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L^2 EXPANSIONS IN SERIES OF FRACTIONAL PARTS

Abstract. In this paper we consider the problem of L^2 expansions in series of fractional parts for functions $f \in L^2([0,1],\mathbb{R})$ which are odd with respect to the point $\frac{1}{2}$. Sufficient conditions are given, but we also prove by the Banach-Steinhaus theorem that this is not always possible.

1. Introduction

Let :

$$P(x) = -\frac{1}{\pi}\sum_{m\geq 1}\frac{\sin(2\pi m x)}{m} = \begin{cases} x-[x]-\frac{1}{2} & \text{if } x \notin Z \\ 0 & \text{if } x \in Z \end{cases} \tag{1}$$

In a paper of 1936 [5] H. Davenport considered the formal identity

$$\sum_{n\geq 1}\alpha(n)P(nx) = -\frac{1}{\pi}\sum_{n\geq 1}\frac{\sum_{d/n} d\alpha(d)}{n}\sin(2\pi n x) \tag{2}$$

which is obtained by substituting into (2) the expansion of P given in (1) and collecting together the terms for which the product mn has the same value. Davenport considered the cases in which $\alpha(n)$ has the values $\frac{\mu(n)}{n}$, $\frac{\lambda(n)}{n}$ and $\frac{\Lambda(n)}{n}$, where, as usual, μ,λ and Λ denote, respectively, the Möbius, Liouville and Von Mangoldt funtions.
In [5] Davenport proved the following results:

i) The functions $\sum_{n=1}^{N}\frac{\mu(n)}{n}P(nx)$ and $\sum_{n=1}^{N}\frac{\lambda(n)}{n}P(nx)$ are uniformly bounded in N and x.

ii) The identities

$$\sum_{n\geq 1}\frac{\mu(n)}{n}P(nx) = -\frac{1}{\pi}\sin(2\pi x)$$

$$\sum_{n\geq 1}\frac{\lambda(n)}{n}P(nx) = -\frac{1}{\pi}\sum_{n\geq 1}\frac{\sin(2\pi n^2 x)}{n^2}$$

hold almost everywhere.

In a subsequent paper [6], Davenport, using Vinogradov's method and the Siegel-Walfisz theorem, showed that the identities ii) hold everywhere and that the two series converge uniformly in $\mathbb{R}$. The proof of the weaker properties ii) (the almost everywhere convergence) depends on L^2 convergence, namely on the estimate [5]:

$$\| R_N \|^2 = \int_0^1 |\sum_{n=1}^{N} \frac{\mu(n)}{n} P(nx) + \frac{1}{\pi} \sin(2\pi x)|^2 dx = O(\frac{\log(N)}{N})$$

The aim of the present paper is just to consider the formal identity (2) for general $\alpha(n)$ and from the point of view of L^2-convergence. In order to state in a more precise way the problems studied and the results obtained we give the following definitions.
Le us define:

$$L_0^2 = \{ f \in L^2([0,1], \mathbb{R}),\ f(x) = -f(1-x) \ \ a.e.\ on\ [0,1]\}$$

that is L_0^2 is the subspace of the functions $f \in L^2([0,1], \mathbb{R})$ which are odd with respect to the point $\frac{1}{2}$. It is easy to see that L_0^2 is a closed subspace of $L^2([0,1], \mathbb{R})$ and so it is a Hilbert space.
Moreover, if n is a nonnegative integer, we denote with $M^+(n)$ the greatest prime which divides n and, as usual, we set $\Psi(x, y) = \sum_{\{n \le x, M^+(n) \le y\}} 1$.
We shall consider the following problems:

α) Is it possible to apply the Gram-Schimdt procedure to the functions $P(nx)$ and, if $(\varphi_n(x))_n$ is the resulting orthonormal system, is it true that

$$f(x) = \sum_{n \ge 1} b(n) \varphi_n(x)$$

in the L^2 sense for every $f \in L_0^2$?

β) Is it possible to give sufficient conditions for expanding (in the L^2 sense) in series of fractional parts the functions belonging to certain subsets of L_0^2 ?

γ) Is it possible to expand (in the L^2 sense) every $f \in L_0^2$ in series of fractional parts ?

Theorem 1 below answers positively to question α), while in Theorem 2 we show that it is possible to give a positive answer to question β).
On the contrary, the answer to question γ) is negative. This is proved in Theorem 3, whose proof depends on the Banach-Steinhaus theorem and on an asymptotic formula with an estimate of the error term (see Lemma 1) for the difference $\Psi(x, y) - \Psi(\frac{x}{2}, y)$ with y fixed.

REMARK 1. The identity

$$\sum_{n\geq 1} \frac{\mu(n)}{n} P(nx) = -\frac{1}{\pi} \sin(2\pi x) \quad x \in \mathbb{R} \tag{3}$$

has interesting applications. For example, it was used in [4] to prove a result of linear independence for the fractional parts, namely the fact that the determinant:

$$\det(P(\frac{md}{q})) \neq 0 \qquad 0 \leq d \leq m \leq \frac{q-1}{2} \tag{4}$$

Actually, for the proof of (4), it is enough to know that (3) holds for rational x and this, in turn, is equivalent to the result $L(1,\chi) \neq 0$ (here χ is a caracter $mod\ q$ and $L(s,\chi)$ is the corresponding L function). Moreover it can be shown that also the weaker result that (3) holds in the L^2 sense can be used to prove results like the following one.
Let $h : [0,1] \to \mathbb{R}$ be continuous and of bounded variation. Let us also suppose that $h(x) = h(1-x),\ \forall x \in [0,1]$ and that $\int_0^1 f(x)dx = 0$. Then the following implication holds:

$$\sum_{a=1}^{d} h(\frac{a}{d}) = 0 \ \ \forall\, d \geq 1 \Rightarrow h(x) = 0, \quad \forall\, x \in [0,1]$$

It is well known (see [3] and [2]) that this property is false if we drop the bounded variation condition.

2. Notation and results

If a and b are positive integers (a,b) will denote the greatest common divisor of a and b, and $d|n$ means that d divides n. With $\tau(n)$ we will indicate the number of divisors of n, as usual.
We have obtained the following results.

THEOREM 1. *The following properties hold:*

i) The functions $f_n(x) = P(nx)$, $n = 1, 2, \ldots$ *,are linearly independent.*

ii) The set of functions $f_n(x) = P(nx)$, $n = 1, 2, \ldots$ *is complete in* L_0^2.

iii) Let $(\varphi_n(x))$ *be the orthonormal system obtained from* $f_n(x) = P(nx)$ *by the Gram-Schmidt procedure. Then we have*

$$\| \sum_{n=1}^{N} b(n)\varphi_n(x) - f(x) \|_{L^2} \to 0 \qquad when \ \ N \ \to \ +\infty \tag{5}$$

for every $f \in L_0^2$*, where the* $(b(n))$ *are the Fourier coefficients of* $f(x)$ *with respect to the system* $(\varphi_n(x))$.

Theorem 1 solves the problem of expanding every function $f \in L_0^2$ in series of linear combination of fractional parts, since we have

$$\varphi_n(x) = \sum_{h=1}^{n} \gamma_{h,n} P(hx)$$

where the $\gamma_{h,n}$ are suitable coefficients ([8], p.305).
As for the problem of the expansion of functions $f \in L_0^2$ in series of fractional parts we have the following theorems.

THEOREM 2. *Let* $f(x) \sim \sum_{k\geq 1} a(k)\sin(2\pi kx)$ *be the Fourier series of* $f \in L_0^2$. *Set*

$$R_N(x) = \sum_{n=1}^{N} \alpha(n) P(nx) - f(x)$$

with $\alpha(n) \in \mathbb{R}$. *Then we have*

$$\| R_N \|^2 = \frac{1}{2} \sum_{m\geq 1} |\frac{1}{\pi} \sum_{\substack{n|m \\ n\leq N}} \alpha(n) \frac{n}{m} + a(m)|^2 \tag{6}$$

If $\| R_N \| \to 0$ *when* $N \to \infty$ *we must have*

$$\alpha(n) = -\pi \sum_{d|n} \frac{\mu(d)}{d} a(\frac{n}{d}). \tag{7}$$

Finally, let

$$a^*(n) = \sum_{\substack{h\geq 1 \\ h\equiv 0(n)}} |a(h)|^2.$$

If the condition

$$\sum_{n\geq 1} \tau(n)\sqrt{a^*(n)}\, \frac{1}{n} < +\infty \tag{8}$$

holds then

$$\| R_N \| = \| f - \sum_{n=1}^{N} \alpha(n) P(nx) \| \to 0 \tag{9}$$

when $N \to +\infty$, *with* $\alpha(n)$ *as in* (7).

THEOREM 3. *There exist functions* $f \in L_0^2$ *such that*

$$\sup_N \| R_N \| = +\infty \tag{10}$$

where $R_N(x) = \sum_{n=1}^{N} \alpha(n)P(nx) - f(x)$ *and the* $\alpha(n)$ *are given by* (7).

REMARK 2. Condition (8) of Theorem 2 is not a very restrictive one. It is certainly satisfied, for instance, if there exists $(b(n)) \subset \ell^2$ such that $b(n+1) \leq b(n)$, $|a(n)| \leq b(n)$, $\forall n \geq 1$, since in this case we have

$$a^*(n) = \sum_{h\geq 1} |a(hn)|^2 \leq \sum_{h\geq 1} \left(\frac{1}{n} \sum_{r=1}^{n} b^2((h-1)n+r) \right) = \frac{\| b \|^2}{n}$$

This means that (8) holds in particular for all bounded variation functions, since in this case we have $a(n) = O(\frac{1}{n})$ ([1], vol.I, p. 72).
Another example is given by the class Λ_β of functions which satisfy a Lipschitz condition of order $\beta > 0$: in this case we have ([1], vol.I, p.215)

$$(\sum_{k\geq n} |a_k|^2)^{\frac{1}{2}} = O(\frac{1}{n^\beta})$$

which gives $a^*(n) = O(\frac{1}{n^{2\beta}})$ and (8) holds again.

It should be noted that under condition (8) we can assert that the formal identity (2) holds almost everywhere for a subsequence. This is an obvious corollary of (9), (7) and Möbius inversion formula.

REMARK 3. The proof of Theorem 3 is based on Banach-Steinhaus theorem and so we do not give an explicit example.

The contrast between (5) of Theorem 1 and (10) of Theorem 3 can be explained as follows.
If $f \in L^2_0$, the minimum value of the difference

$$\| f - \sum_{n=1}^{N} c_n P(nx) \|$$

can be obtained directly by solving the linear system ([10], p. 83)

$$\sum_{n=1}^{N} a_{m,n} c_n = b_m \quad , \quad m = 1, ..., N \tag{11}$$

where

$$a_{m,n} = \int_0^1 P(nx)P(mx)dx = \frac{1}{12} \frac{(n,m)^2}{nm}$$

and

$$b_m = \int_0^1 P(mx) f(x) dx$$

or equivalently by considering the sum

$$\sigma_N(x) = \sum_{n=1}^{N} b(n)\varphi_n(x) = \sum_{n=1}^{N} b(n) (\sum_{h=1}^{n} \gamma_{h,n} P(hx)) = \sum_{n=1}^{N} c_n P(nx)$$

which appears in (5) of Theorem 1. Generally the $\alpha(n)$ given by (7) of Theorem 2 are not the solution of system (11): for example it is easy to check directly that when $f(x) = -\frac{1}{\pi}\sin(2\pi x)$ we have

$$\alpha(n) = \frac{\mu(n)}{n} \quad , \qquad b_m = \begin{cases} \frac{1}{2\pi^2} & \text{if } m = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the numbers $\mu(n)/n$ do not satisfy condition (11), and consequently the difference $\| f - \sum_{n=1}^{N} \alpha(n)P(nx) \|$ is not the minimal one.
Usually, if the function is not a very irregular one (see condition (8) of Theorem 2) the sum $S_N(x) = \sum_{n=1}^{N} \alpha(n)P(nx)$ will be a good approximation to $\sigma_N(x)$ and it will happen that $\| S_N - \sigma_N \| \to 0$ when $N \to +\infty$. But there exist very irregular functions $f \in L_0^2$ for which we have

$$\sup_N \| R_N \| = \sup_N \| S_N - \sigma_N \| = +\infty$$

although we think that is not easy to give an explicit example.
Let us now prove the results stated.

Proof of Theorem 1. Proof of property *i)*.
We want to prove that the relation

$$\sum_{k=1}^{n} c_k P(kx) = 0 \quad a.e.\ on\ [0,1] \quad c_k \in \mathbb{R} \tag{12}$$

implies $c_k = 0\ \ \forall k = 1, ..., n$. If $n = 1$ the implication is obviously true. Suppose now that the equality (12) holds with $n \geq 2$.
If $0 \leq x < \frac{1}{n}$ we have $[kx] = 0$ for every $k \leq n$ and from (12) follows

$$\sum_{k=1}^{n} c_k(kx - \frac{1}{2}) = 0 \qquad a.e.\ \ on\ \ [0, \frac{1}{n})$$

from which we obtain

$$\sum_{k=1}^{n} c_k = 0 \tag{13}$$

If $\frac{1}{n} \leq x < \frac{1}{n-1}$ we have

$$[kx] = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \leq n-1 \end{cases}$$

and so from (12) follows

$$\sum_{k=1}^{n-1} c_k(kx - \frac{1}{2}) + c_n(nx - \frac{3}{2}) = 0 \qquad a.e.\ \ on\ \ [\frac{1}{n}, \frac{1}{n-1})$$

which in turn implies

$$\sum_{k=1}^{n-1} c_k + 3c_n = 0 \tag{14}$$

Relations (13) and (14) give $c_n = 0$ and the desidered implication follows by induction. This gives property *i)*.

Proof of property *ii)*.

Take $f \in L_0^2$ and set $\delta(n) = \int_0^1 f(x)P(nx)dx$. Since

$$P(x) = -\frac{1}{\pi}\sum_{k\geq 1}\frac{\sin(2\pi kx)}{k}$$

we have

$$\delta(n) = -\frac{1}{\pi}\sum_{k\geq 1}\frac{1}{k}\int_0^1 \sin(2\pi knx)\, f(x)dx = -\frac{1}{2\pi}\sum_{k\geq 1}\frac{c(kn)}{k} \tag{15}$$

where the $c(n)$ are the Fourier coefficients of f. The relation (15) can be inverted and we have

$$c(n) = -2\pi\sum_{k\geq 1}\mu(k)\frac{\delta(nk)}{k} \tag{16}$$

provided the sufficient condition

$$\sum_{k\geq 1}\frac{\tau(k)|c(nk)|}{k} < +\infty \quad \forall n \geq 1 \tag{17}$$

is satisfied ([7], Theorem 270). But (17) is certainly true by Schwarz inequality, and so (16) holds. If we suppose $\delta(n) = 0 \;\; \forall n \geq 1$ from (16) follows $c(n) = 0 \;\; \forall n \geq 1$, but this means that $f \in L_0^2$ has all the Fourier coefficients equal to zero. The completeness of the trigonometrical system implies $f(x) = 0 \;\; a.e.$ and property *ii)* follows.

Proof of property *iii)*.

If $(\varphi_n(x))$ is the orthonormal system obtained from $P(nx)$ by the Gram-Schmidt procedure we have ([8], p. 305)

$$\varphi_n(x) = \sum_{h=1}^{n}\gamma_{h,n}P(hx)$$

and viceversa

$$P(nx) = \sum_{k=1}^{n}\gamma'_{h,n}\varphi_k(x)$$

where the $\gamma_{h,n}$ and $\gamma'_{h,n}$ are suitable coefficients. This implies, by property *ii)*, that the orthonormal system $(\varphi_n(x))$ is complete in L_0^2 and so Parseval identity holds. This proves (5) and concludes the proof of Theorem 1.
Let us now prove Theorem 2.

□

Proof of Theorem 2. Formula (6) is simply Parseval identity for $R_N(x) = \sum_{n=1}^{N} \alpha(n)P(nx) - f(x)$ in fact we have

$$(18)\qquad \begin{aligned} c_N(m) &= \sqrt{2}\int_0^1 R_N(x)\sin(2\pi mx)dx = \\ &= -\frac{1}{\sqrt{2}}\left(\frac{1}{\pi}(\sum_{\substack{n|m\\ n\le N}}\alpha(n)\frac{n}{m}) + a(m)\right) \end{aligned}$$

To justify (18) we note that

$$\sqrt{2}\int_0^1 P(nx)\sin(2\pi mx)dx = -\frac{\sqrt{2}}{\pi}\sum_{k\ge 1}\frac{1}{k}\int_0^1 \sin(2\pi knx)\sin(2\pi mx) =$$

$$= \begin{cases} \frac{-n}{\pi\sqrt{2}m} & \text{if } n|m \\ 0 & \text{otherwise} \end{cases}$$

Since we have $\| R_N \|^2 = \sum_{m\ge 1}|c_N(m)|^2$ from (18) we obtain (6). Let us prove (7).

If $\| R_N \| \to 0$ from (6) it follows immediatly that

$$\frac{1}{\pi}(\sum_{n|m}\alpha(n)\frac{n}{m}) = -a(m) \quad \forall m \ge 1$$

from which we obtain (7), since the Dirichlet inverse of $\frac{1}{n}$ is $\frac{\mu(n)}{n}$.
Let us now suppose that condition (7) holds: in this case (6) becomes

$$(19)\qquad \| R_N \|^2 = \frac{1}{2}\sum_{m=N+1}^{\infty}|\frac{1}{\pi}\sum_{\substack{n|m\\ n\le N}}\alpha(n)\frac{n}{m} + a(m)|^2$$

when $N \to +\infty$. From (19) it follows

$$(20)\qquad \| R_N \| \to 0 \Longleftrightarrow \sum_{m=N+1}^{\infty}|\sum_{\substack{n|m\\ n\le N}}\alpha(n)\frac{n}{m}|^2 \to 0$$

since $(a(m)) \in \ell^2$. But we also have

$$(21)\qquad -\frac{1}{\pi}\sum_{\substack{n|m\\ n\le N}}\alpha(n)\frac{n}{m} = \sum_{\substack{d|m\\ d\le N}}a(d)\frac{d}{m}(\sum_{\substack{k|(m/d)\\ k\le (N/d)}}\mu(k))$$

From (20) and (21) follows

$$\text{(22)} \qquad \| R_N \| \to 0 \Longleftrightarrow \sum_{m=N+1}^{\infty} |a_N^{'}(m)|^2 \to 0$$

when $N \to \infty$, where

$$a_N^{'}(m) = \frac{1}{m} \sum_{\substack{d|m \\ d \le N}} da(d) \Big(\sum_{\substack{k|(m/d) \\ k \le (N/d)}} \mu(k) \Big).$$

Let us now prove that (8) implies (9). First we note that

$$\text{(23)} \qquad |a_N^{'}(n)| \le \sum_{h|n} |a(h)| \tau(\frac{n}{h}) \frac{h}{n} \equiv \gamma(n)$$

uniformly in N. From the definition of $\gamma(n)$ in (23) follows easily

$$\begin{aligned}
\sum_{n\ge1} \gamma^2(n) &= \sum_{n\ge1} \sum_{\substack{h|n \\ k|n}} |a(h)||a(k)| \tau(\frac{n}{h}) \tau(\frac{n}{k}) \frac{hk}{n^2} \le \\
\text{(24)} \qquad &\le \Big(\sum_{m\ge1} \frac{\tau^2(m)}{m^2}\Big) \left(\sum_{h,k\ge1} |a(h)||a(k)| \tau(\frac{k}{(h,k)}) \tau(\frac{h}{(h,k)}) \frac{(h,k)^2}{hk} \right) = \\
&= \Big(\sum_{m\ge1} \frac{\tau^2(m)}{m^2}\Big) \sum
\end{aligned}$$

say, if we remember that $\tau(nm) \le \tau(n)\tau(m)$. If $(h,k) = \delta$ so that $h = r\delta$, $k = s\delta$, $(r,s) = 1$ we also have

$$\begin{aligned}
\sum &\le \sum_{r,s\ge1} \frac{\tau(r)\tau(s)}{rs} \sum_{\delta\ge1} |a(r\delta)||a(s\delta)| \le \\
\text{(25)} \qquad &\le \sum_{r,s\ge1} \frac{\tau(r)\tau(s)}{rs} \Big(\sum_{\delta\ge1} |a(r\delta)|^2\Big)^{\frac{1}{2}} \Big(\sum_{\delta\ge1} |a(s\delta)|\Big)^{\frac{1}{2}} = \\
&= \left(\sum_{r\ge1} \frac{\tau(r)\sqrt{a*(r)}}{r} \right)^2 < +\infty
\end{aligned}$$

if condition (9) holds. From (24) and (25) follows that the series $\sum_{n\ge1} \gamma^2(n)$ is convergent, but this implies that condition (22) is satisfied, since from (23) it follows obviously that

$$\sum_{n=N+1}^{\infty} |a_N^{'}(n)|^2 \le \sum_{n=N+1}^{\infty} \gamma^2(n) \to 0$$

when $N \to +\infty$. This proves (10) and concludes the proof of Theorem 2.

$\square$

Proof of Theorem 3. We need the following two lemmas.

LEMMA 1. *Let $p_1 < p_2 < ... < p_n$ be the prime numbers up to p_n, with $p_1 = 2$. If $M^+(k)$ denotes the greatest prime divisor of the integer k, with the convention $M^+(1) = 1$, set*

$$\Psi(x, p_n) = \sum_{\substack{k \le x \\ M^+(k) \le p_n}} 1$$

Then we have

$$\Psi(x, p_n) - \Psi(\frac{x}{2}, p_n) = c(n) \ln^{n-1}(x) + O(\ln^{n-2}(x)) \quad \text{when } x \to +\infty \tag{26}$$

where

$$c(n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{(n-1)!} \prod_{k=2}^{n} (\ln(p_k))^{-1} & \text{if } n \ge 2 \end{cases}$$

Proof. Formula (26) is obviously true if $n = 1$, because in this case we have

$$\Psi(x, p_1) - \Psi(\frac{x}{2}, p_1) = 1 \qquad \forall x \ge 2$$

Now we will prove that if (26) holds for the integer n then it holds also for $n + 1$ and the lemma will follow by induction. We have

$$\Psi(y, p_{n+1}) = \sum_{p_1^{k_1} \ldots p_n^{k_n} p_{n+1}^{k_{n+1}} \le y} 1 = \sum_{k_{n+1}=0}^{[\ln y / \ln p_{n+1}]} \Psi\left(\frac{y}{p_{n+1}^{k_{n+1}}}, p_n\right) \tag{27}$$

with $k_j \ge 0$ for $j = 1, ..., n + 1$. From (27) with $y = x$ and $y = \frac{x}{2}$ we obtain

$$\Psi(x, p_{n+1}) - \Psi(\frac{x}{2}, p_{n+1}) = \sum_{k_{n+1}=0}^{[\ln \frac{x}{2} / \ln p_{n+1}]} \left(\Psi(\frac{x}{p_{n+1}^{k_{n+1}}}, p_n) - \Psi(\frac{x}{2 p_{n+1}^{k_{n+1}}}, p_n) \right) +$$

$$+ \sum_{[\ln \frac{x}{2} / \ln p_{n+1}] < k_{n+1} \le [\ln x / \ln p_{n+1}]} \Psi(\frac{x}{p_{n+1}^{k_{n+1}}}, p_n) = \sum_1(x) + \sum_2(x), \tag{28}$$

where $\sum_2(x)$ is zero if the sum is empty. Consider now $\sum_1(x)$: by induction hypothesis we have

$$\sum_1(x) = c(n) \sum_{k_{n+1}=0}^{[\ln \frac{x}{2} / \ln p_{n+1}]} \ln^{n-1}\left(\frac{x}{p_{n+1}^{k_{n+1}}}\right) + O\left(\ln^{n-1} x\right) =$$

$$= c(n) \sum_{h=0}^{n-1} (-1)^h \binom{n-1}{h} \ln^{n-1-h} x \cdot \ln^h p_{n+1} \left(\sum_{k_{n+1}=0}^{[\ln \frac{x}{2} / \ln p_{n+1}]} k_{n+1}^h \right) + O\left(\ln^{n-1} x\right) \tag{29}$$

where, if $n = 1$ the above expression must obviously be interpreted as

$$\sum_1(x) = \frac{c(1)}{\ln p_2}\ln x + O(1)$$

We now recall that

$$\begin{aligned} &1^h + 2^h + ... + (m-1)^h = \\ (30) \qquad &= \frac{1}{h+1}\left(m^{h+1} + \binom{h+1}{1}B_1 m^h + \binom{h+1}{2}B_2 m^{h-1} + ...\right) \end{aligned}$$

where the B_j are the Bernoulli numbers ([9], p. 65). If we use the expression (30) in (29) we obtain

$$\sum_{k_{n+1}=1}^{[\ln(\frac{x}{2})/\ln p_{n+1}]} k_{n+1}^h = \frac{1}{h+1}\left(\frac{\ln x}{\ln p_{n+1}}\right)^{h+1} + O(\ln^h x) \quad when \;\; x \to +\infty$$

and if we substitute this in (29) we have

$$\begin{aligned} \sum_1(x) &= \frac{c(n)}{\ln p_{n+1}}\left(\sum_{h=0}^{n-1}(-1)^h\binom{n-1}{h}\frac{1}{h+1}\right)\ln^n x + O(\ln^{n-1} x) = \\ (31) \qquad &= \frac{c(n)}{n\ln p_{n+1}}\ln^n x + O(\ln^{n-1} x) \end{aligned}$$

in view of the identity

$$\sum_{h=0}^{n-1}(-1)^h\binom{n-1}{h}\frac{1}{h+1} = \frac{1}{n}\sum_{h=0}^{n-1}(-1)^h\binom{n}{h+1} = \frac{1}{n}$$

which holds since

$$\sum_{h=0}^{n-1}(-1)^h\binom{n}{h} = (1-1)^n = 0$$

Let us now consider the sum $\sum_2(x)$ which appears in (28).
Since $0 < \ln 2/\ln p_{n+1} < 1$, $\forall n \geq 1$ the sum, if it is not empty, contains at most one term, namely $k_{n+1} = [\ln x/\ln p_{n+1}]$: for this value of k_{n+1} we obviously have $(x/p_{n+1}^{k_{n+1}}) \leq p_{n+1}$ which implies that

$$(32) \qquad \sum_2(x) = \Psi\left(\frac{x}{p_{n+1}^{k_{n+1}}}, p_n\right) \leq \Psi(p_{n+1}, p_n) = O(1)$$

uniformly in x. From (28), (31) and (32) it follows that

$$\Psi(x, p_{n+1}) - \Psi(\frac{x}{2}, p_{n+1}) = \frac{c(n)}{n\ln p_{n+1}}\ln^n x + O(\ln^{n-1} x)$$

which proves the lemma.

□

LEMMA 2. *Let*

$$S(x) = \sum_{\substack{\frac{x}{2} < h,k < x \\ (h,k)=1}} \frac{1}{hk}.$$

Then we have

$$S(x) = \frac{6}{\pi^2}(\ln 2)^2 + o(1) \tag{33}$$

when $x \to +\infty$.

Proof. If we set

$$\sigma(x) = \sum_{\frac{x}{2} < h,k < x} \frac{1}{hk}$$

we can write

$$\sigma(x) = \sum_{n \le x} \frac{1}{n^2} S(\frac{x}{n})$$

and, by Möbius inversion formula, we have

$$S(x) = \sum_{n \le x} \frac{\mu(n)}{n^2} \sigma(\frac{x}{n})$$

Since it is

$$\sigma(x) = \left(\sum_{\frac{x}{2} < h < x} \frac{1}{h} \right)^2 = (\ln 2)^2 + O(\frac{1}{x})$$

we have

$$S(x) = \sum_{n \le \sqrt{x}} \frac{\mu(n)}{n^2} (\ln 2)^2 + o(1) = \left(\sum_{n \ge 1} \frac{\mu(n)}{n^2} \right) (\ln 2)^2 + o(1) \tag{34}$$

when $x \to +\infty$. From 34 follows (33) if we remember that

$$\sum_{n \ge 1} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$$

□

We return to the proof of Theorem 3. It is based on the Banach-Steinhaus theorem: for $N \ge 1$ we consider the bounded linear transformation

$$\Lambda_N \begin{cases} \ell^2 \to \ell^2 \\ a \to \Lambda_N(a) = a' \end{cases}$$

defined by

$$a'(n) = \begin{cases} 0 & \text{if } 1 \le n \le N \\ \frac{1}{n} \sum_{\substack{d|n \\ d \le N}} d a(d) \left(\sum_{\substack{k|(n/d) \\ k \le (N/d)}} \mu(k) \right) & \text{if } n > N \end{cases} \tag{35}$$

where we have set $a = (a(n)) \in \ell^2$ and $a' = (a'(n))$. The linearity of Λ_N is obvious: let us prove that is also bounded. We have

$$|a'(n)| \le \frac{N}{n} \sum_{\substack{d|n \\ d \le N}} |a(d)| \le \frac{N}{n} \left(\sum_{d \le N} |a(d)|^2 \right)^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \le N^{\frac{3}{2}} \frac{\| a \|}{n}$$

from which the conclusion is immediatly obtained by squaring and summing over n. We will now prove that the family of linear transformation Λ_N is not uniformly bounded. Let $p_1 < p_2 < ... < p_n < ...$ be the sequence of prime numbers and let $M^+(n)$ the greatest prime divisor of the integer n. We define

$$a_N(n) = \begin{cases} 1 & \text{if } \frac{N}{2} < n \le N \;\; and \;\; M^+(n) \le p_r \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

for p_r and N fixed.
From formula (35) follows that in this case $a'_N = \Lambda_N(a_N)$ is given by

$$a'_N(n) = \begin{cases} 0 & \text{if } 1 \le n \le N \\ \frac{1}{n} \sum_{\substack{d|n \\ \frac{N}{2} < d \le N}} d \, a_N(d) & \text{if } n > N \end{cases}$$

since the inner sum $\sum_{\substack{k|(n/d) \\ k \le (N/d)}} \mu(k)$ reduces to $\mu(1) = 1$.
Consider now

$$\| \Lambda_N(a_N) \|^2 = \sum_{n=N+1}^{\infty} |a'_N(n)|^2$$

we have

$$\| \Lambda_N(a_N) \|^2 = \sum_{\frac{N}{2} < d_1, d_2 \le N} a_N(d_1) a_N(d_2) \sum_{\{n \ge N+1, \; n \equiv 0 \; (d_1), \; n \equiv 0 \; (d_2)\}} \frac{1}{(n/d_1)} \frac{1}{(n/d_2)} =$$

$$= \sum_{\frac{N}{2} < d_1, d_2 \le N} \frac{a_N(d_1) a_N(d_2)}{d_1 d_2} (d_1, d_2)^2 \left(\sum_{m > \frac{N(d_1 d_2)}{d_1 d_2}} m^{-2} \right)$$

If $\frac{N}{2} < d_1 \le N$ and $\frac{N}{2} < d_2 \le N$ we have $\frac{N}{d_1 d_2}(d_1, d_2) < 4$ and it follows

$$\| \Lambda_N(a_N) \|^2 \ge \overline{c_1} \sum_{\frac{N}{2} < d_1, d_2 \le N} \frac{a_N(d_1) a_N(d_2)}{d_1 d_2} (d_1, d_2)^2 \tag{37}$$

where $\overline{c_1} = \sum_{m\geq 4} m^{-2}$. From formula (33) of Lemma 2 we have

$$S(\frac{N}{d}) = \sum_{\substack{\frac{N}{2d}<h,k\leq\frac{N}{d}\\(h,k)=1}} \frac{1}{hk} \geq \overline{c}_2 > 0 \tag{38}$$

where $\overline{c}_2$ is an absolute constant, if $\frac{N}{d}$ is sufficiently large, say $\frac{N}{d} > m$. Let

$$f_r(n) = \begin{cases} 1 & \text{if } M^+(n) \leq p_r \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\Psi(x, p_r) = \sum_{\substack{n\leq x\\M^+(n)\leq p_r}} 1 = \sum_{n\leq x} f_r(n)$$

Consider now formula (37): collecting together the terms with the same greatest common divisor we obtain, taking $p_r > m$,

$$\begin{aligned} \| \Lambda_N(a_N) \|^2 &\geq \overline{c}_1 \sum_{d=1}^{N} \left(\sum_{\substack{\frac{N}{2d}<h,k\leq\frac{N}{d}\\(h,k)=1}} \frac{a_N(hd)a_N(kd)}{hk} \right) \geq \\ &\geq \overline{c}_1 \sum_{\frac{N}{p_r}\leq d\leq\frac{N}{m}} f_r(d) S(\frac{N}{d}) \geq \overline{c}_1\overline{c}_2 \sum_{\frac{N}{p_r}\leq d\leq\frac{N}{m}} f_r(d) \end{aligned} \tag{39}$$

if we remember definition (36) and formula (38). From formula (26) of Lemma 1 follows

$$\begin{aligned} \sum_{\frac{N}{p_r}\leq d\leq\frac{N}{m}} f_r(d) &\geq \sum_{k=[\ln m/\ln 2]+1}^{[\ln p_r/\ln 2]-1} \left(\sum_{\frac{N}{2^{k+1}}<d\leq\frac{N}{2^k}} f_r(d) \right) = \\ &= \sum_{k=[\ln m/\ln 2]+1}^{[\ln p_r/\ln 2]-1} \left(\Psi(\frac{N}{2^k}, p_r) - \Psi(\frac{N}{2^{k+1}}, p_r) \right) = \\ &= \sum_{k=[\ln m/\ln 2]+1}^{[\ln p_r/\ln 2]-1} \left(c(r)\ln^{r-1}(\frac{N}{2^k}) + O(\ln^{r-2}(\frac{N}{2^k})) \right) = \quad (40) \\ &= c(r)\ln^{r-1} N \left([\frac{\ln p_r}{\ln 2}] - [\frac{\ln m}{\ln 2}] - 1 \right) + O(\ln^{r-2} N) \quad (41) \end{aligned}$$

for $N \to +\infty$, where $c(r)$ is the constant specified in (26) of Lemma 1. From (39) and (40) follows

$$\frac{\| \Lambda_N(a_N) \|^2}{\| (a_N) \|^2} \geq \overline{c}_1\overline{c}_2 \frac{c(r)\ln^{r-1} N \left([\frac{\ln p_r}{\ln 2}] - [\frac{\ln m}{\ln 2}] - 1 \right) + O(\ln^{r-2} N)}{c(r)\ln^{r-1} N + O(\ln^{r-2} N)} \tag{42}$$

since, by the definition (36) of $a_N(n)$ and by lemma 1 we have

$$\| (a_N) \|^2 = c(r) \ln^{r-1} N + O(\ln^{r-2} N)$$

From (42) we obtain

$$\liminf_{N\to+\infty} \frac{\| \Lambda_N(a_N) \|^2}{\| (a_N) \|^2} \geq \overline{c}_1\overline{c}_2 \left([\frac{\ln p_r}{\ln 2}] - [\frac{\ln m}{\ln 2}] - 1 \right) = L(r) \tag{43}$$

where $\overline{c}_1$,$\overline{c}_2$ and m are absolute constants. Since $\sup_r L(r) = +\infty$ formula (43) proves that the linear transformation Λ_N are not uniformly bounded and this implies, by the Banach-Steinhaus theorem, the existence of a sequence $a = (a(n)) \in \ell^2$ such that

$$\sup_N \| \Lambda_N(a) \| = +\infty \tag{44}$$

If $f(x) \sim \sum_{n\geq 1} a(n) \sin(2\pi nx)$ and $R_N(x) = \sum_{n=1}^{N} \alpha(n) P(nx) - f(x)$, where the coefficients $\alpha(n)$ are given by (7) of Theorem 2, it is easy to see that

$$\| \Lambda_N(a) \|^2 \leq 4 \| R_N \|^2 + 2 \left(\sum_{n=N+1}^{\infty} |a(n)|^2 \right) \tag{45}$$

since

$$\| R_N \|^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} | - a^{'}(n) + a(n)|^2$$

and

$$\| \Lambda_N(a) \|^2 = \sum_{n=N+1}^{\infty} |a^{'}(n)|^2$$

if we remember (20), (22) and (35). From (44) and (45) follows (10). This concludes the proof of Theorem 3.

□

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