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NON-PRIMITIVE LINEAR SYSTEMS ON SMOOTH ALGEBRAIC CURVES AND A GENERALIZATION OF MARONI THEORY

Abstract. Let X be a smooth curve of genus g and M, L special spanned line bundles with $h^0(X, L) = 2$ and $h^0(X, M \otimes L^*) = h^0(X, M) - 2 > 0$. Generalizing Maroni theory for trigonal curves we study the existence of such triples (X, M, L) for certain numerical invariants and classify all spanned line bundles, R, on any such X with $h^0(X, M \otimes R^*) > 0$. We construct smooth curves with non-primitive special linear systems with prescribed numerical invariants.

1. Introduction

Let X be a smooth connected projective curve of genus g. We introduce the following definition.

DEFINITION 1. *Take $M \in Pic^d(X)$, $L \in Pic^k(X)$ with M and L spanned, $h^0(X, L) = 2$ and $h^0(X, M) \geq 3$. We will say that the pair (M, L) is a Maroni pair if $h^0(X, M \otimes L^*) = h^0(X, M) - 2$.*

The terminology comes from the case $k = 3$, $M \cong \omega_X$; indeed we will see how to use the classical theory of Maroni of special linear systems on trigonal curves in our set-up. With the terminology of [6], Definition in 1.1, a Maroni pair (M, L) is essentially a linear series M of type $2 + 1$ with respect to the pencil L (see 1). This is the first unknown case, because D. Eisenbud gave a complete classification of all pairs (M, L) with $h^0(X, M \otimes L^* = h^0(X, M) - 1$ (see [7], Cor. 5.2, or [5], Lemma 1.2.1).

DEFINITION 2. *Take $M \in Pic^d(X)$, $L \in Pic^k(X)$ with M and L spanned, $h^0(X, L) = 2$ and $h^0(X, M) \geq 3$. Let $W \subseteq H^0(X, M)$ be a linear subspace spanning M and with $r := dim(W) - 1 \geq 2$. We will say that (M, W, L) is a weak Maroni triple if $dim(W(-D)) = r - 1$ for every $D \in |L|$. Now we drop the assumption $h^0(X, L) = 2$ and take a linear subspace V of $H^0(X, L)$ with $dim(V) = 2$ and V spanning L; if $dim(W(-D)) = r - 1$ for every $D \in |V|$, then we will say that (M, W, L, V) is a very weak Maroni quadruple.*

In section 2 we will study curves with a Maroni pair. We will give existence the-

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orems for assigned invariants g, $deg(L)$, $h^0(X,M)$ and the so-called Maroni invariant of a Maroni pair (see Remarks 2 and 3). Here is a sample of our results on this topic.

PROPOSITION 1. *Assume $char(K) = 0$. Fix integers g, d, k, r, m with $g \geq 5$, $r \geq 3$, $k \geq 3$, $0 \leq m \leq r-1$, $r-m$ odd, $2g-2 \geq d \geq k(r+m-1)/2$ and $0 \leq g \leq 1+(k-1)(d-k(r-m-1)/2)-k-m(k^2-k)/2$.*
Then there exist a smooth curve X of genus g and a very weak Maroni quadruple (M,W,L,V) on X with invariants d, k, r and m and such that the morphism h_W is birational. If $m \neq r-1$ or $m = r-1$ and $k(r-1) \leq d \leq k(r-1)+2$ we may find a smooth curve X and a very weak Maroni quadruple (M,W,L,V) such that $h_W(X)$ has exactly $1+(k-1)(d-k(r-m-1)/2)-k-m(k^2-k)/2-g+\epsilon$ ordinary nodes as singularities, $0 \leq \epsilon \leq 1$ and $\epsilon = 1$ if and only if $m = r-1$ and $d = k(r-1)+2$.

See Propositions 2 and 3 for other results.

For any curve X with a Maroni pair (M,L) such that the induced morphism $h_M : X \to \mathbf{P}(H^0(X,M))$ is birational we will give a partial classification of all $R \in Pic(X)$ with R spanned and $h^0(X, M \otimes R^*) > 0$ (see Theorem 2).

Now we recall the following classical definition ([3]).

DEFINITION 3. *Let X be a smooth projective curve and $M \in Pic(X)$. M is said to be primitive if both M and $\omega_X \otimes M^*$ are spanned by their global sections.*

By Riemann - Roch $M \in Pic(X)$ is primitive if and only if M is spanned and $h^0(X, M(P)) = h^0(X,M)$ for every $P \in X$.

In the last part of section 3 we will use the method of [2] to construct pairs (X,L), L spanned, such that not only $L^{\otimes s}$ is not primitive, but $\omega_X \otimes (L^{\otimes s})^*$ has base locus containing c distinct points $P_1,\dots,P_c$ of X. We want $h^0(X,L^{\otimes s}) = s+1$ and hence we will obtain $h^0(X, L^{\otimes s}(P_1+...+P_i)) = s+1+i$ for all integers i with $1 \leq i \leq c$. This condition seems to be a very restrictive condition for a pair (X,L). For instance it is never satisfied for $s \geq 2$ and any L if X is a general curve of genus g or a general k-gonal curve ([3], [4], [5], [6]). Notice that $h^0(X,L^{\otimes s}) = s+1$ implies $h^0(X,L^{\otimes t}) = t+1$ for $1 \leq t \leq s$. In section 3 we will prove the following result.

THEOREM 1. *Fix integers g, k and s with $s \geq 1$, $k \geq 4$, $g \geq s(k-1)$ and $g \geq 2k+2$. Assume the existence of integers a, w with $s < a \leq 2s+2$, $0 < w \leq [k/2]+1, 2[(k+a)/2] > w(a-1-s)$ and $w(a-1-s) \leq ak-a-k+1-g < ([k/2]+1)([(a+1)/2]+1)$. Then there exist a smooth curve X of genus g and $L \in Pic^k(X)$ with $h^0(X,L) = 2$, $h^0(X,L^{\otimes t}) = t+1$ for every integer t with $1 \leq t \leq s$, L spanned and $\omega_X \otimes (L^{\otimes s})^*$ has base locus containing $w(s+1)$ distinct points of X.*

From Theorem 1 taking a = s+2 we obtain the following corollary.

COROLLARY 1. *Fix integers g, k and s with $s \geq 1$, $k \geq 4$, $g \geq s(k-1)+1$ and $g \geq 2k+2$. Fix an integer w with $1 \leq w \leq 1+[k/2]$ and assume $w \leq sk-s+k-1-g$. Then there exist a smooth curve X of genus g and $L \in Pic^k(X)$ with $h^0(X,L^{\otimes t} = t+1$ for every integer t with $1 \leq t \leq s$, L spanned and $\omega_X \otimes (L^{\otimes s})^*$ has base locus containing $w(s+1)$ distinct points of X.*

In the first part of section 3 we will use the same ideas to obtain results related to Theorem 1 but concerning line bundles, L, with just one point as the base locus of $\omega_X \otimes (L^{\otimes s})^*$ (see Corollary 2 and to construct pairs (X, L, D) with D effective divisor of degree $b \geq 1$, $L \in Pic(X)$, L spanned, $h^0(X, L^{\otimes t}) = t + 1$ for every integer t with $1 \leq t \leq s$, with $h^0(X, L^{\otimes s}(D)) = s + 2$ and $L^{\otimes s}(D)$ spanned, i.e. with $h^0(X, L^{\otimes s}(D'))$ = s+1 for every effective (or zero) divisor strictly contained in D (see (**) at page 12). The proofs of all our existence results use the deformation theory of nodal curves on Hirzebruch surfaces.

2. Maroni pairs

We work over an algebraically closed field $\mathbf{K}$. In this section we study properties of curves admitting a Maroni pair (M, L) and related to the pair (M, L). Let X be a smooth connected projective curve of genus g. For any spanned line bundle F on X, let

$$h_F : X \to \mathbf{P}(H^0(X, F))$$

be the associated morphism; if $W \subseteq H^0(X, F)$ is a linear subspace spanning F, let

$$h_W : X \to \mathbf{P}(W)$$

be the associated morphism. For any linear subspace W of $H^0(X, F)$ and any effective divisor D on X, set

$$W(-D) := W \cap H^0(X, I_D \otimes F).$$

For any $L \in Pic(X)$ and any linear subspace W of $H^0(X, L)$, let $|W|$ be the associated linear system of effective divisors on X. If $W = H^0(X, L)$ we will often write $|L|$ instead of $|H^0(X, L)|$.

REMARK 1. Let (M, W, L, V) be a very weak Maroni quadruple. If $W = H^0(X, M)$ then $dim(W(-D)) = dim(W(-D'))$ for all D, $D' \in |L|$. Assume $u := h^0(X, L) - 1 \geq 2$, i.e. assume that (M, L) is not a Maroni pair. Let N be the subsheaf of $M \otimes L^*$ spanned by $H^0(X, M \otimes L^*)$, say $N(D) \cong M \otimes L^*$ with D an effective divisor. Since

$$h^0(X, N) + h^0(X, L) \geq h^0(X, M) + 1 \geq h^0(X, N \otimes L) + 1,$$

by a lemma of Eisenbud ([5], Lemma 1.2.1) there is $R \in Pic(X)$ with R spanned, $h^0(X, R) = 2$, $N \cong R^{\otimes(r-2)}$ and $L \cong R^{\otimes u}$. Thus $M \cong R^{\otimes(r+u-2)}(D)$. Since M is spanned, $u \geq 2$ and $h^0(X, M) = r + 1$, we obtain $D = \emptyset$ and $u = 2$. Furthermore, $h^0(X, R^{\otimes t}) = t + 1$ for every integer t with $1 \leq t \leq r$. Viceversa, for any such R the pair $(R^{\otimes r}, R^{\otimes 2})$ induces a very weak Maroni quadruple with $W := H^0(X, M)$.

REMARK 2. Let (M, L) be a Maroni pair. If h_M is birational, then the condition $h^0(X, M \otimes L^*) = h^0(X, M) - 2$ means that for a general $D \in |L|$ the set $h_M(D)$ is contained in a line of $\mathbf{P}(H^0(X, M))$. It is easy to check that the same is true even if h_M is not birational (see e.g. ($\ddagger$) at page 9) and that the condition $h^0(X, M \otimes L^*) < h^0(X, M) - 1$ gives the uniqueness of this line. Call $S \subset \mathbf{P}(H^0(X, M))$ the union of these lines. By [11], § 2, S is a minimal degree rational normal surface (not the Veronese surface of $\mathbf{P}^5$), i.e. $deg(S) = h^0(X, M) - 2$ and either S is the cone over a rational normal curve of a hyperplane of $\mathbf{P}(H^0(X, M))$ or it is isomorphic to a Hirzebruch surface F_m. In the latter case we will call m the Maroni invariant of the pair (M, L). In the former case we will call $h^0(X, M) - 2$ the Maroni invariant of (M, L). The integer $r := h^0(X, M) - 1$ is called the dimension of (M, L). S is called the scroll associated to the pair (M, L). Instead of m and r we may use two integers e_1, e_2 with $e_1 \geq e_2 \geq 0$, $e_1 + e_2 = r - 1$ and $e_1 - e_2 = m$. Notice that $r - m$ is always odd.

Remark 3. Let (M, L) be a Maroni pair with invariants e_1 and e_2. It is easy to check that $h^0(X, M \otimes (L^*)^{\otimes(e_1)}) \neq 0$ and $h^0(X, M \otimes (L^*)^{\otimes(e_1+1)}) = 0$. In particular we have $deg(M) \geq e_1(deg(L))$ with equality if and only if $M \cong L^{\otimes(e_1)}$.

Remark 4. We will often use the dimension of some cohomology groups of certain line bundles on a Hirzebruch surface F_s, $s \geq 0$. We will give here the general formulas. We have $F_s \cong \mathbf{P}(O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-s))$. Let $u : F_s \to \mathbf{P}^1$ be a ruling (which is unique if $s > 0$). We have $Pic(F_s) \cong \mathbf{Z}^{\oplus 2}$ and we take as base of $Pic(F_s)$ a fiber, f, of the ruling u and a section, h, of the ruling with minimal self-intersection. We have $h^2 = -s$, $h \cdot f = 1$ and $f^2 = 0$. The canonical line bundle of F_s is $O_{F_s}(-2h + (-s-2)f)$. Hence by Serre duality we have

$$h^1(F_s, O_{F_s}(th + zf)) = h^1(F_s, O_{F_s}((-t-2)h + (-z-s-2)f)).$$

We have $h^0(F_s, O_{F_s}(th + zf)) = 0$ if $t < 0$, because $|f|$ is base point free and $(th + zf) \cdot f = t$. Hence to compute the cohomology groups of all line bundles on F_s it is sufficient to compute $h^0(F_s, O_{F_s}(th + zf))$ for all integers t, z with $t \geq 0$ and $h^1(F_s, O_{F_s}(th+zf))$ for all integers t, z with $t \geq -1$. We have $h^0(F_s, O_{F_s}(th+zf)) < 0$ if $z < 0$, while if $ts > z \geq 0$, h is a base component of the linear system associated to $h^0(F_s, O_{F_s}(th + zf))$ and h occurs as a base component at least (and, as we will see, exactly) with multiplicity e, where e is the minimal integer such that $(t-e)s \leq z$. Hence to compute all values of $h^0(F_s, O_{F_s}(th + zf))$ (resp. $h^1(F_s, O_{F_s}(th + zf))$) it is sufficient to compute the ones with $t \geq 0$ and $z \geq st$ (resp. $t \geq -1$). We claim that for all integers t, z with $t \geq 0$ we have

$$u_*(O_{F_s}(th + zf)) = \bigoplus_{0 \leq i \leq t} O_{\mathbf{P}^1}(z - is). \tag{1}$$

Set $E := O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(-s)$. We have $F_s = \mathbf{P}(E)$. By [8], Prop. II.7.11 (a), for every integer $t \geq 0$ we have

$$u_*(O_{F_s}(th)) \cong S^t(E) \cong \bigoplus_{0 \leq i \leq t} O_{\mathbf{P}^1}(-is).$$

Hence (1) follows the projection formula ([8], Ex. II.V.1 (d)). By (1) and the Leray spectral sequence of u we obtain that if $t \geq 0$ we have $h^1(F_s, O_{F_s}(th + zf)) = 0$ if and only if $z \geq ts - 1$. Since $h^0(\mathbf{P}^1, O_{\mathbf{P}^1}(-1)) = h^1(\mathbf{P}^1, O_{\mathbf{P}^1}(-1)) = 0$, we have $R^1u_*(O_{F_s}(-h+zf)) = u_*(O_{F_s}(-h+zf)) = 0$ for every z. Hence the Leray - Spectral sequence of u gives $h^1(F_s, O_{F_s}(-h + zf)) = 0$ for every integer z. By (1) we obtain $h^0(F_s, O_{F_s}(th + zf)) = \sum_{0 \leq i \leq t}(z - is + 1) = (t + 1)(2z - ts + 2)/2$ if $t \geq 0$ and $z \geq ts - 1$. It is easy to check using (1) that if $t > 0$ the linear system $|th + tsf|$ is base point free and its general member is smooth and irreducible, while if $z > ts$ and $t > 0$ then $O_{F_s}(th + zf))$ is very ample; alternatively, see [8], V.2.17 and V.2.18. Now fix integers k, x with $k \geq 0$, $x \geq sk$, $(k, x) \neq (0, 0)$ and any $D \in |kh + xf|$. By the adjunction formula we have

$$\omega_D \cong \omega_{F_s}(D)|D \cong O_D((k-2)h + (x-2-s)f).$$

Thus

$$2p_a(D) - 2 = -sk(k-2) + x(k-2) + k(x-2-s),$$

i.e.

$$\begin{aligned} p_a(D) &= 1 + kx - k - x - sk(k-1)/2 \\ &= 1 + (k-1)x - k - sk(k-1)/2 \\ &= 1 + sk(k-1)/2 + k(x-ks) - (x-ks) - k. \end{aligned}$$

Proof of Proposition 1. Let $S \subset \mathbf{P}^r$ be a minimal degree surface (not the Veronese surface if $r = 5$) with Maroni invariant m. Thus $deg(S) = r - 1$ and S is a cone if and only if $m = r - 1$. First assume $0 \leq m < r - 1$, i.e. $S \cong F_m$. We use the notation of Remark 4 with $s = m$. The linear system $|h + ((r + m - 1)/2)f|$ on S is very ample and it induces the complete linear system associated to the fixed embedding of S into $\mathbf{P}^r$. Set $x := d - k(r - m - 1)/2$. Thus $x \geq km$. Hence the linear system $|kh + xf|$ is base point free and a general member of $|kh + xf|$ is a smooth curve of genus $q := 1 + kx - x - k - m(k^2 - k)/2$ (see Remark 4). By [12], Remark 3.2, there is an integral curve $C \in |kh + xf|$ whose only singularities are $q - g$ ordinary nodes. Let $\pi : X \to C$ be the normalization. Thus X is a smooth curve of genus g. The pencil $|f|$ induces $|L \in Pic^k(X)$ with $h^0(X, L) \geq 2$. Set $M := \pi^*(O_C(1))$. Hence $M \in Pic^d(X)$, M is spanned, $h^0(X, M) \geq r + 1$, $h^0(X, M \otimes L^*) = r - 1$ if $h^0(X, M) = r + 1$ and h_M is birational with nodal image. Set $W := \pi^*(H^0(\mathbf{P}^r, O_{\mathbf{P}^r}(1)))$ and $V := \pi^*(|f|)$. We have $dim(W(-D)) = r - 1$ for every divisor $D \in |V|$ and hence (M, W, L, V) is a very weak Maroni pair. Now assume $m = r - 1$, i.e. S a cone. Let $\delta : F_{r-1} \to S$ be the minimal resolution. We repeat the previous construction in F_{r-1} and obtain a nodal curve $C \in |kh + df|$. The curve $\delta(C) = h_W(X) \subset \mathbf{P}^r$ is nodal if and only if it has not a worst singularity at the vertex, v, of the cone S. A necessary condition for this is $C \cdot h \leq 2$, i.e. $d \leq k(r - 1) + 2$. If $d = k(r - 1)$, $C \subset F_{r-1}$ and C is nodal, then $\delta(C)$ is nodal because $v \notin \delta(C)$. If $d = k(r - 1) + 1$, $C \subset F_{r-1}$ and C is nodal, then $\delta(C)$ is nodal because $v \in \delta(C)_{reg}$. If $d = k(r - 1) + 2$ we may find a nodal curve $C \subset F_{r-1}$ intersecting transversally h and hence with $\delta(C)$ nodal at v.

□

Remark 5. Let (M, L) be a Maroni pair on X with invariants d, k, r and m. The proof of Proposition 1 shows that the morphism h_M is very ample if and only if it is birational and either $m < r-1$ and $g = 1+(k-1)(d-k(r-m-1)/2)-k-m(k^2-k)/2$ or $m = r - 1$, $k(r - 1) \leq d \leq k(r - 1) + 1$ and g is as above.

Remark 6. Let (M, L) be a Maroni pair with invariants d, k, r and m. Assume h_M birational. The proof of Proposition 1 gives that $M \otimes (L^*)^{\otimes t}$ is base point free with $h^0(X, M \otimes (L^*)^{\otimes t}) = r + 1 - 2t$ if $0 \leq t \leq (r - 1 - m)/2$.

Proposition 2. *With the notations and assumptions of Remark 6, make the further assumptions*

$$d \geq k(r + m - 1)/2 - m + 2$$

and

$$1+(k-1)(d-k(r-m-1)/2)-k-m(k^2-k)/2-g \le (k-1)(d-k(r-m-1)/2-2-m)-m(k-1)(k-2)/2.$$

Then $h^0(X,L)=2$, i.e. (M,W,L) is a weak Maroni triple.

Proposition 3. *With the notations and assumptions of Remark 5, make the further assumptions*

$$d \ge 1+(k+1)(r+m-1)/2-2m$$

and

$$1+(k-1)(d-k(r-m-1)/2)-k-m(k^2-k)/2-g \le (k-2)(d-k(r-m-1)/2-1-m-(r+m-1)/2) - m(k-2)(k-3)/2.$$

Then $h^0(X,L)=2$ and $h^0(X,M)=r+1$, i.e. (M,L) is a Maroni pair.

Proof of Propositions 2 and 3. We use the notation introduced in the proof of Proposition 1. In particular $d=(kh+xf)\cdot(h+((r+m-1)/2)f)=-mk+x+k(r+m-1)/2$. Since $d \ge k(r+m-1)/2$, we have $x \ge km$. Under the assumptions of Proposition 2 we need to check the equality $h^0(X,L)=2$; for Proposition 3 we need to check the equality $h^0(X,M)=r+1$. We have $h^0(S,O_S(1))=r+1$. We have $h^0(S,O_S(1)(-C))=r+1$ because $k>1$. We have

$$\begin{aligned} h^1(S,O_S(1)(-C)) &= h^1(S,O_S((1-k)h+((r+m-1)/2-x)f) \\ &= h^1(S,O_S((k-3)h+(x-(r+m-1)/2-2-m)f) \\ &\quad \text{(Serre duality).} \end{aligned}$$

By Remark 4 we have $h^1(S,O_S((k-3)h+(x-(r+m-1)/2-2-m)f)=0$ if and only if $x \ge -1+m(k-3)+(r+m-1)/2+2+m$, i.e. if and only if

$$\begin{aligned} d &\ge k(r-m-1)/2-1+m(k-3)+(r+m-1)/2+2+m \\ &= (k+1)(r+m-1)/2-2m+1. \end{aligned}$$

Hence we have $h^0(C,O_C(1))=r+1$ in the case of Proposition 3 by our assumption on d. We have $h^0(C,O_C(f))=2$ if and only if $h^1(S,O_S(-kh-(x-1)f))=0$, i.e. by Serre duality if and only if $h^1(S,O_S((k-2)h+(x-3-m)f))=0$ and this is true if an only if $x-3-m+1 \ge m(k-2)$ (see Remark 4), i.e. if and only if $d \ge k(r+m-1)/2-m+2$. By the adjunction formula we have $\omega_S \cong |(-2h+(-2-m)f|$ and $\omega_C \cong O_C((k-2)h+(x-2-m)f)$ (see Remark 4). Since $h^0(S,\omega_S)=h^1(S,\omega_S)=0$, the restriction map $H^0(S,O_S((k-2)h+(x-2-m)f)) \to H^0(C,\omega_C)$ is bijective. Since C is nodal, the linear system $H^0(X,\omega_X)$ is induced by $H^0(S,I_{Sing(C)}((k-2)h+(x-2-m)f))$. We have $card(Sing(C))=q-g$. Hence the equality $h^0(X,L)=2$ (resp. $h^0(X,M)=r+1$) is equivalent to the linear independence of the conditions imposed by $Sing(C)$ to the linear system $|(k-2)h+(x-3-m)f|$ (resp. $|(k-3)h+(x-2-m-(r+m-1)/2)f|$. There is a nodal curve $D \in |kh+xf|$ with geometric genus 0 and such that D is the flat limit inside $|kh+xf|$ of a flat family of nodal irreducible curves with geometric genus g (and hence with $q-g$ ordinary nodes as only singularities) ([12], 2.2, 2.11, 2.13 and 2.14); here the proofs are quite simple because ω_S^* is ample). With the terminology of [12] this flat family is obtained taking a rational irreducible nodal curve $D \in |kx + xf|$ and smoothing inside $|kh + xf|$ exactly g of the nodes of D; any subset, B, of $Sing(D)$ with $card(B) = g$ may be taken as such set of "unassigned"' nodes, while the set $Sing(D)\backslash B$ is the limit of the nodes of the nearby nodal curves with geometric genus g. We would like to take as C a general member of this flat family which smooths exactly the unassigned nodes of D. By semicontinuity to use that curve C it is sufficient to show the existence of a subset Γ of $Sing(D)$ with $card(\Gamma) = q - g$ and such that Γ imposes independent conditions to $|(k-2)h + (x-3-m)f|$ (resp. $|(k-3)h + (x-2-m-(r+m-1)/2)f|$) for Proposition 2 (resp. 3). Since D is rational, we have $h^0(S, I_{Sing(D)}((k-2)h + (x-2-m)f)) = 0$. Thus

$$\begin{aligned} &h^0(S, I_{Sing(D)}((k-2)h + (x-3-m)f)) \\ &= h^0(S, I_{Sing(D)}((k-3)h + (x-2-m-(r+m-1)/2)f)) = 0. \end{aligned}$$

Hence for every integer $z \leq h^0(S, O_S((k-2)h+(x-3-m)f)) = (k-1)(2x-mk-4)/2$ (resp. $w \leq h^0(S, O_S((k-3)h+(x-2-m-(r+m-1)/2)f))) = (k-2)(2x-mk-r-1)/2)$ there is a subset Φ of $Sing(D)$ with $card(\Phi) = z$ (resp. $card(\Phi) = w$) and imposing independent conditions to $H^0(S, O_S((k-2)h+(x-3-m)f))$ (resp. $h^0(S, O_S((k-3)h+(x-2-m-(r+m-1)/2)f))))$. Hence if $q-g := card(Sing(C)) \leq h^0(S, O_S((k-2)h+(x-3-m)f))$ (resp. $q-g \leq h^0(S, O_S((k-3)h+(x-2-m-(r+m-1)/2)f)))$ there is a subset A of $Sing(C)$ with $card(A) = q-g$ and $h^1(S, I_A((k-2)h+(x-3-m)f)) = 0$ (resp. $h^1(S, I_A((k-3)h + (x-2-m-(r+m-1)/2)f)) = 0$).
The values of the h^0 's explain the upper bound of g in the statements of Propositions 2 and 3.

□

REMARK 7. Fix a minimal degree surface $S \subset \mathbf{P}^r$ (not the Veronese surface) with Maroni invariant m. In the case $m = r-1$ we must work on the minimal desingularization, F_{r-1}, of the cone S, but we leave the details to the reader. Since $|h + ((r+m-1)/2)f| \cong |O_S(1))|$, we have $h^0(S, O_S(1)(-tf)) = r+1-2t$ if $0 \leq t \leq (r-m+1)/2$ and $h^0(S, O_S(1)(-tf)) = max\{0, (r+m+1)/2 - t\}$ if $t > (r-m+1)/2$. Now assume that we are in the case considered in Proposition 3, i.e. assume $h^0(X, M) = r+1$. We obtain the same values of $h^0(X, M \otimes (L^*)^{\otimes t})$, $t \geq 0$, i.e. we obtain $h^0(X, M \otimes (L^*)^{\otimes(t-1)}) - h^0(X, M \otimes (L^*)^{\otimes t}) = 2$ if $1 \leq t \leq (r-m+1)/2$ and $h^0(X, M \otimes (L^*)^{\otimes(t-1)}) - h^0(X, M \otimes (L^*)^{\otimes t}) = 1$ if $t > (r-m+1)/2$ and $h^0(X, M \otimes (L^*)^{\otimes(t-1)}) > 0$, explaining the meaning of the Maroni invariant m. Indeed, these equalities are true because the assumption $h^0(X, M) = r+1$ forces the set $Sing(C)$ to impose independent conditions to a suitable linear system which is a subsystem of the ones needed for the inequalities considered here.

REMARK 8. Let X be a smooth curve of genus g with a Maroni pair (M, L) with invariants d, k, r and m and such that the morphism h_M is birational. Then the triple (X, M, L) arises as in the proof of Proposition 1, except that C may be not nodal, but just an integral curve in $|kh + xf|$ with geometric genus g.

REMARK 9. By [12], Prop. 2.11, the family, Γ, of nodal curves considered in the proof of Proposition 1 is smooth and equidimensional of dimension $dim(|kh + xf|) - (q-g)$. If $q-g$ is quite small, it is even easy to check that Γ is irreducible and that its closure in $|kh + xf|$ contains all irreducible curves with geometric genus $|g|$.

Lemma 1. *Let (M, L) be a Maroni pair with $r := h^0(X, M) - 1 \geq 4$. Let N be the subsheaf of $M \otimes L^*$ spanned by $H^0(X, M \otimes L^*)$.*
Then either (N, L) is a Maroni pair or $N \cong L^{\otimes(r-2)}$. In the latter case there is an effective (or empy) divisor D with $M \cong L^{\otimes(r-1)}(D)$ and $h^0(X, O_X(D)) = 1$.

Proof. We have $h^0(X, N) = H^0(X, M \otimes L^*) = r - 1 \geq 3$. Since $h^0(X, N) > h^0(X, N \otimes L^*)$ and $h^0(X, N) - h^0(X, N \otimes L^*) \leq h^0(X, M) > h^0(X, M \otimes L^*) = 2$, either (N, L) is a Maroni pair or we may apply a Lemma of Eisenbud ([5], Lemma 1.2.1, or, in arbitrary characteristic, [7], Cor. 5.2) and obtain $N \cong L^{\otimes(r-2)}$. In the latter case the definition of N gives $M \otimes L^* = N(D)$ with $h^0(X, O_X(D)) = 1$, concluding the proof. □

Remark 10. Fix a Maroni pair (M, L) with h_M birational. Since $h^0(X, M) = r + 1$, we may use Remark 7 to apply several times Lemma 1.

Remark 11. Let (M, L) be a Maroni pair on X and F a subsheaf of M with F spanned. We have $1 \leq h^0(X, F) - h^0(X, F \otimes L^*) \leq h^0(X, M) - h^0(X, M \otimes L^*) = 2$. If $h^0(X, F) - h^0(X, F \otimes L^*) = 1$ we may apply [5], Lemma 1.2.1, or [7], Cor. 5.2, and obtain $F \cong L^{\otimes f}$ with $f := deg(F)/deg(L) \in \mathbb{N}$. If $h^0(X, F) - h^0(X, F \otimes L^*) = 2$ and $h^0(X, F) \geq 3$, then (F, L) is a Maroni pair.

Definition 4. *Let (M, L) be a Maroni pair on X and F a subsheaf of M. The level, $s(F, M)$ of F in (M, L) is the maximal integer $s \geq 0$ such that $h^0(X, M \otimes F^* \otimes (L^*)^{\otimes s}) \neq 0$. By DEfinition 2 we have $s(O_X, M) = e_1$.*

Theorem 2. *Let (M, L) be a Maroni pair on X with invariants d, k, r and m with h_M birational and $R \in Pic(X)$ with R spanned and $h^0(X, M \otimes R^* > 0$. Then one of the following cases occurs:*

- *(i) $R \cong L^{\otimes y}$ for some integer y with $y \leq (r + m - 1)/2$;*
- *(ii) there is an integer $t > 0$ and an effective divisor D (or empty) with $R(D) \cong M \otimes (L^*)^{\otimes t}$, $h^0(X, D) = 1$ and $h^0(X, L^{\otimes t}(D)) = h^0(X, L^{\otimes t})$, i.e. D is the fixed divisor of $L^{\otimes t}(D)$.*

Proof. Let $S \subset \mathbf{P}^r$ be the minimal degree ruled surface associated to M. For simplicity we assume $m < r - 1$; in the case $m = r - 1$ we just use F_{r-1} instead of S. For any subscheme Z of $\mathbf{P}^r$, let $\langle Z \rangle$ be its linear span. For any effective divisor B of X let $\langle B \rangle$ be the intersection of all hyperplanes of $\mathbf{P}^r$ whose pull-back is a divisor of $|M|$ containing B. Assume R not isomorphic to $L^{\otimes y}$ for some $y \leq (r + m - 1)/2$.

(a) First assume $h^0(X, M \otimes R^*) = 1$. Hence for a general $E \in |R|$ the linear span $\langle E \rangle|$ is a hyperplane H. The hyperplane H corresponds to a divisor $B + E \in |M|$ with B effective. We have $\dim(\langle B \rangle) = r - 1 - \dim(|R|) \leq r - 2$.
Claim: $\langle B \rangle \cap S$ is finite.
Proof of the Claim: Assume $\langle B \rangle \cap S$ not finite. Hence it contains a curve $\Delta \in |\epsilon h + \alpha f|$ with $0 \leq \epsilon \leq 1$, $0 \leq \alpha \leq (r + m - 1)/2$ and $\alpha + \epsilon \neq 0$. Since R is spanned, the divisor of X induced by Δ is contained in B. Hence $h_M(E)$ is contained in a curve belonging to $|(1 - \epsilon)h + ((r + m - 1)/2 - \alpha)f|$, contradicting the assumption $h^0(X, M \otimes R^*) = 1$.

(b) Now assume $h^0(X, M \otimes R^*) > 1$. Hence for a general $E \in |R|$ the linear span $\langle E \rangle$

has dimension at most $r-2$. Since $|R|$ is not composed with $|L|$, for a general $P \in X$ there are divisors $E_P \in |R|$ and $F \in |L|$ with $P \in E_P$, $P \in F$ and $deg(F) - deg(F \cap E_P) \geq 2$. By the generality of P we have $\dim(\langle E_P \rangle) \leq r-2$. Set $F_P := F - (E_P \cap F)$ and tale $Q \in F_P$. Since $h^0(X, M \otimes L^*) = r-1$, a general hyperplane H in $\mathbf{P}^r$ containing Q and $\langle E_P \rangle$ contains F. Hence for every $B \in |M \otimes R^*|$ with $B \geq Q$ we find $B \geq F_P$. Hence a general hyperplane in $\mathbf{P}^r$ containing B contains F. Since $|R|$ is spanned, this implies $F \subset B$. Since Q is general on X, we find $|B|$ composed with $|L|$, i.e. $|B| = t|L| + D$ with D fixed divisor of $|B|$. Hence $t = \dim(|M \otimes R^*|) = \dim(|M \otimes R^*(-D)|)$. Thus $h^0(X, O_X(D)) = 1$ and D is a fixed divisor of $|L^{\otimes t}(D)|$. We have $R \cong M \otimes (L^*)^{\otimes t}(-D)$, i.e. we are in case *(ii)*.

□

DEFINITION 5. *Let (M, L) be a Maroni pair and F a subsheaf of M spanned by $H^0(X, F)$. Let N be the subsheaf of $M \otimes F^*$ spanned by $H^0(X, M \otimes F^*)$. Hence there is an effective divisor (or $\emptyset$) with $N(D) \cong M \otimes F^*$ and $h^0(X, N) = h^0(X, M \otimes F^*)$. The line bundle $M \otimes N^* \cong F(D)$ will be called the primitive hull of F in M. We have $deg(M \otimes N^*) = deg(F) + deg(D)$. If $F = M \otimes N^*$ we will say that F is primitive in M.*

Motivated by Theorem 2 we raise the following questions:

(Q1) What are the integers $h^0(X, L^{\otimes t})$ for $2 \leq t \leq s(L, M)$, i.e. what are the part of the scrollar invariants of L related to M ?

(Q2) For what integers t with $1 \leq t \leq s(L, M)$ is $L^{\otimes t}$ primitive in M ? If $L^{\otimes t}$ is not primitive, what is the degree of its primitive hull in M, i.e. what is the degree of the base locus of the linear system $|M \otimes L^*)^{\otimes t}|$?

First assume h_M birational. For questions (Q1) and (Q2) a key integer is the Maroni invariant m of the pair (M, L) (Remark 6). If h_M is not birational, we may use (‡) below to reduce questions (Q1) and (Q2) (at least if $char(\mathbf{K}) = 0$) to the case in which h_M is birational.

(‡) Assume $char(\mathbf{K}) = 0$. We always assume $k \geq 3$, because if X is hyperelliptic everything is obvious. Let (M, L) be a Maroni pair on X such that the morphism h_M is not birational. Thus there is (C, α, M') with C smooth curve, α: $X \to C$ morphism with $deg(\alpha) > 1$ and $M' \in Pic(C)$ with $M \cong \alpha^*(M')$ and $\alpha^*(H^0(C, M')) = H^0(X, M)$. Thus M' is spanned, $h^0(C, M') = h^0(X, M)$ and $deg(M') = deg(M)/deg(\alpha)$. In general the pair (C, α) is not unique, up to an automorphism of C, unless $deg(h_M)$ is prime, but it is unique if we impose that α does not factor in a non-trivial way through a smooth curve. We assume this condition; in particular in general we do not take as C the normalization of $h_M(X)$. There is a push-forward map $\alpha_! : Pic(X) \to Pic(C)$ with $deg(\alpha_!(R)) = deg(R)$ for every $R \in Pic(X)$; the map $\alpha_!$ is defined sending $O_X(P)$ into $O_C(\alpha(P))$ and then using additivity; the map defined in this way preserves linear equivalence of divisors because $Pic^0(C)$ is an abelian variety and hence there is no non-constant rational map from $\mathbf{P}^1$ into $Pic^0(C)$; hence the map between divisors defined in this way induces a map $\alpha_! : Pic(X) \to Pic(C)$. Call N the subsheaf of $M \otimes L^*$ spanned by $H^0(X, M \otimes L^*)$. Since $H^0(X, N)$ may be seen as a subspace of $H^0(X, M)$, h_N factors through α, i.e. there exists $N' \in Pic(C)$ with $N \cong \alpha^*(N')$ and $\alpha^*(H^0(C, N')) = H^0(X, N)$. Thus N'

is spanned, $h^0(C, N') = h^0(X, M \otimes L^*)$ and $deg(N') = deg(N)/deg(\alpha)$.

(i) First assume that for a general $A \in |L|$ we have $card(\alpha(A)) = k$. For every $R \in Pic(X)$ we have $h^0(X, R) \le h^0(C, \alpha_!(R))$ because the map inducing $\alpha_!$ preserves linear equivalence. If R is spanned, then $\alpha_!(R)$ is spanned because for every $Q \in C$ there is $D \in |R|$ not intersecting $\alpha^{-1}(Q)$. Set $L' := \alpha_!(L) \in Pic^k(C)$. Take a general $A \in |L|$, say $A = P_1 + \ldots + P_k$. Since $h^0(X, M \otimes L^*) < h^0(X, M) - 1$, there is $D \in |M|$ containing P_1 but not P_2. Since every $D \in |M|$ containing $\{P_1, P_2\}$ contains every P_i with $i \ge 3$ and $\alpha^*(H^0(C, M')) = H^0(X, M)$, every $D' \in |M'|$ containing $\{\alpha(P_1), \alpha(P_2)\}$ contains every $\alpha(P_i)$, $i \ge 3$. Thus L' is spanned and $h^0(C, M' \otimes L'^*) = h^0(C, M') - 2$.

(i1) Here we assume $h^0(C, L') = 2$. Thus (M', L') is a Maroni pair on C with $deg(M') = deg(M)/deg(\alpha)$ and $deg(L') = deg(L)$. We may work on C instead of working on X.

(i2) Here we assume $h^0(C, L') \ge 3$. Fix a general $D \in |L|$ and set $\alpha(D) \in |L'|$. Since $h^0(X, M) = h^0(C, M')$, we have $h^0(X, M(-D)) = h^0(C, M'(-D'))$. Thus $h^0(C, M') = r + 1$, $h^0(C, M' \otimes L'^*) = r - 1$ and M', L' are spanned. By Remark 1 we have $h^0(C, L') = 3$ and there is $R \in Pic(C)$, R spanned, with $h^0(C, R) = 2$, $L' \cong R^{\otimes 2}$ and $M' \cong R^{\otimes r}$. Set $T := \alpha^*(R)$. Thus $M \cong T^{\otimes r}$. Since $h^0(X, M) = r + 1$ and $r \ge 2$ we have $h^0(X, T^{\otimes 2}) = 3 = h^0(C, R^{\otimes 2})$. Since $h^0(X, M) = h^0(C, M')$ and the morphism associated to $|M'|$ is induced by the composition of the morphism $C \to \mathbf{P}^1$ induced by $|R|$ and of the degree r Veronese embedding $v_r : \mathbf{P}^1 \to \mathbf{P}^r$, we have $h_M = v_r \circ h_T$. By construction for every $E \in |R^{\otimes 2}|$ we have $\alpha^{-1}(E) \in |T^{\otimes 2}|$. Hence there is an effective divisor B with $L(B) \cong T^{\otimes 2}$. Since $h^0(X, L) = 2$, we have $B \ne \emptyset$. Since L is spanned and the map induced by $|T^{\otimes 2}|$ is the composition of h_T and the degree 2 Veronese embedding $v_2 : \mathbf{P}^1 \to \mathbf{P}^2$, every fiber of h_T is a fiber of h_L. Since $B \ne \emptyset$, this implies $L = T$, contradicting the assumption $card(\alpha(A)) = k$ for a general $A \in |L|$.

(ii) Now assume that for a general $A \in |L|$ we have $card(\alpha(A)) < k$. Since $char(\mathbf{K}) = 0$, by [1], Th. 4.1, the morphisms h_M and h_L factor through a common covering. Indeed, taking any such covering and then applying again the monodromy argument in [1] and the minimality condition of α, we see easily that h_L factors through α, i.e. we see the existence of a spanned $L' \in Pic(C)$ with $\alpha^*(L') = L$ and $\alpha^*(H^0(C, L')) = H^0(X, L)$. Hence $h^0(C, L') = 2$. Thus (M', L') is a Maroni pair on C with $deg(M') = deg(M)/deg(\alpha)$ and $deg(L') = deg(L)/deg(\alpha)$. Again, we may work on C with respect to the Maroni pair (M', L').

3. Non-primitive line bundles

At the end of this section we will prove Theorem 1 and hence Corollary 1. First, we will consider the following problem. Let X be a smooth projective curve and fix $L \in Pic(X)$ with L spanned and $h^0(X, L) = 2$. Study the pairs (s, D) with s positive integer, D effective divisor on X, $h^0(X, L^{\otimes s}) = s + 1$ and $h^0(X, L^{\otimes s}(D)) \ge s + 2$. Taking D minimal we may (and will) even assume $h^0(X, L^{\otimes s}(D)) = s + 2$ and that $L^{\otimes s}(D)$ is spanned. If $D \in |L|$ this is just a question on the scrollar invariants of L. If $0 < deg(D) < deg(L)$ and X is a general k-gonal curve with $k := deg(L)$, this question was solved in [4], Prop. 1.1, and we were inspired by that result to start our study of some more general cases.

(††) We fix any such X, L, s and D and set $k := deg(L)$, $b := deg(D) > 0$. For all integers t with $1 \le t \le s$ the morphism $h_{L^{\otimes t}}$ has degree k and $h_{L^{\otimes t}}(X)$ is a rational normal curve of $\mathbf{P}^t$. Fix a rational normal curve A of $\mathbf{P}^s$ and see $\mathbf{P}^s$ as a hyperplane of $\mathbf{P}^{s+1}$. Fix $\mathbf{v} \in (\mathbf{P}^{s+1} \backslash \mathbf{P}^s)$ and let T be the cone with vertex $\mathbf{v}$ and base A; for $s = 1$ we have a degenerate (simpler) situation because $T \cong \mathbf{P}^2$. Set $B := h_{L^{\otimes t}(D)}(X)$. By assumption either the morphism $h_{L^{\otimes t}(D)}$ is birational or it factors through a curve of genus $q > 0$. We will always assume $h_{L^{\otimes t}(D)}$ birational; the proof of (‡) at page 9 may be useful when $h_{L^{\otimes t}(D)}$ is not birational. Thus $B \subset \mathbf{P}^{s+1}$ is a curve of degree $sk + b$ with $B \subset T$ and B has multiplicity b at $\mathbf{v}$. Let $\pi : F_s \to T$ be the blowing - up of T at $\mathbf{v}$; indeed, F_s is isomorphic to the Hirzebruch surface with invariant s considered in Remark 4; this is true even in the case $s = 1$ in which case $T \cong \mathbf{P}^2$. With the notation of Remark 4 we have $A \cong \pi^-1(A) \in |h + sf|$. Call C the strict transform of B in F_s. Thus $C \in |kh + (ks + b)f|$. X is isomorphic to the normalization of C because we assumed that $h_{L^{\otimes t}(D)}$ is birational. The pencil $|f|$ induces L. Viceversa, the normalization, X', of any integral curve $C \in |kh + (ks + b)f|$ gives a solution (X', L') with $L' \in Pic(X)$, $deg(L') = k$ and L' spanned if the following three conditions are satisfied:

(C1) $h^0(X', L') = 2$;

(C2) $h^0(X', L'^{\otimes s}) = s + 1$;

(C3) call D' the degree b effective divisor of X' induced by the length b scheme $h \cap C$; we have $h^0(X', L'^{\otimes s}(D')) = s + 2$.

Remark 12. Obviously (C2) implies (C1). Since $|kh + (ks + b)f|$ is spanned, $L'^{\otimes s}(D')$ is spanned. Thus if $h^0(X', L'^{\otimes s}) = s + 1$ and $h^0(X', L'^{\otimes s}(D')) = s + 2$, then for every effective divisor D'' strictly contained in D' we have $h^0(X', L'^{\otimes s}(D'') = s+1$.

Remark 13. Take the set-up and notation of Remark 4 and (††). By Remark 4 we have $p_a(C) = 1 + k^2s/2 - sk/2 - k + bk - b$. Thus $g \le 1 + k^2s/2 - sk/2 - k + bk - b$ and $g = 1 + k^2s/2 - sk/2 - k + bk - b$ if and only if C is smooth.

Remark 14. We use the notation of (††). The curve $B = \pi(C)$ has multiplicity b at $\mathbf{v}$. Thus B may be nodal only if $b \le 2$. If $b = 0$, then $\mathbf{v} \notin B$. If $b = 1$, then $\mathbf{v} \in B_{reg}$. If $b = 2$ the curve B has an ordinary node at $\mathbf{v}$ if and only if C intersects transversally h. As we will see in Remark 15, this is the general case.

Remark 15. Let $Y \subset F_s$ be the general union of k general curves of type $|h + sf|$ and b general fibers of the ruling. Thus Y is a nodal curve and $card(Sing(Y)) = 1 + k^2s/2 - sk/2 - k + bk - b + k + b - 1$. For every irreducible component, D, of Y we have $Y \cdot \omega_{F_s} < 0$. Hence we may apply [12], Prop. 2.11, and obtain the following result. Fix an integer g with $0 \le g \le 1 + k^2s/2 - sk/2 - k + bk - b$ and $S \subset Sing(Y)$ with $card(S) = 1 + k^2s/2 - sk/2 - k + bk - b - g$ and such that for each irreducible component D of Y we have $(Sing(Y) \backslash S) \cap D \neq \emptyset$. Then we assign the nodes in S and smooth the nodes of $Sing(Y) \backslash S$. As a general such smoothing we obtain a nodal curve C with geometric genus g. Since every irreducible component of Y intersects S, C is irreducible. Thus for any admissible numerical datum (k, b, g) we may find a nodal irreducible $C \in |kh + (ks + b)f|$ whose normalization has genus g. Since Y is transversal to h, the general partial smoothing, C, of Y is tranversal to h; if $b \geq 4$ we may even assume that $h \cap C$ is formed by b general points of $h \cong \mathbf{P}^1$. Thus the image $B \subset T$ of such curve has an ordinary point of multiplicity b at $\mathbf{v}$. Instead of Y we may take the nodal curve $T' = T_1 \cup \ldots \cup T_k$ with $T_i \in |h+sf|$ for $i < k$, $T_k \in |h+(s+b)f|$ and each T_j general.

(*) Here we will compute $h^0(C, O_C(f))$ and $h^0(C, O_C(sf))$ in most interesting cases. To study $h^0(X', L'^{\otimes s}(D'))$ we will compute $h^0(C, O_C(h+sf))$. Since $O_C(h+sf) \cong \pi^*(O_B(1))$, $h^0(C, O_C(h+sf)) = s+2$ if and only if the morphism $C \to \mathbf{P}^{s+1}$ with image B is induced by a complete linear system of C. We are really interested to study the corresponding problem for X' and the case of C will be only an intermediate step. Twisting by $O_{F_s}(f)$ (resp. $O_{F_s}(sf)$ the exact sequence

$$0 \to O_{F_s}(-kh-(ks+b)f) \to O_{F_s} \to O_C \to 0 \tag{2}$$

we obtain $h^0(C, O_C(f)) = 2$ (resp. $h^0(C, O_C(f)) = s+1$) if and only if $h^1(F_s, O_{F_s}(-kh -(ks+b-1)f)) = 0$ (resp. $h^1(F_s, O_{F_s}(-kh-(ks+b-s)f)) = 0$). Hence by Remark 4 we obtain $h^0(C, O_C(f)) = 2$ (resp. $h^0(C, O_C(f)) = s+1$) if and only if $ks+b-3-s-s(k-2) \geq -1$ (resp. $ks+b-2-2s-s(k-2) \geq -1$). Thus we have $h^0(C, O_C(f)) = 2$ if either $s \geq 2$ or $s = 1$ and $b > 0$, while $h^0(C, O_C(f)) = s+1$ if and only if $b > 0$. Twisting (2) by $O_{F_s}(h+sf)$ and using Serre duality we obtain $h^0(C, O_C(h+sf)) = h^0(F_s, O_{F_s}(h+sf))$ (i.e. $h^0(C, O_C(h+sf)) = s+2$) if and only if $h^1(F_s, O_{F_s}((k-3)h+(ks+b-2-2s)f)) = 0$, i.e. by Remark 4 if and only if $ks+b-2-2s \geq s(k-3)-1$. This inequality is always satisfied.

(**) We fix the integers g, s, k and b. Take an integral nodal curve $C \subset F_s$ with $C \in |kh+(ks+b)f|$, $k \geq 3$, and normalization, X', of genus g. The existence of such curve follows from [12], Remark 3.2. Obviously, we need to assume $g \leq p_a(C)$, i.e. $g \leq 1+k^2s/2-sk/2-k+bk-b$. We assume $h^0(C, O_C(f)) = 2$, $h^0(C, O_C(sf)) = s+1$ and $h^0(C, O_C(h+sf)) = s+2$, i.e. by (*) we assume $b > 0$. Let L' be the degree k spanned line bundle associated to $|f|$. As in the proof of Propositions 2 and 3 we may describe when (C1), (C2) and (C3) are satisfied for the pair (X', L'). For Condition (C1) (resp. (C2), resp. (C3)) we will assume $g \geq k+3$ (resp. $g \geq sk-s+1$, resp. $g \geq sk-s+b$) because we look for pairs (X', L') or triples (X', L', D') with $h^1(X', L') \geq 2$ (resp. $h^1(X', L'^{\otimes s}) > 0$, resp. $h^1(X', L'^{\otimes s}(D')) > 0$). We may repeat the proof of Propositions 2 and 3 just taking $m = s$, $x = ks+b$ and $S := F_s$. Condition (C1) (resp. (C2), resp. (C3)) is equivalent to $h^1(F_s, I_{Sing(C)}((k-2)h+(ks+b-3-s)f)) = 0$ (resp. $h^1(F_s, I_{Sing(C)}((k-2)h+(ks+b-2-2s)f)) = 0$, resp. $h^1(F_s, I_{Sing(C)}((k-3)h+(ks+b-2-2s)f)) = 0$). Thus for (C1) (resp. (C2), resp. (C3)) it is necessary to have $card(Sing(C)) \leq h^0(F_s, O_{F_s}((k-2)h+(ks+b-3-s)f))$ (resp. $h^0(F_s, O_{F_s}((k-2)h+(ks+b-2-2s)f))$, resp. $h^0(F_s, O_{F_s}((k-3)h+(ks+b-2-2s)f))$) and these conditions are sufficient if C is a general partial smoothing inside F_s of a rational nodal curve (proof of Propositions 2 and 3 and semicontinuity). We have $card(Sing(C)) = 1+k^2s/2-sk/2-k+bk-b-g$. By Remark 4 we have $h^0(F_s, O_{F_s}((k-2)h+(ks+b-3-s)f)) = (k-1)(ks+2b-4)/2$ since $ks+b-3-s \geq s(k-2)-1$. By Remark 4 we have $h^0(F_s, O_{F_s}((k-2)h+(ks+b-2-2s)f)) = (k-1)(ks+2b-2-2s)/2 if ks+b-2-2s \geq s(k-2)-1$. By Remark 4 we have $h^0(F_s, O_{F_s}((k-3)h+(ks+b-2-2s)f)) = (k-2)(ks+2b-2-s)/2$. Hence to have (C1) (resp. (C2), resp. (C3)) it is sufficient to assume $g \geq k+3$ (resp. $sk-s+2$, resp. $sk-s+3+b$). Notice that the condition $card(Sing(C)) \leq$

$h^0(F_s, O_{F_s}((k-2)h+(ks+b-3-s)f)) = (k-1)(ks+2b-4)/2$ (resp. $card(Sing(C)) \le h^0(F_s, O_{F_s}((k-2)h+(ks+b-2-2s)f)) = (k-1)(ks+2b-2-2s)/2$, resp. $card(Sing(C)) \le h^0(F_s, O_{F_s}((k-3)h+(ks+b-2-2s)f)) = (k-2)(ks+2b-2-s)/2$ is a necessary condition for (C1) (resp. (C2), resp. (C3)) for any integral curve with only ordinary nodes and ordinary cusps as singularities. Taking $length(Sing(C))$ instead of $card(Sing(C))$ and a suitable scheme-structure for $Sing(C)$ (the adjoint ideal as ideal sheaf) such conditions are necessary for any integral curve which is a flat limit of curves with only ordinary nodes and ordinary cusps as singularities and with the same geometric genus.

Remark 16. Take (X', L') arising as in (††) from a curve $C \in |kh + (ks+1)f|$, i.e. with $b = 1$, and satisfying Conditions (C1) and (C2). Thus $deg(D') = 1$. Since $|kh+(ks+1)f|$ is base point free, $L'^{\otimes s}(D')$ is spanned. Since D' is effective, $deg(D') = 1$ and $h^0(X', L'^{\otimes s}) = s+1$ (Condition (C2)) we have $h^0(X', L'^{\otimes s}(D')) = s+2$, i.e. Condition (C3) holds.
Taking $b = 1$ in (**) we obtain the following existence theorem.

Corollary 2. *Fix integers g, k, s with $s \ge 1$, $k \ge 3$ and $sk - s + 2 \le g \le k^2s/2 - sk/2$. Then there exist a smooth curve X with genus g and $L \in Pic^k(X)$ with $h^0(X, L) = 2$, $h^0(X, L^{\otimes t}) = t+1$ for every integer t with $1 \le t \le s$, L spanned and $L^{\otimes s}$ not primitive.*

Proof. By (**) we need only to check the condition $h^1(F_s, I_{Sing(C)}((k-3)h + (ks + 1 - 2s)f)) = 0$. This can be done as in the proof of Proposition 1 taking as C a partial smoothing with geometric genus g of a rational nodal curve $D \in |kh + (ks+1)f|$. □

Remark 17. Take L as in Corollary 2, i.e. with $L^{\otimes s}$ is not primitive, and $P \in X$ such that $h^0(X, L^{\otimes s}(P)) \ge s+2$. Since $deg(L) > 1$ and $L^{\otimes(s+1)}$ is spanned, we have $h^0(X, L^{\otimes(s+1)}) \ge s+3$.

Proof of Theorem 1. We divide the proof into two steps.
Step 1). Fix an integer a with $s < a \le 2s+2$ and $g \le ak - a - k + 1$. Fix an integer w with $0 < w < k$. Assume the existence of an integral nodal curve $Y \subset \mathbf{P}^1 \times \mathbf{P}^1$ of type (k, a) such that the set $S := Sing(Y)$ has the following properties. There are w lines of type $(1,0)$ on $\mathbf{P}^1 \times \mathbf{P}^1$, say $L_1, \dots, L_w$, such that $card(L_i \cap S) = a - 1 - s$, while S is "sufficiently general with this restriction"; more precisely, setting $S'' := S \backslash S \cap L_1 \cup \dots \cup L_w)$, we will need $h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_{S''}(k-2-w, a-2-s)) = 0$; in particular, calling g the geometric genus of Y, we need $0 \le ak - a - k + 1 - g - w(a-1-s) = card(S'') \le (k-1-w)(a-1-s)$, i.e. we need

$$w(a-1-s) \le ak - a - k + 1 - g \le (k-1)(a-1-s). \tag{3}$$

The second inequality in (3) is equivalent to the inequality $g \ge s(k-1)$. Hence both inequalities in (3) are assumed to be satisfied in the set-up of Theorem 1. Since no L_i is contained in Y, we need $2(a-1-s) \le a$, i.e. here we use the condition $a \le 2s+2$. Let X be the normalization of Y. Since $h^1(\mathbf{P}^1 \times \mathbf{P}^1, O_{\mathbf{P}^1 \times \mathbf{P}^1}) = 0$, the canonical divisor of $\mathbf{P}^1 \times \mathbf{P}^1$ has type $(-2,-2)$ and Y is nodal, the complete canonical system of X is induced by $H^0(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2))$. We claim that, with the assumption for S'' just introduced, we have $h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2-y)) = 0$ for $0 \le y \le s$, but $H^0(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2-s))$ has $L_1 \cup \ldots \cup L_w$ as base locus. To check the first part of the claim it is sufficient to check it for $y = s$, i.e. it is sufficient to prove $h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2-s)) = 0$. Since $card(S \cap L_i) = a - 1 - s$, each L_i is in the base locus of $H^0(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2-s))$ and $h^0(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2-s)) = H^0(\mathbf{P}^1 \times \mathbf{P}^1, I_{S''}(k-2-w, a-2-s))$. Since $h^0(\mathbf{P}^1 \times \mathbf{P}^1, O_{\mathbf{P}^1 \times \mathbf{P}^1}(k-2, a-2-s)) = h^0(\mathbf{P}^1 \times \mathbf{P}^1, O_{\mathbf{P}^1 \times \mathbf{P}^1}(k-2-w, a-2-s)) + w(a-1-s)$ and $card(S \backslash S'') = w(a-1-s)$, we have $h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_S(k-2, a-2-s)) = h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_{S''}(k-2-w, a-2-s))$. Obviously we need $w \leq k-1$ and $card(S'') \leq (k-1-w)(a-1-s)$. Viceversa, if $card(S'') \leq (k-1-w)(a-1-s)$, then for a general S'' we have $h^1\mathbf{P}^1 \times \mathbf{P}^1, I_{S''}(k-2-w, a-2-s)) = 0$. Thus we obtain $h^1(X, L^{\otimes y}) = g - yk + y$ for $0 \leq y \leq s$, i.e. $h^0(X, L^{\otimes y}) = y+1$, but $H^0(X, \omega_X \otimes (L^{\otimes s})^*)$ has base locus containing the $w(s+1)$ counterimages in X of the points $Y_{reg} \cap (L_1 \cup \ldots \cup L_w)$.

Step 2). Here we will prove the existence of an integral nodal curve Y as in Step 1 for parameters g, k, s, a and w. By [12], Remark 3.2, for every triple (α, β, γ) of integers with $\alpha > 0$, $\beta > 0$, and $0 \leq \gamma \leq \alpha\beta - \alpha - \beta + 1$ there exists an irreducible nodal curve $Z \subset \mathbf{P}^1 \times \mathbf{P}^1$ of type (α, β) and with geometric genus γ, i.e. with exactly $\alpha\beta - \alpha - \beta + 1 - \gamma$ ordinary nodes as singularities. Furthermore we may find such curve which is transversal to $L_1 \cup \ldots \cup L_w$ and to any fixed in advance reduced curve. Take smooth curves Z, W with Z of type $([k/2], [(a+1)/2])$, W of type $([(k+1)/2], [a/2])$ and Z intersecting transversally W. We assume that both Z and W intersect transversally $L_1, \ldots, L_w$ but that they have exactly $a - 1 - s$ common points on each L_i, $1 \leq i \leq w$; we will discuss at the end of the proof when this is possible. Let $\mu : \Pi \to \mathbf{P}^1 \times \mathbf{P}^1$ be the blowing - up of the $w(a-1-s)$ points $Z \cap W \cap L_1 \cup \ldots \cup L_w)$ and let Z' (resp. W') be the strict transform of Z (resp. W) in Π. Call E_j, $1 \leq j \leq w(a-1-s)$, the exceptional divisors of μ. Since $a \leq 2s+2$ and $w < k$, we have $w(a-1-s) < [k/2][a/2] + [(k+1)/2][(a+1)/2]$. Since $card(Z \cap W) = [k/2][a/2] + [(k+1)/2][(a+1)/2] > w(a-1-s)$, $Z' \cup W'$ is a connected nodal curve with $Sing(Z' \cup W') = Z' \cap W'$. We take a partial smoothing of $Z' \cup W'$ in which we smooth $ak - a - k - g - w(a-1-s)$ nodes of $Z' \cup W'$ and call Y' the general such curve obtained in this way; here we use $card(Sing(Z' \cap W')) > ak - a - k - g - w(a-1-s)$, i.e. $[k/2][a/2] + [(k+1)/2][(a+1)/2] > ak - a - k - g$, to obtain a connected curve; this inequality is satisfied under the assumptions of Theorem 1 because $a \leq 2s+2$ and $g \geq s(k-1)$; here to apply [12], 2.11, we need $\omega_\Pi \cdot Z' < 0$ and $\omega_\Pi \cdot W' < 0$, i.e. $\omega_{\mathbf{P}^1 \times \mathbf{P}^1} \cdot Z < -w(a-1-s)$ and $\omega_{\mathbf{P}^1 \times \mathbf{P}^1} \cdot W < -w(a-1-s)$, i.e. $2([(k+a)/2] > w(a-1-s)$. Set $Y := \mu(Y')$. Since $Y' \cdot E_j = 2$ for every j and Y' intersects transversally every exceptional divisor E_j (for general Y' near $Z' \cup W'$), Y' has exactly $ak - a - k + 1 - g$ ordinary nodes and $Z \cap W \cap L_1 \cup \ldots \cup L_w) \subseteq Sing(Y)$.

Now we discuss the condition "$card(Z \cap W \cap L_i)) = a - 1 - s$ for all integers i with $1 \leq i \leq w$". Take Z as above and intersecting transversally $L_1 \cup \ldots \cup L_w$. We fix a set $S \subset Z \cap (L_1 \cup \ldots \cup L_w)$ with $card(S \cap L_i) = a - 1 - s$ for every i. Call (α, β) the type of Z; we have $\alpha = [k/2]$, but to handle W we will need to consider also the che case $\alpha = [(k+1)/2]$. Since $O_Z(Z) \cong O_Z(\alpha, \beta)$ is the normal bundle of Z in $\mathbf{P}^1 \times \mathbf{P}^1$, the subscheme $Hilb(\mathbf{P}^1 \times \mathbf{P}^1, S)$ of the Hilbert scheme $Hilb(\mathbf{P}^1 \times \mathbf{P}^1)$ of $\mathbf{P}^1 \times \mathbf{P}^1$ formed by the curves containing S has tangent space $H^0(Z, O_Z(\alpha, \beta)(-S))$ at Z and $Hilb(\mathbf{P}^1 \times \mathbf{P}^1, S)$ is smooth at Z if $H^1(Z, O_Z(\alpha, \beta)(-S)) = 0$ ([10], 1.4). The exact sequence

$$0 \to O_{\mathbf{P}^1 \times \mathbf{P}^1}(-t, 0) O_{\mathbf{P}^1 \times \mathbf{P}^1}(\alpha - t, \beta) \to O_Z(\alpha - t, \beta) \to 0 \tag{4}$$

shows that we have

$$h^0(Z, O_Z(\alpha - t, \beta)) = h^0(\mathbf{P}^1 \times \mathbf{P}^1, O_{\mathbf{P}^1 \times \mathbf{P}^1}(\alpha - t, \beta)) = h^0(Z, O_Z(\alpha, \beta)) - t(\beta + 1)$$

and

$$h^1(Z, O_Z(\alpha - t, \beta)) = 0$$

if $0 < t \leq \alpha + 1$. Since $S \subset L_1 \cup \ldots \cup L_w$, we have $H^1(Z, O_Z(\alpha, \beta)(-S)) = 0$ if $w \leq \alpha + 1$, i.e. if $w \leq [k/2] + 1$. Furthermore, if $H^1(Z, O_Z(\alpha, \beta)(-S)) = 0$, then moving the curve Z in $Hilb(\mathbf{P}^1 \times \mathbf{P}^1)$ we obtains curves near Z which intersect $L_1 \cup \ldots \cup L_w$ in a subset near S and formed by $a - 1 - s$ general points of each L_i; we stress: if $H^1(Z, O_Z(\alpha, \beta)(-S)) = 0$ we obtain in this way $w(a - 1 - s)$ general points of $L_1 \cup \ldots \cup L_w$ with the only restriction that each L_i contains $a - 1 - s$ of these general points. We just checked that the condition $H^1(Z, O_Z(\alpha, \beta)(-S)) = 0$ is satisfied if $w \leq [k/2] + 1$. We do the same for W. Since we just checked that $H^1(Z, O_Z(Z)(-S)) = H^1(Z, O_W(W)(-S)) = 0$ if $w \leq [k/2] + 1$, we obtain Z and W with $card(Z \cap W \cap L_i)) = a - 1 - w$ for all integers i with $1 \leq i \leq w$. Furthermore, again by [10], 1.5, we find such S, Z and W with Z transversal to W. Remember that to prove Theorem 1 it is sufficient to show the existence of C as in Step 1 with $h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_{S''}(k - 2 - w, a - 2 - s)) = 0$. We may work in Π and hence use again [12], §2, i.e. Severi theory of partial smoothing with assigned and unassigned nodes because $\omega_\Pi \cdot Z' < 0$ and $\omega_\Pi \cdot W' < 0$. Hence by semicontinuity it is sufficient to show the existence of $B \subseteq (Z \cap (W \backslash (L_1 \cup \ldots \cup L_w))$ with $card(B) = ak - a - k + 1 - w(a - 1 - s) - g$ and $h^1(\mathbf{P}^1 \times \mathbf{P}^1, I_B(k - 2 - w, a - 2 - s)) = 0$. Now we fix W but not Z. The restriction map $H^0(\mathbf{P}^1 \times \mathbf{P}^1, O_{\mathbf{P}^1 \times \mathbf{P}^1}([(k+1)/2], [a/2]))) \to H^0(W, O_W([(k+1)/2], [a/2])))$ is surjective and hence for every integer $b < ([k/2] + 1)([(a + 1)/2] + 1) - w(a - 1 - s)$ and any general $B \subset W$ with $card(B) = b$ there is Z containing $B \cup S$. Since W is not in the base locus of $H^0(\mathbf{P}^1 \times \mathbf{P}^1, I_B(k - 2 - w, a - 2 - s))$, we conclude.

□

Remark 18. We believe that the assumption $ak - a - k + 1 - g < ([k/2] + 1)([(a+1)/2] + 1)$ in the statement of Theorem 1 can be weakend with small variations of our construction. This was not necessary to obtain Corollary 1 (and hence the ubiquity of non-primitive linear series) because in Corollary 1 there is the assumption $g \geq s(k - 1) + 1$ which by Riemann - Roch is quite natural if one look for pairs (X, L) with $h^1(X, L^{\otimes s}) \neq 0$ and $h^0(X, L^{\otimes s}) = s + 1$. The same types of inequalities (say $g \geq s(k - 1)$ or $g \geq s(k - 1) + \epsilon$ with ϵ small) are not sufficient to carry over (without any other assumptions) the last part of our proof of Theorem 1 if a is very near to $2s + 2$.

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