Rend. Sem. Mat. Univ. Pol. Torino
Vol. 59, 1 (2001)

C. Buşe

EXPONENTIAL STABILITY FOR PERIODIC EVOLUTION FAMILIES OF BOUNDED LINEAR OPERATORS

Abstract. We prove that a q-periodic evolution family

$$\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$$

of bounded linear operators is uniformly exponentially stable if and only if

$$\sup_{t>0} \left|\left| \int_0^t e^{-i\mu\xi} U(t,\xi) f(\xi) d\xi \right|\right| = M(\mu, f) < \infty$$

for all $\mu \in \mathbb{R}$ and $f \in P_q(\mathbb{R}_+, X)$, (that is f is a q-periodic and continuous function on $\mathbb{R}_+$).

Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. We denote by $||\cdot||$, the norms of vectors and operators. Let $A \in \mathcal{L}(X)$ and $\mathbb{R}_+$, the set of the all non-negative real numbers. It is known, see e.g. [1] that if the Cauchy Problem

$$\dot{x}(t) = Ax(t) + e^{i\mu t} x_0, \quad x(0) = 0,$$

has a bounded solution on $\mathbb{R}_+$ for every $\mu \in \mathbb{R}$ and any $x_0 \in X$ then the homogenous system $\dot{x} = Ax$, is uniformly exponentially stable. The hypothesis of the above result can be written in the form:

$$\sup_{t>0} \left|\left| \int_0^t e^{-i\mu\xi} e^{\xi A} x_0 d\xi \right|\right| < \infty, \quad \forall \mu \in \mathbb{R}, \forall x_0 \in X.$$

This result cannot be extended for C_0-semigroups (cf. [14], Example 3.1). However, Neerven (cf. [11], Corollary 5) shown that if $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on X and

$$\sup_{\mu\in\mathbb{R}} \sup_{t>0} \left|\left| \int_0^t e^{i\mu\xi} T(\xi) x_0 d\xi \right|\right| < \infty, \quad \forall x_0 \in X, \tag{1}$$

then $\omega_1(\mathbf{T}) < 0$. For details concerning $\omega_1(\mathbf{T})$, we refer to [12] or [9], Theorem A IV.1.4. Moreover, under the hypothesis (1), it results that the resolvent $R(z, A_{\mathbf{T}}) =$

$(z - A_{\mathbf{T}})^{-1}$ of the infinitesimal generator of $\mathbf{T}$, exists and is uniformly bounded on $\mathbf{C}_+ := \{\lambda \in \mathbf{C} : \text{ Re } (\lambda) > 0\}$, see [11]. Combining this with a result of Gearhart [6], (see also Huang [7], Weiss [15] or Pandolfi [13] for other proofs and generalizations), it results that if X is a complex Hilbert space and (1) holds, then $\mathbf{T}$ is uniformly exponentially stable, i.e. its growth bound $\omega_0(\mathbf{T})$ is negative. A similar problem for q-evolution families of bounded linear operators seems to be an open question. In the general case, when X is a Banach space the last results is not true, see e.g. [2], Example 2. However, a weakly result, announced before, holds.

1. Definitions. Preliminary results

Let $q > 0$ and $\Delta = \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$. A mapping $\mathcal{U} : \Delta \to \mathcal{L}(X)$ would be called q-periodic evolution family of bounded linear operators on X, iff:

- **(i)** $U(t, s) = U(t, r)U(r, s)$ for all $t \geq s \geq r \geq 0$;
- **(ii)** $U(t, t) = Id$, (Id is the identity on X), for all $t \geq 0$;
- **(iii)** for all $x \in X$, the map $(t, s) \mapsto U(t, s)x : \Delta \to X$, is continuous;
- **(iv)** $U(t + q, s + q) = U(t, s)$ for all $t \geq s \geq 0$.

The operator $\mathcal{U}(t, s)$ was denoted by $U(t, s)$.

If A is a linear operator on X, $\sigma(A)$ will denote the *spectrum of* A, and if $T \in \mathcal{L}(X)$, $r(T)$ will denote the *spectral radius* of T.

The following two lemmas, which would be used later, are essentially known (see [4], Ch.V, Theorem 1.1, Corollary 1.1 or [5], Theorem 6.6).

LEMMA 1. *A q-periodic evolution family $\mathcal{U}$ on X has exponential growth, that is, there exust $\omega \in \mathbb{R}$ and $M > 1$ such that*

$$||U(t, s)|| \leq Me^{\omega(t-s)} \quad \forall t \geq s \geq 0. \tag{2}$$

We recall that the evolution family $\mathcal{U}$ is called *exponentially stable* if there are $\omega < 0$ and $M > 1$ such that (2) holds. Let $V = U(q, 0) \in \mathcal{L}(X)$.

LEMMA 2. *A q-periodic evolution family $\mathcal{U}$ is exponentially stable if and only if $r(V) < 1$.*

For the proofs of these lemmas we refer to [3].

Let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a strongly continuous semigroup on X and $A_{\mathbf{T}}$ its infinitesimal generator. In [14], Proposition 3.3, it is shown that if

$$\sup_{t>0} || \int_0^t e^{i\mu\xi} T(\xi) d\xi || < \infty, \quad \forall x \in X, \forall \mu \in \mathbb{R}$$

then

$$\sigma(A_{\mathbf{T}}) \subset \mathbf{C}_- := \{z \in \mathbf{C} : \text{ Re } (z) < 0\}.$$

The discret version of this result is the following:

LEMMA 3. *Let $T \in \mathcal{L}(X)$. If*

$$\sup_{n\in\mathbf{N}} ||\sum_{k=0}^{n} e^{i\mu k}T^k|| = M_\mu < \infty \quad \forall \mu \in \mathbb{R},$$

then $r(T) < 1$.

We mention that the result in Lemma 3 is also known and is, for instance, consequence of the uniform ergodic theorem ([8], Theorems 2.1 and 2.7). For reasons of self-containedness we give the proof of Lemma 3 in detail.

Proof. We will use the identity:

$$\sum_{k=0}^{n} e^{i\mu k}T^k(e^{i\mu}T - Id) = e^{i\mu(n+1)}T^{n+1} - Id. \tag{3}$$

From (3) it follows:

$$||e^{i\mu(n+1)}T^{n+1}|| \leq 1 + M_\mu(1 + ||T||) \quad \forall n \in \mathbf{N}, \tag{4}$$

that is $r(T) \leq 1$. Suppose that $1 \in \sigma(T)$. Then for all $m = 1, 2, \cdots$, there exists $x_m \in X$ with $||x_m|| = 1$ and $(Id - T)x_m \to 0$ as $m \to \infty$, (see [9], Proposition 2.2, p. 64). From (4) it results that $T^k(Id - T)x_m \to 0$ as $m \to \infty$, uniformly for $k \in \mathbf{N}$. Let $N \in \mathbf{N}$, $N > 2M_0$ and $m \in \mathbf{N}$ such that

$$||T^k(Id - T)x_m|| \leq \frac{1}{2N}, \quad k = 0, 1, \cdots N.$$

Then

$$\begin{aligned} M_0 &\geq \|x_m + \sum_{k=1}^{N}(x_m + \sum_{j=0}^{k-1} T^j(T - Id)x_m)\| \\ &= \|(N+1)x_m + \sum_{k=1}^{N}\sum_{j=0}^{k-1} T^j(T - Id)x_m\| \\ &\geq (N+1) - \tfrac{N(N+1)}{4N} > \tfrac{N}{2} > M_0. \end{aligned}$$

This contradiction concludes that $1 \notin \sigma(T)$. Now, it is easy to show that $e^{i\mu} \notin \sigma(T)$ for $\mu \in \mathbb{R}$, that is, $r(T) < 1$.

□

2. Uniform exponential stability

Let us consider the following spaces:

- $BUC(\mathbf{I}, X), \mathbf{I} \in \{\mathbb{R}, \mathbb{R}_+\}$ is the Banach space of all X-valued bounded uniformly continuous functions on $\mathbf{I}$, with the sup-norm.
- $AP(\mathbf{I}, X)$ is the linear closed hull in $BUC(\mathbf{I}, X)$ of the set of all functions

$$t \mapsto e^{i\mu t}x : \mathbf{I} \to X, \quad \mu \in \mathbf{R}, \quad x \in X.$$

- $P_q(\mathbf{I}, X)$ is the set of all continuous functions $f : \mathbf{I} \to X$ such that $f(t+q) = f(t)$, for any $t \in \mathbf{I}$ and some $q > 0$.

THEOREM 1. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a q-periodic evolution family on the Banach space X. If*

$$\sup_{t>0} \|\int_0^t e^{-i\mu\xi} U(t,\xi) f(\xi) d\xi\| < \infty, \quad \forall \mu \in \mathbb{R}, \forall f \in P_q(\mathbb{R}_+, X), \tag{5}$$

then $\mathcal{U}$ is exponentially stable.

Proof. Let $V = U(q,0)$, $x \in X$, $n = 0, 1, \cdots$ and $g \in P_q(\mathbb{R}_+, X)$, such that

$$g(\xi) = \xi(q-\xi)U(\xi, 0)x, \quad \forall \xi \in [0, q].$$

From (5), for $t = (n+1)q$, we obtain:

$$\sup_{n\in\mathbf{N}} \|\sum_{k=0}^{n} \int_{kq}^{(k+1)q} U((n+1)q, \xi) e^{-i\mu\xi} g(\xi) d\xi\| < \infty, \quad \forall \mu \in \mathbb{R}. \tag{6}$$

In the view of definition of q-periodic evolution family **(iv)**, it follows:

$$U(pq+q, pq+u) = U(q,u), \quad \forall p \in \mathbf{N}, \quad \forall u \in [0,q]$$

and

$$U(pq, jq) = U((p-j)q, 0) = V^{p-j}, \quad \forall p \in \mathbf{N}, \forall j \in \mathbf{N}, \quad p \geq j.$$

Now, for every $k = 0, 1, \cdots$, we have:

$$\begin{aligned}
&\int_{kq}^{(k+1)q} U((n+1)q, \xi) e^{-i\mu\xi} g(\xi) d\xi = \\
=\ & \int_{kq}^{(k+1)q} U((n+1)q, (k+1)q) U((k+1)q, \xi) e^{-i\mu\xi} g(\xi) d\xi \\
=\ & V^{n-k} \int_0^q U((k+1)q, u+kq) e^{-i\mu(u+kq)} g(kq+u) du \\
=\ & e^{-i\mu kq} V^{n-k} \int_0^q e^{-i\mu u} U(q,u) g(u) du \\
=\ & e^{-i\mu kq} V^{n-k} \int_0^q e^{-i\mu u} u(q-u) U(q,u) U(u,0) x du \\
=\ & e^{-i\mu kq} (\int_0^q e^{-i\mu u} u(q-u) du) V^{n-k+1} x \\
=\ & M(\mu, q) e^{-i\mu(n+1)q} e^{i\mu(n-k+1)q} V^{n-k+1} x,
\end{aligned}$$

where

$$M(\mu, q) = \int_0^q u(q-u) e^{-i\mu u} du \neq 0.$$

We return in (6) and obtain

$$\sup_{n\in\mathbf{N}} \|\sum_{j=0}^{n+1} e^{i\mu jq} V^j\| < \infty,$$

that is, $r(V) < 1$ and $\mathcal{U}$ is exponentially stable.

□

REMARK 1. It is clear that the converse statement from Theorem 1 is also true. Moreover if we denote by $P_2^0(\mathbb{R}_+, X)$ the set of all functions $f \in P_2(\mathbb{R}_+, X)$ for which $f(0) = 0$, then (5) holds (with $P_2(\mathbb{R}_+, X)$ replaced by $P_2^0(\mathbb{R}_+, X)$ if and only if the family $\mathcal{U}$ is exponentially stable).

COROLLARY 1. *A q-periodic evolution family $\mathcal{U}$ on X is uniformly exponentially stable if and only if*

$$\sup_{t>0} \| \int_0^t U(t,\xi) f(\xi) d\xi \| < \infty, \quad \forall f \in AP(\mathbb{R}_+, X).$$

For the other proofs of Corollary 1, see e.g. [2] and [14]. In the end we give a result about evolution families on the line. In this context,

$$\mathcal{U} = \{U(t,s) : t \geq s \in \mathbb{R}\}$$

will be a q-periodic evolution family on $\mathbb{R}$. We shall use the same notations as in Section 1, with $\mathbb{R}_+$ replaced by $\mathbb{R}$ and variables such as s and t taking any value in $\mathbb{R}$. Let us consider the evolution semigroup $\mathbf{T}_{ap}$ associated to $\mathcal{U}$ on the space $AP(\mathbb{R}, X)$. This semigroup is strongly continuous, see Naito and Minh ([10], Lemma 2).

COROLLARY 2. *Let $\mathcal{U} = \{U(t,s), t \geq s\}$ be a q- periodic evolution family of bounded linear operators on X and $\mathbf{T}_{ap}$ the evolution semigroup associated to $\mathcal{U}$ on the space $AP(\mathbb{R}, X)$. Then $\mathcal{U}$ is uniformly exponentially stable if and only if*

$$\sup_{t\geq 0} \| (\int_0^t e^{i\mu\xi} T_{ap}(\xi) f d\xi)(t) \| < \infty \quad \forall \mu \in \mathbb{R}, \quad \forall f \in P_q(\mathbb{R}_+, X).$$

Proof. For $t > 0$, we have

$$\begin{aligned} (\int_0^t e^{i\mu\xi} T_{ap}(\xi) f)(t) = & \int_0^t e^{i\mu\xi} U(t, t-\xi) f(t-\xi) d\xi \\ = & \; e^{i\mu t} \int_0^t e^{-i\mu\tau} U(t,\tau) f(\tau) d\tau. \end{aligned}$$

Now, from Theorem 1, it follows that the restriction $\mathcal{U}_0$ of $\mathcal{U}$ to the set $\{(t,s) : t \geq s \geq 0\}$ is uniformly exponentially stable. Let $N > 0$ and $\nu > 0$ such that

$$\|U(t,s)\| \leq N e^{-\nu(t-s)}, \quad \forall t \geq s \geq 0.$$

Then for all real numbers u and v with $u \geq v$, we have

$$\|U(u,v)\| = \|U(u+nq, v+nq)\| \leq N e^{-\nu(u-v)},$$

where $n \in \mathbf{N}$ is such that $v + nq \geq 0$, that is, $\mathcal{U}$ is uniformly exponentially stable.

□

References

[1] Balint S., *On the Perron-Bellman theorem for systems with constants cofficients*, Ann. Univ. Timişoara **21** 1-2 (1983), 3–8.

[2] Buşe C., *On the Perron-Bellman theorem for evolutionary processes with exponential growth in Banach spaces*, New-Zealand Journal of Mathematics **27** (1998), 183–190.

[3] Buşe C., *Asymptotic stability and Perron condition for periodic evolution families on the half line*, Evolution Equations and Semigroups (preprint), http://ma1serv.mathematik.uni-karlsruhe.de/evolve-l/index.html.

[4] Daletckii Ju.L. and Krein M.G., *Stability of solutions of differential equations in Banach spaces*, Am. Math. Soc. **30**, Providence, RI 1974.

[5] Daners D. and Koch Medina P., *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes 1992.

[6] Gearhart L., *Spectral theory for contraction semigroups on Hilbert spaces*, Trans. Amer. Math. Soc. **236** (1978), 385–394.

[7] Huang F., *Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces*, Ann. Diff. Eq. **1** (1985), 43–56.

[8] Krengel U., *Ergodic theorems*, De Gruyter 1985.

[9] Nagel R. (ed.), *One-parameter semigroups of positive operators*, Springer Lect. Notes in Math. **1184** (1986).

[10] Naito T. and Minh N.V., *Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations*, J. Diff. Eq. **152** (1999), 358–376.

[11] van Neerven J.M.A.M., *Individual stability of C_0-semigroups with uniformly bounded local resolvent*, Semigroup Forum **53** (1996), 155–161.

[12] Neubrander F., *Laplace transform and asymptotic behaviour on strongly continous semigroups*, Houston Math. J. **12** (1986), 549–561.

[13] Pandolfi L., *Some properties of distributed control systems with finite-dimensional input space*, SIAM J. Control and Optimization **30** 4 (1992), 926–941.

[14] Reghiş M. and Buşe C., *On the Perron-Bellman theorem for C_0-semigroups and periodic evolutionary processes in Banach spaces*, Italian J. Pure and Appl. Math. **4** (1998), 155–166.

[15] Weiss G., *Weak L^p-stability of a linear semigroup on a Hilbert space implies exponential stability*, J. Diff. Eq. **76** (1988), 269–285.

AMS Subject Classification: 47D06, 34D05.

Constantin BUŞE
Department of Mathematics
West University of Timişoara
Bd. V. Parvan 4
Timişoara ROMÂNIA
e-mail: `buse2002f@netscape.net`

Lavoro pervenuto in redazione il 02.11.1999 e in forma definitiva il 30.03.2000.