m-1. \end{cases} \tag{26}$$

This makes sense as f_k have compact support, $\forall k = 0,1,\dots,m-1$ and $u \in \mathcal{S}(\mathbb{R}^n)$, $\forall t$ fixed.
Now we consider the Cauchy Problem (26) as an ordinary differential problem in t, depending on the parameter ξ. A solution to problem (26) is given by:

$$\hat{u}(t,\xi) = \sum_{j=0}^{m-1} \hat{f}_j(\xi) F_j(t,\xi), \tag{27}$$

where $F_j(t,\xi)$, $j = 0,1\dots,m-1$, satisfy the ordinary Cauchy Problem on t depending on the parameter $\xi \in \mathbb{R}^n$:

$$\begin{cases} P(D_t,\xi)F_j = 0 \\ D_t^k F_j(0,\xi) = \delta_{jk} \quad k = 0,1,\dots,m-1 \end{cases} \tag{28}$$

where δ_{jk} denote the Kronecker delta.
The solution of (28) exists and is unique by the Cauchy theorem for ordinary differential equations, and the function $\hat{u}$ defined in (27) gives indeed a solution to the Cauchy Problem (26), as is easy to check. Now we want to estimate $|D_x^\alpha u(t,x)|$ or, equivalently, $|\hat{u}(t,\xi)|$ to obtain generalized Gevrey estimates with respect to the space variables.
By assumption $\hat{f}_j(\xi) \in G_0^{r\mathcal{P}}(\mathbb{R}^n)$, so, in view of Theorem 4,(1), there are constants $\epsilon_j, C_j > 0$ $(j = 0,1,\dots,m-1)$ such that for every $\xi \in \mathbb{R}^n$:

$$|\hat{f}_j(\xi)| \le C_j \exp(-\epsilon_j |\xi|_{\mathcal{P}}^{\frac{1}{r}}) \le C \exp(-\epsilon |\xi|_{\mathcal{P}}^{\frac{1}{r}}),$$

taking:

$$C = \max\{C_j, \ j = 0,\dots,m-1\},$$
$$\epsilon = \max\{\epsilon_j, \ j = 0,\dots,m-1\}.$$

To estimate F_j we use the following lemma (for the proof see for example Hörmander[9], Lemma 12.7.7).

LEMMA 5. *Let $L(D) = D^m + \sum_{j=0}^{m-1} a_j D^j$ be an ordinary differential operator with constant coefficients $a_j \in \mathbb{C}$. Write $\Lambda = \{\lambda \in \mathbb{C} : L(\lambda) = 0\}$ and assume:*

$$\begin{aligned} &\max_{\lambda\in\Lambda} |\lambda| \le A, \\ &\max_{\lambda\in\Lambda} |\Im\lambda| \le B \quad \textit{for } \lambda \in \Lambda. \end{aligned} \tag{29}$$

Then the solutions $v_j(t)$, $j = 0, 1, \ldots, m-1$ of the Cauchy Problems:

$$\begin{cases} L(D)v_j = 0 \\ (D^k v_j)(0) = \delta_{jk}, \quad k = 0, \ldots, m-1 \end{cases} \tag{30}$$

satisfy the following estimates:

$$\begin{aligned} &|D^N v_j(t)| \le 2^m (A+1)^{N+m+1} e^{(B+1)|t|}, \\ &N = 0, 1, \ldots,\ t \in \mathbb{R}. \end{aligned} \tag{31}$$

We now apply the estimates of Lemma 5 for $N = 0$ to the functions $F_j(t,\xi)$ in (28), $j = 0, 1, \ldots, m-1$, taking ξ as a parameter. If $P(D)$ is $(s, \mathcal{P})$-hyperbolic, then $\exists C' > 0$ such that the roots of $P(\lambda)$ satisfy:

$$|\Im\lambda| \le C' |\xi|_{\mathcal{P}}^{\frac{1}{s}},$$

consequently we may take $B = C'|\xi|_{\mathcal{P}}^{\frac{1}{s}}$.
Now we determine A. Let's consider the characteristic polynomial of P:

$$P(\lambda,\xi) = \lambda^m + \sum_{j=0}^{m-1} a_j(\xi)\lambda^j$$

where $a_j(\xi)$ is a polynomial of degree at most equal to $m - j$. So there are constants C_j such that:

$$|a_j(\xi)| \le C_j (1 + |\xi|)^{m-j}.$$

It follows easily that for $\epsilon > 0$ sufficiently small the zeros of $P(\lambda,\xi)$ cannot belong to the region $\{(1 + |\xi|) < \epsilon|\lambda|\}$ and must necessarily satisfy:

$$|\lambda| \le \epsilon^{-1}(1 + |\xi|)\ . \tag{32}$$

So we can take:

$$A = \epsilon^{-1}(1 + |\xi|) \tag{33}$$

and estimate for a suitable $C > 0$:

$$|F_j(t,\xi)| \le (\epsilon^{-1}(1 + |\xi|) + 1))^{m+1} C \exp(C(|t| + 1)|\xi|_{\mathcal{P}}^{\frac{1}{s}}) \tag{34}$$

By summing up the estimates for $\hat{f}_j, F_j$ we get the following estimates for $\hat{u}$:

$$
\begin{aligned}
|\hat{u}(t,\xi)| &\le \sum_{j=0}^{m-1} |\hat{f}_j(\xi)||F_j(t,\xi)| \\
&\le C \sum_{j=0}^{m-1} \exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}}) \exp(C(1+|t|)|\xi|_{\mathcal{P}}^{\frac{1}{s}}) \ .
\end{aligned} \tag{35}
$$

By assumption, $r < s$, and so $\frac{1}{r} > \frac{1}{s}$ implies that:

$$
\lim_{|\xi|\to+\infty} \frac{|\xi|_{\mathcal{P}}^{\frac{1}{s}}}{|\xi|_{\mathcal{P}}^{\frac{1}{r}}} = 0
$$

Then there exist positive constants $C_1' = C_1'(|t|),\ C_2' = C_2'(|t|)$ such that:

$$
C(1+|t|)|\xi|_{\mathcal{P}}^{\frac{1}{s}} - \epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}} \le -C_1'|\xi|_{\mathcal{P}}^{\frac{1}{r}} + C_2' \ .
$$

Hence we get the following estimate for $\hat{u}$:

$$
|\hat{u}(t,\xi)| \le C'' \exp(-C_1'|\xi|_{\mathcal{P}}^{\frac{1}{r}}).
$$

So we have obtained that $u \in G^{r\mathcal{P}}$ for any $t \in \mathbb{R}$ in view of Theorem 4,2). We observe that the constants C_1', C'' may depend on t, but are locally bounded, for $|t| \le T,\ \forall T > 0$.

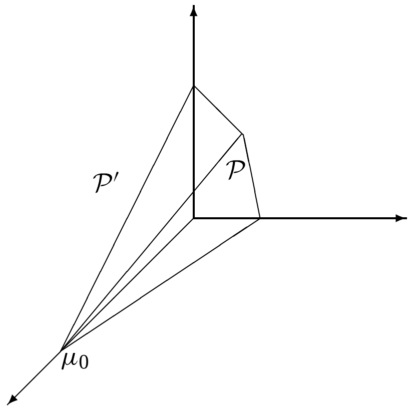
□

REMARK 6. We have supposed that $r > s$ to get the result of regularity. In the case $r = s$, the regularity is only local in time, as evident from the previous computations.

4. Regularity with respect to the time variable

We know that the solution of the Cauchy Problem is in $C^\infty([-T,T], G^{r\mathcal{P}}(\mathbb{R}^n))$, $\forall T > 0$; now we will discuss its regularity with respect to the time variable in generalized Gevrey classes. To do so, it is necessary to extend the polyhedron to $(n+1)$ variables, that is possible by means of the following proposition.

PROPOSITION 7. *Given a complete polyhedron $\mathcal{P}$ in $\mathbb{R}^n$, we define $\mathcal{P}'$ as the convex hull in $\mathbb{R}^{n+1}$ of the vertices of $\mathcal{P}$ plus the vector $(\mu_0, 0, \ldots, 0)$ with $\mu_0 \in \mathbb{Q}_+,\ 0 < \mu_0 \le \mu$, cf. figure. Then $\mathcal{P}'$ is a complete polyhedron in $\mathbb{R}^{n+1}$ with the same formal order μ of $\mathcal{P}$.*



The proof is trivial and follows immediately from the definition of complete polyhedra and of formal order. Let's observe that it is not possible to construct $\mathcal{P}'$ with a smaller formal order. Of course, one could take more than one additional vertex to build $\mathcal{P}'$; the construction in Proposition 7 represents the cheapest procedure, which could be easily iterated to extend $\mathcal{P}$ to $(n+m)$ dimensions, $\forall m$.

DEFINITION 7. *We call $\mathcal{P}'$ in Proposition 7 an extension of $\mathcal{P}$ in $\mathbb{R}^{n+1}$.*
If the further vertex has coordinates $(\mu, 0, \ldots, 0)$ with μ denoting the formal order of $\mathcal{P}$, we say that $\mathcal{P}'$ is the maximal extension of $\mathcal{P}$ in $\mathbb{R}^{n+1}$.

PROPOSITION 8. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ and let $\mathcal{P}'$ be an extension of $\mathcal{P}$ to $\mathbb{R}^{n+1}$ by the additional vertex $(\mu_0, 0, \ldots, 0)$. Then for any $\alpha' = (\alpha_0, \alpha) \in \mathbb{R}^{n+1}_+$:*

$$k(\alpha', \mathcal{P}') = k(\alpha, \mathcal{P}) + \frac{\alpha_0}{\mu_0} = k(\alpha, \mathcal{P}) + k(\alpha_0, \mathcal{R}) \tag{36}$$

where $\mathcal{R}$ denotes the one-dimensional polyhedron $[0, \mu_0]$ in $\mathbb{R}$.

Proof. Writing $\alpha_0 = \frac{\alpha_0}{\mu_0}\mu_0$, we now compute $k(\alpha', \mathcal{P}') = k((\alpha_0, \alpha), \mathcal{P}')$.
Let us write:

$$\alpha = k(\alpha, \mathcal{P}) \sum_{i=1}^{n(\mathcal{P})} t_i s_i$$

$$s_i \in \mathcal{V}(\mathcal{P}), \quad \sum_{i=1}^{n(\mathcal{P})} t_i = 1 \quad 0 \le t_i \le 1, \quad i = 1, \ldots, n(\mathcal{P});$$

$$(\alpha_0, 0, \ldots, 0) = \frac{\alpha_0}{\mu_0}(\mu_0, 0, \ldots, 0) = t_0 k(\alpha_0, \mathcal{R}) s_0,$$

$$t_0 = 1, \qquad k(\alpha_0, \mathcal{R}) = \frac{\alpha_0}{\mu_0} \qquad s_0 = (\mu_0, 0, \ldots, 0).$$

We want to find $k(\alpha', \mathcal{P}')$ such that:

$$(\alpha_0, \alpha) = k(\alpha', \mathcal{P}') \left(\sum_{i=1}^{n(\mathcal{P})} t'_i s_i + t'_0 s_0 \right)$$

$$t'_0 + \sum_{i=1}^{n(\mathcal{P})} t'_i = 1 \quad 0 \leq t'_i \leq 1 \, .$$

On the other hand we have:

$$\begin{aligned}(\alpha_0, \alpha) &= \sum_{i=1}^{n(\mathcal{P})} k(\alpha, \mathcal{P}) t_i s_i + \frac{\alpha_0}{\mu_0} t_0 s_0 \\ &= \left(k(\alpha, \mathcal{P}) + \frac{\alpha_0}{\mu_0} \right) \left\{ \sum_{i=1}^{n(\mathcal{P})} t'_i s_i + t'_0 s_0 \right\}\end{aligned}$$

with:

$$t'_i = \frac{k(\alpha, \mathcal{P}) t_i}{\left(k(\alpha, \mathcal{P}) + \frac{\alpha_0}{\mu_0} \right)}, \quad t'_0 = \frac{\frac{\alpha_0}{\mu_0}}{\left(k(\alpha, \mathcal{P}) + \frac{\alpha_0}{\mu_0} \right)} t_0$$

$$t'_0 + \sum_{i=1}^{n(\mathcal{P})} t'_i = 1 \, .$$

So $k(\alpha', \mathcal{P}') = k(\alpha, \mathcal{P}) + \frac{\alpha_0}{\mu_0}$ is univocally determined as s_0 is orthogonal to $\mathcal{P}$.

□

We will prove first a theorem of regularity of the Cauchy Problem with respect to the time variable in the particular case when the coefficients $a_j(\xi)$ satisfy the condition:

$$|a_j(\xi)| \leq C |\xi|_{\mathcal{P}}^{m-j} \quad j = 0, 1, \ldots, m-1 \tag{37}$$

and then a theorem for general $a_j(\xi)$ which requires a further discussion on the relation between the euclidean norm in $\mathbb{R}^{n+1}$ and the weight associated to the polyhedron.

THEOREM 5. *Under the assumptions of Theorem 2, if* (37) *is satisfied, then the solution u of the Cauchy Problem* (2) *is of class $G^{r\mathcal{P}'}(\mathbb{R}^{n+1})$ where $\mathcal{P}'$ denotes the maximal extension of $\mathcal{P}$ to $\mathbb{R}^{n+1}$.*

Proof. We have to test the regularity of u with respect to the time variable. Let us go back to the proof of Theorem 2. From (27) we have:

$$|D_t^N \hat{u}(t, \xi)| \leq \sum_{j=0}^{m-1} |\hat{f}_j(\xi)| |D_t^N F_j(t, \xi)| \, .$$

By Lemma 5 , we can estimate:

$$|D_t^N F_j(t,\xi)| \le 2^m (A+1)^{N+m+1} \exp((B+1)|t|). \tag{38}$$

By the hypothesis of multi-quasi-hyperbolicity for $P(D)$, as before we may take:

$$B = C_1 |\xi|_{\mathcal{P}}^{\frac{1}{s}} .$$

To determine A we use the hypothesis (37) that implies:

$$|P(\lambda,\xi) - \lambda^m| = |\sum_{j=0}^{m-1} a_j(\xi)\lambda^j| \le C \sum_{j=0}^{m-1} |\xi|_{\mathcal{P}}^{m-j} |\lambda|^j < \frac{|\lambda|^m}{2}$$

in the region $\{(\lambda,\xi) \in \mathbb{C} \times \mathbb{R}^n : |\xi|_{\mathcal{P}} < \epsilon|\lambda|\}$ for a sufficiently small $\epsilon > 0$. Consequently, $|P(\lambda,\xi)| > \frac{|\lambda|^m}{2}$ and the zeros of $P(\lambda,\xi)$ can't be in this region, so they must satisfy:

$$|\lambda| \le C|\xi|_{\mathcal{P}} , \quad \text{for } C > 0.$$

So we can take $A = C|\xi|_{\mathcal{P}}$ and estimate:

$$\begin{aligned} |D_t^N F_j(t,\xi)| &\le 2^m (C|\xi|_{\mathcal{P}})^{N+m+1} \exp(C_0(|t|+1)|\xi|_{\mathcal{P}}^{\frac{1}{s}}) \\ &\le C(C'|\xi|_{\mathcal{P}})^N \exp(C_1(|t|+1)|\xi|_{\mathcal{P}}^{\frac{1}{s}}) . \end{aligned}$$

Hence:

$$\begin{aligned} |D_t^N \hat{u}(t,\xi)| &\le \sum_{j=0}^{m-1} |\hat{f}_j(\xi)| |D_t^N F_j(t,\xi)| \\ &\le C(C'|\xi|_{\mathcal{P}})^N \exp\{(C_1(|t|+1)|\xi|_{\mathcal{P}}^{\frac{1}{s}}) - \epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}}\}. \end{aligned}$$

Arguing as in the proof of Theorem 2, we obtain for a suitable $\epsilon_1 > 0$:

$$|D_t^N \hat{u}(t,\xi)| \le C(C'|\xi|_{\mathcal{P}})^N \exp(-\epsilon_1 |\xi|_{\mathcal{P}}^{\frac{1}{r}}). \tag{39}$$

Now we pass to consider the Fourier antitransform of $\hat{u}$ with respect to the space variables:

$$u(t,x) = \mathcal{F}^{-1}_{\xi \mapsto x} \hat{u}(t,\xi)$$

and estimate for $\alpha' = (\alpha_0, \alpha)$:

$$\begin{aligned}
|D^{\alpha'}u(t,x)| &= |D_t^{\alpha_0}D_x^{\alpha}u(t,x)| = |D_t^{\alpha_0}\mathcal{F}^{-1}(\xi^{\alpha}\hat{u}(t,\xi))| \\
&= |D_t^{\alpha_0}[(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\xi^{\alpha}\hat{u}(t,\xi)d\xi]| \\
&\leq (2\pi)^{-n}\int_{\mathbb{R}^n} |\xi^{\alpha}||D_t^{\alpha_0}\hat{u}(t,\xi)|d\xi \\
&\leq C(2\pi)^{-n}\int_{\mathbb{R}^n} |\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})}(C'|\xi|_{\mathcal{P}})^{\alpha_0}\exp(-\epsilon_1|\xi|_{\mathcal{P}}^{\frac{1}{r}})d\xi \\
&\leq C(2\pi)^{-n}\int_{\mathbb{R}^n} (C|\xi|_{\mathcal{P}})^{\mu(k(\alpha,\mathcal{P})+\alpha_0)}\exp(-\epsilon_1|\xi|_{\mathcal{P}}^{\frac{1}{r}})d\xi \\
&\leq C^{|\alpha|+\alpha_0+1}(\mu k(\alpha,\mathcal{P})+\alpha_0)^{r(\mu k(\alpha,\mathcal{P})+\alpha_0)}
\end{aligned}$$

where we have used (11) and we have followed the arguments of the proof of Theorem 4 (2).
Letting $\mathcal{P}'$ be the maximal extension of $\mathcal{P}$ in $\mathbb{R}^{n+1}$, by Proposition 8:

$$k(\alpha',\mathcal{P}') = k(\alpha,\mathcal{P}) + \frac{\alpha_0}{\mu} .$$

So we can conclude that for a suitable constant $C > 0$:

$$|D^{\alpha'}u(t,x)| \leq C^{|\alpha'|+1}(\mu k(\alpha',\mathcal{P}'))^{r\mu k(\alpha',\mathcal{P}')}, \qquad \forall \alpha' \in \mathbb{Z}_+^{n+1} \tag{40}$$

that means $u \in G^{r\mathcal{P}'}(\mathbb{R}^{n+1})$ as we wanted to prove.

□

REMARK 7. If $r = s$ the regularity is only local in time and u is of class $G^{s\mathcal{P}'}$ only in the set $|t| < \frac{\epsilon}{C_1}$ as to satisfy condition (39).

THEOREM 6. *Under the assumptions of Theorem 2 the solution u of the Cauchy Problem is of class $G^{r\mathcal{P}'}(\mathbb{R}^{n+1})$, where $\mathcal{P}'$ is the extension of $\mathcal{P}$ to $\mathbb{R}^{n+1}$ obtained adding the vertex:*

$s_0 = (\mu_0, 0, \ldots, 0),\ \mu_0 = \mu^{(0)}$,
$\mu^{(0)} = \mu^{(0)}(\mathcal{P}) = \min\{m_j : m_j e_j \in \mathcal{V}(\mathcal{P}),\ j = 1, \ldots, n\} = \min_{\gamma \in \mathcal{V}(\mathcal{P})\setminus\{0\}} |\gamma|$.

Since $\mu_0 < \mu$ but in the elliptic case, the present result of regularity is weaker than the one expressed by Theorem 5 under the additional assumption (37).

Proof. We proceed as in the proof of Theorem 5 to estimate:

$$|D_t^N\hat{u}(t,\xi)| \leq \sum_{j=0}^{m-1} |\hat{f}_j(\xi)D_t^N F_j(t,\xi)| . \tag{41}$$

From Lemma 5 we have:

$$|D_t^N F_j(t,\xi)| \le C(A+1)^{N+m+1} \exp(1+B)|t| \tag{42}$$

where:

$$B = C|\xi|_{\mathcal{P}}^{\frac{1}{s}}$$

by the hypothesis of multi-quasi-hyperbolicity, and now:

$$A = C'(1+|\xi|)\ .$$

So arguing as in the previous proof we can estimate:

$$|D_t^{\alpha_0}\hat{u}(t,\xi)| \le C_2^{\alpha_0+1}(1+|\xi|)^{\alpha_0}\exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}})$$

and passing to the Fourier antitransform with respect to ξ:

$$\begin{aligned} |D^{\alpha'}u(t,x)| &= |D_t^{\alpha_0}D_x^{\alpha}u(t,x)| \\ &\le (2\pi)^{-n}C^{\alpha_0+1}\int_{\mathbb{R}^n}|\xi^{\alpha}|(1+|\xi|)^{\alpha_0}\exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}})d\xi. \end{aligned}$$

Using the inequalities:

$$|\xi^{\alpha}| \le |\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})}$$

$$(1+|\xi|) \le |\xi|_{\mathcal{P}}^{\frac{\mu}{\mu_0}}$$

we obtain:

$$|D^{\alpha'}u(t,x)| \le (2\pi)^{-n}C^{\alpha_0+1}\int_{\mathbb{R}^n}|\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})}|\xi|_{\mathcal{P}}^{\frac{\mu}{\mu_0}\alpha_0}\exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}})d\xi\ .$$

Now we consider the vector $\alpha' = (\alpha_0,\alpha) \in \mathbb{R}^{n+1}$ and we define the extension $\mathcal{P}'$ of $\mathcal{P}$ to $\mathbb{R}^{n+1}$ as the convex hull of $\mathcal{P}\bigcup\{(\mu_0,0,\dots,0)\}$.
By Proposition 8:

$$k(\alpha',\mathcal{P}') = k(\alpha,\mathcal{P}) + k(\alpha_0,\mathcal{R}) = k(\alpha,\mathcal{P}) + \frac{\alpha_0}{\mu_0}$$

and therefore we can get the estimate:

$$\begin{aligned} |D^{\alpha'}u(t,x)| &\le (2\pi)^{-n}C^{\alpha_0+1}\int_{\mathbb{R}^n}|\xi|_{\mathcal{P}}^{\mu k(\alpha',\mathcal{P}')}\exp(-\epsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}})d\xi \\ &\le C_1C_2^{|\alpha'|}(\mu k(\alpha',\mathcal{P}'))^{r(\mu k(\alpha',\mathcal{P}'))}. \end{aligned}$$

We have obtained that $u \in G^{r\mathcal{P}'}(\mathbb{R}^{n+1})$ as we wanted to prove.

□

REMARK 8. Analogously to Theorem 5 if $r = s$ the regularity is only local in time and u is of class $G^{s\mathcal{P}'}(\mathbb{R}^n)$ only in the set $|t| < \epsilon$, with $\epsilon > 0$ depending on the initial data.

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