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TOWARDS AN ANALYTICAL MECHANICS OF DISSIPATIVE MATERIALS

Abstract. A Lagrangian-Hamiltonian variational formulation is proposed for the thermoelasticity of heat conductors and its generalization to anelasticity - described by means of internal-state variables- by using a gauge-theoretical technique (introduction of an additional variable of state - the gradient of thermacy - that renders the system apparently Hamiltonian). Projecting the equations resulting from the Euler-Lagrange equations and the equations deduced from the application of Noether's theorem back on the original space provides all local balance equations of the dissipative theory, including the entropy equation and the equation of canonical momentum in material space (which are not strict conservation laws). A canonical structure clearly emerges for the anelasticity of conductors in finite strains.

1. Introduction

A recurrent dream of many mathematical physicists is to construct a variational formulation for all field equations of continuum physics *including in the presence of dissipative effects.* We all know that this is not possible unless one uses special tricks such as introducing complex-valued functions and adjoint fields (e.g., for heat conduction). But we do present here a variational and canonical formulation for the nonlinear continuum theory of *thermoelastic conductors* and then generalize this to the case of anelastic conductors of heat. This is made possible through the introduction of a rather old notion, clearly insufficiently exploited, that of *thermacy* introduced by Van Danzig (cf. [9]), a field of which the time derivative is the thermodynamical temperature. It happens that we used such a notion in relativistic studies in the late 60s-early 70s, (Pre-general exam Seminar at Princeton University, Spring 1969; [10], [11]), a time at which we found that thermacy is nothing but the *Lagrange multiplier* introduced to account for *isentropy* in a Lagrangian variational formulation. But , completely independently and much later, Green and Naghdi ([4]) formulated a strange "thermoelasticity without dissipation". Dascalu and I ([1]) identified thermacy as the unknowingly used notion by Green and Naghi (unaware of works in relativistic variational formulations), and we formulated the corresponding *canonical balance laws of momentum and energy* - of interest in the design of fracture criteria - which, contrary to the expressions of the classical theory, indeed present *no* source of dissipation and canonical momentum, e.g., no thermal source of quasi-inhomogeneities (cf. [2]). In recent works ([14], [21]), we have shown the consistency between the expressions of intrinsic dissipation and source of canonical momentum in dissipative continua. This is developed within the framework of so-called *material* or *configurational forces*, "Eshelbian mechanics", that world of forces which, for instance, drive structural rearrangements and material defects of different types on the material

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manifold (for these notions see [12], [13], [5] and [7]). The road to the analytical continuum mechanics was explored in particular by P.Germain (1992) in [3], but not in a variational framework.

Herebelow we first present a consistent variational formulation for thermoelastic conductors of heat, which, with the use of Noether's theorem, delivers all equations of interest, that is, the balance of linear momentum, the equation of entropy, the balance of canonical momentum, and the energy equation, all in the apparently"dissipationless form". But these equations can be transformed to those of the classical theory of (obviously thermally dissipative) nonlinear elastic conductors (cf. [20]). Therefore, we have a good starting point for a true canonical formulation of dissipative continuum mechanics. The possible extension of the formulation to anelastic conductors of heat is also presented when the anelastic behavior is accounted for through the introduction of internal variables of state. Elements of the present work were given in a paper by Kalpakides and Maugin ([6]).

2. Direct Variational formulation and its results.

We use classical elements of field theory as enunciated in several books (e.g. [12], [14], [21]). The reader is referred to these works for the abstract equations.

Consider Hamiltonian-Lagrangian densities (per unit volume of the undeformed configuration K_R of nonlinear continuum mechanics given by the following general expression:

$$L = \bar{L}\,(\mathbf{v}, \mathbf{F}, \theta, \beta; \mathbf{X}) = K\,(\mathbf{v}; \mathbf{X}) - W\,(\mathbf{F}, \theta, \beta; \mathbf{X})\,, \tag{1}$$

where

$$K(\mathbf{v}; \mathbf{X}) = \frac{1}{2}\rho_0\,(\mathbf{X})\,\mathbf{v}^2, \quad \theta = \dot{\gamma}, \quad \beta = \nabla_{\mathbf{R}}\gamma.$$

Here, K is the kinetic energy, W is the free energy density, ρ_0 is the mass density at the reference configuration, a superimposed dot denotes time differentiation at constant fixed material point $\mathbf{X}$, ∇_R denotes the material gradient, the scalar function β is called the *thermacy*, and $\mathbf{v}$ and $\mathbf{F}$ are the physical velocity and direct deformation gradient such that

$$\mathbf{v} = \left.\frac{\partial\chi}{\partial t}\right|_X, \qquad \mathbf{F} = \left.\frac{\partial\chi}{\partial\mathbf{X}}\right|_t \equiv \nabla_R\chi,$$

if

$$\mathbf{x} = \chi\,(\mathbf{X}, t)\,, \qquad det\,\mathbf{F} > 0,$$

is the smooth placement of $\mathbf{X}$ at Newtonian time t. The explicit dependence of ρ_0 and W on $\mathbf{X}$ indicates material inhomogeneity (direct smooth dependence on the material point $\mathbf{X}$).

In the Lagrangian density (1), the *basic fields* are the *placement* $\mathbf{x}$ and the thermacy γ, both being assumed sufficiently smooth functions of the *space-time parametrization* $(\mathbf{X}, t)$, which is the one favored in the Piola-Kirchhoff formulation of nonlinear continuum mechanics (cf.Truesdell and Noll, 1965). Notice that L is not an explicit function of $\mathbf{x}$ by virtue of Galilean invariance (translations in physical space of placements). Neither is it an explicit function of γ itself, this implying a sort of *gauge invariance* very similar to that of electrostatic for the electric potential. Since the focus is on field equations rather than on boundary conditions and initial conditions, the density (1) may be integrated over a Newtonian space-time volume of infinite extent with proper limit behavior of the various involved functions at infinity in space and at time limits. According to the general field theory, in the absence of external sources (these would be explicit functions of the fields themselves), the **field equations** , i.e, the Euler-Lagrange equations (cf. eqns. (A.7) in Maugin, 1999a) associated with χ and γ, are immediately given by

$$\left.\frac{\partial\,\mathbf{p}}{\partial\,t}\right|_X - div_R\mathbf{T} = \mathbf{0}, \tag{2}$$

$$\left.\frac{\partial\,S}{\partial\,t}\right|_X + \nabla_R.\mathbf{S} = 0, \tag{3}$$

wherein

$$\begin{aligned} \mathbf{p} \,:=\, \rho_0\mathbf{v} = \frac{\partial\,L}{\partial\,\dot{\chi}}, \quad \mathbf{T} \,:=\, \frac{\partial\,W}{\partial\,\mathbf{F}} = -\frac{\partial\,L}{\partial\,(\nabla_R\chi)}, \\ S := -\frac{\partial\,W}{\partial\,\theta} = \frac{\partial\,L}{\partial\dot{\gamma}}, \quad \mathbf{S} \,:= -\frac{\partial\,W}{\partial\,\beta} = \frac{\partial\,L}{\partial\,(\nabla_R\gamma)}, \end{aligned} \tag{4}$$

are, respectively, the *linear momentum vector* in physical space, the *first Piola-Kirchhoff stress*, the *entropy density* (by appealing to the axiom of local state and assuming that entropy density has the same general functional definition as in thermostatics), and, accordingly, the *entropy flux* in material form.

Invoking now Noether's theorem (cf. eqns. (A.11) in Maugin, 1999a) for the Lagrangian (1) with respect to the space-time parametrization $(\mathbf{X}, t)$, we obtain the following two, respectively co-vectorial and scalar, equations:

$$\left.\frac{\partial\,P_L^{th}}{\partial\,t}\right|_X - \left(div_R\mathbf{b}^{th}\right)_L = \left(\mathbf{f}^{inh}\right)_L, \tag{5}$$

and

$$\left.\frac{\partial\,H}{\partial\,t}\right|_X - \nabla_R.\,\mathbf{U} = 0, \tag{6}$$

where we have defined the *canonical momentum* (material-covariant) vector $\mathbf{P}^{th}$ of the present approach, the corresponding *canonical material stress* tensor $\mathbf{b}^{th}$, *the material force* of true inhomogeneities $\mathbf{f}^{inh}$, the *Hamiltonian density* (total energy density) H, and the material Umov-Poynting energy-flux vector $\mathbf{U}$ by [15] (compare the general definitions given in eqns. (A.16), (A.17), (A.14) and (A.15)).

$$\mathbf{P}^{th} = -\nabla_R\chi.\frac{\partial L}{\partial\,\mathbf{v}} - \nabla_R\gamma\,\frac{\partial\,L}{\partial\dot{\gamma}} = -\mathbf{p}.\mathbf{F} - S\beta = \mathbf{P}^{\mathbf{mech}} - \mathbf{S}\beta, \tag{7}$$

$$\begin{aligned} \mathbf{b}^{th} \;:\; &= \left\{ b^K_{.L} := -\left(L\delta^K_L - \left(\gamma_{,L}\frac{\partial\,L}{\partial\,\gamma_{,K}} + \chi_{,L}\frac{\partial\,L}{\partial\,\chi_{,K}}\right)\right)\right. \\ &= \left.-\left(L\delta^K_L - S^K\beta_{\mathbf{L}} + \mathbf{T}^{\mathbf{K}}_{.\mathbf{i}}\mathbf{F}^{\mathbf{i}}_{.\mathbf{L}}\right)\right\}, \end{aligned} \tag{8}$$

$$\mathbf{f}^{inh} := \left.\frac{\partial\,L}{\partial\,\mathbf{X}}\right|_{\exp l} = \left(\frac{\mathbf{v}^2}{2}\right)(\nabla_R\rho_0) - \left.\frac{\partial\,W}{\partial\,\mathbf{X}}\right|_{\exp l},$$

(9)
$$H = \dot{\gamma}\frac{\partial L}{\partial \dot{\gamma}} + \mathbf{v}.\frac{\partial L}{\partial \mathbf{v}} - L \ = S\theta + 2K - L = K + E,$$

$$U^K = -\left(\dot{\gamma}\frac{\partial L}{\partial \gamma_{,K}} + v^i \frac{\partial L}{\partial \chi^i_{.K}}\right) = T^K_{.i} v^i - S^K \theta,$$

where we have defined the *mechanical canonical* (material) momentum $\mathbf{P}^{mech}$ and the *internal energy* per unit reference volume by

(10)
$$\mathbf{P}^{mech} = -\mathbf{p}.\mathbf{F}, \quad E = W + S\theta.$$

For the first of these sometimes referred to as the *pseudomomentum,* see, for instance [12], [13], eq. (10) is the usual Legendre transformation of thermodynamics between internal and free energies. As a matter of fact, the definition (9) contains two Legendre transformations, one related to mechanical fields, and the other to thermal ones.

If we assume, as in standard continuum thermodynamics, that entropy and heat flux are related by the usual relation

(11)
$$\mathbf{S} = \mathbf{Q}/\theta,$$

we have

(12)
$$\mathbf{Q} = -\theta \frac{\partial W}{\partial \beta},$$

and eqn. (6 takes on the classical form of the energy-conservation equation (cf. Maugin and Berezovski, 1999)

(13)
$$\left.\frac{\partial H}{\partial t}\right|_{\mathbf{X}} - \nabla_R.(\mathbf{T}.\mathbf{v} - \mathbf{Q}) = 0.$$

Summing up, we have deduced from the Hamiltonian-Lagrangian density (1) all field equations, balance laws and constitutive relations for the theory of materially inhomogeneous, finitely deformable, thermoelastic conductors of heat. As a matter of fact, eqns. (2) and (13) are the local balance equations of linear momentum (in physical space) and energy, respectively. This is completed by the balance equation of mass which here trivially reads

(14)
$$\left.\frac{\partial \rho_0}{\partial t}\right|_X = 0.$$

These are all formally identical to those of the classical thermoelasticity of conductors (e.g., as recalled in [18]). Another balance law if that of *moment of momentum* (in physical space). This is deduced from (1) by considering the action of the infinitesimal rotational component of the connected group SO(3) in physical space. A classical derivation yields then (in components in order to avoid any confusion in notation)

(15)
$$\frac{\partial W}{\partial F^{[i}_{.K}} F^{\,j]}_{.L} = 0 \quad or \quad T^K_{.[i} F^{\,j]}_{.L} = 0,$$

as the action of this group is inoperative on the material vector β. Apart from the functional dependence of W, eqn. (15) is also formally identical to the classical counterpart. Only the equation of canonical momentum (5) differs from the one originally obtained in [2] material thermoelasticity. But, abstraction being made of material inhomogeneities, it is the same as the one obtained by direct algebraic manipulations in [1] in the "dissipationless" formulation of thermoelasticity. Indeed, canonical momentum (7) is made of two parts, a strictly mechanical part - which is none other than the pull back, changed of sign, of the physical momentum - and a purely thermal part given by the constitutive behavior. In addition, the canonical stress tensor (8) - also called *Eshelby stress tensor* by the present author - contains a contribution of β because, from its very definition, its captures material gradients of all fields. One should note that the source term in eqn. (5) has no energetic contents. Furthermore, contrary to common use, even the entropy equation (3) is source free so that, surprisingly, in the absence of material inhomogeneities, all equations obtained here are *strict conservation laws*, hence the qualification of "*dissipationless theory*". In this rather strange -we admit it - approach, the entropy flux and heat flux are derived from the free energy, on the same footing as entropy density , and stress (eqns. (4) and (12)).

3. Correspondence with the classical theory

Since eqns. (2), (13), (14) and (15) are formally the same as in the classical theory, the limit where W does not depend on β is trivial for these. What about eqns. (3) and (5) which are of utmost importance for crack and phase-transition front studies (cf. [18]). We need to isolate the contributions of β in order to get some "classical" limit (this means *projecting* onto the classical state space of the thermoelasticity of conductors). First we expand eqn. (5) by accounting for the expressions (7) and (8). After some rearrangements, we obtain the following equation (note that $curl_R \beta = 0$; T =transposed)

(16) $$\left.\frac{\partial\, \mathbf{P}^{mech}}{\partial\, t}\right|_{\mathbf{X}} - div_R \bar{\mathbf{b}}^{mech} = S\, \nabla_R \theta\; + \mathbf{S}.\, (\nabla_R \beta)^T\; + \mathbf{f}^{inh},$$

where $\bar{\mathbf{b}}^{mech} = \bar{\mathbf{b}}^{th} - \mathbf{S} \otimes \beta$. But this is not all because L in $\mathbf{b}^{th}$ still depends on β. We must isolate this dependency by writing

(17) $$\frac{\partial\, W}{\partial\, \mathbf{X}} = \frac{\partial\, W^{mech}}{\partial\, \mathbf{X}} + \frac{\partial\, W}{\partial\, \beta}.\, (\nabla_R \beta)^T = \frac{\partial\, W^{mech}}{\partial\, \mathbf{X}} - \mathbf{S}.\, (\nabla_R \beta)^T\ ,$$

where, in essence, $W^{mech} = W(F, \theta\, ,\, \beta\; =\; 0\, ; X)$. On substituting (17) into the material divergence of $\mathbf{b}^{mech}$, we finally transform (16) to

(18) $$\left.\frac{\partial\, \mathbf{P}^{mech}}{\partial\, t}\right|_{\mathbf{X}} - div_R \mathbf{b}^{mech} = \mathbf{f}^{inh} + \mathbf{f}^{th},$$

where

$$\mathbf{b}^{mech} = -(L\mathbf{1}_R + \mathbf{T}.\mathbf{F}), \quad L = K - W^{mech}(\mathbf{F}, \theta; \mathbf{X}), \quad \mathbf{f}^{th} := S\nabla_R \theta.$$

The last introduced quantity is the material *thermal force of quasi-inhomogeneity* clearly defined by Epstein and Maugin [2] in their general theory of *material uniformity and inhomogeneity*. Thus equation (18) has recovered its "classical" form, the quotation marks here emphasizing that, in fact , while "classical" from our viewpoint, this equation is practically unknown to most people, although it is the one on which thermoelastic generalizations of the *J*-integral of fracture must be based (cf. [18], [21]). As to eqn. (3) we use the following trick. Multiplying (3) by $\theta \neq 0$ and accounting for (11) we obtain the "heat-propagation" equation in the form

$$\theta \frac{\partial S}{\partial t} + \nabla_R . \mathbf{Q} = \mathbf{S} . \dot{\beta}, \tag{19}$$

or else, by integration by parts,

$$\left. \frac{\partial (\theta S)}{\partial t} \right|_{\mathbf{X}} + \nabla_R . \mathbf{Q} = S \dot{\theta} + \mathbf{S} . \dot{\beta}. \tag{20}$$

This equation is interesting by itself because of its structure - especially the right-hand side - which is similar to that of eqn. (16), time derivatives replacing material space derivatives. The "classical" limit is obtained in (19) or (20) by ignoring the β term, i.e., restricting W to W^{mech}. The other terms then acquire their usual significance with **Q** and **S** no longer derivable from a potential. Working then in reverse, one recovers in this approximation the equations

$$\theta \frac{\partial S}{\partial t} + \nabla_R . \mathbf{Q} = 0, \qquad \frac{\partial S}{\partial t} + \nabla_R . \mathbf{S} = \sigma^{th},$$

where $\sigma^{th} = -S . \nabla_R (\ln \theta)$ is the *thermal entropy source*. **S** and $\mathbf{Q} = \mathbf{S}/\theta$ are now given by a constitutive equation obtained by invoking the noncontradiction of the formulation with the second law of thermodynamics, which here locally reads $\sigma^{th} \geq 0$. This yields, for instance, Fourier's law of heat conduction.

4. Accounting for anelasticity

At this point we have effectively formulated a canonical theory of the thermoelasticity of conductors. All field equations, balance laws , and constitutive equations follow from it. The relationship with the "classical" formulation was established. To proceed further, one must envisage the case where nonthermal *dissipative processes* (e.g., anelasticity) are present. Considering the theory of *internal variables of state* to describe these phenomena is a sufficiently general approach as demonstrated in a recent book [15]. The only *a priori* change should be accounting for the dependency of the free energy W on a new set of variables collectively represented by the symbol α The corresponding equation of state reads

$$A + (\partial W / \partial \alpha) = 0.$$

The main problem, however, remains to build the evolution equation of α, normally a relationship between $\dot{\alpha}$ and the thermodynamical force A constrained by the second law of thermodynamics. Thus the very presence of α is related to *dissipative processes* and a priori not amenable by means of a *canonical variational* formulation; $\alpha(X, t)$ is *not* a classical field; neither does it possess inertia, nor is its gradient introduced to account for some nonlocality). But it was recently shown how variables α and θ could play parallel roles in a certain reformulation of the anelasticity of thermoconductors ([14] , [17] (2000)). This is the trend to be followed. In effect, now we propose the following variational formulation in symbolic form:

$$\lim_{\beta \to \mathbf{0}} \delta \int_{E^3 \times T} L\ (\mathbf{v}, \mathbf{F}, \alpha, \theta = \dot{\gamma}, \beta = \nabla_{\mathbf{R}} \gamma; \mathbf{X})\ d^4 X = 0 \tag{21}$$

where L is the Hamiltonian-Lagrangian density per unit reference volume. The *limit* symbolism used in eqn. (21) means that the limit as β goes to zero must be taken *in the equations* resulting from the variational formulation, this applying to both field equations and other consequences of the principle such as the results of the application of Noether's theorem. We claim that in this limit *all equations of the "classical" theory of anelastic conductors of heat* are obtained, including the entropy equation and heat-propagation equation in this quite general case, a rather surprising result, we admit it. The only change compared to (1) is that the free energy W now depends on α, i.e., we have the following general expression

$$L = \bar{L}\,(\mathbf{v}, \mathbf{F}, \alpha, \theta, \beta; \mathbf{X}) = K\,(\mathbf{v}; \mathbf{X}) - W\,(\mathbf{F}, \alpha, \theta, \beta; \mathbf{X})\,.$$

Equations (2), (3), (5) and (6) hold true but for the additional dependence of W on α. The intrinsic dissipation necessary for the expression of the dissipative nature of this variable becomes visible only after performing manipulations of the type of those made in Section 3. We need to isolate the contributions due to the "dissipative" variables in eqns. (3) and (15). Equation (17) is modified due to the dependence on α:

$$\text{(22)}\qquad \begin{aligned} \frac{\partial W}{\partial \mathbf{X}} &= \frac{\partial W^{mech}}{\partial \mathbf{X}} + \frac{\partial W}{\partial \alpha}\,(\nabla_R \alpha)^T + \frac{\partial W}{\partial \beta}\,(\nabla_R \beta)^T \\ &= \frac{\partial W^{mech}}{\partial \mathbf{X}} - A.\,(\nabla_R \alpha)^T - S.\,(\nabla_R \beta)^T \end{aligned}$$

where, in essence, $W^{mech} = W(F, \theta, \alpha = const., \beta = 0; X)$. The equation of canonical momentum first yields

$$\left.\frac{\partial\,\mathbf{P}^{mech}}{\partial\,t}\right|_{\mathbf{X}} - div_R \bar{\mathbf{b}}^{mech} = S\,\nabla_R \theta\, + \mathbf{S}.\,(\nabla_R \beta)^T\, + \mathbf{f}^{inh}.$$

But on substituting from (22) into this equation, it comes

$$\text{(23)}\qquad \left.\frac{\partial\,\mathbf{P}^{mech}}{\partial\,t}\right|_{X} - div_R \mathbf{b}^{mech} = \mathbf{f}^{inh} + \mathbf{f}^{th} + \mathbf{f}^{intr}$$

where

$$\begin{aligned} \mathbf{b}^{mech} &= -\,(L\,\mathbf{1_R} + \mathbf{T}.\mathbf{F}) \\ L &= K - W^{mech}\,(\mathbf{F}, \theta, \alpha = const.; \mathbf{X})\,, \\ \mathbf{f}^{th} : &= S \nabla_R \theta, \\ \mathbf{f}^{intr} : &= A\,(\nabla_R \alpha)^T\,, \end{aligned}$$

The last two introduced quantities are material *forces of quasi-inhomogeneity* due to a nonuniform temperature field (cf. [2]) and to a nonuniform α field, respectively ([14]). The presence of those terms on an equal footing with $\mathbf{f}^{inh}$ means that, insofar as the material manifold is concerned, spatially nonuniform fields of α or θ are equivalent to distributed material inhomogeneities (also continuously distributed defects such as dislocations) ; they are *quasi-plastic effects* (cf. [13]). As to eqn. (3), accounting for the kinetic-energy theorem (obtained by multiplying scalarly eqn. (2) by $\mathbf{v}$ after multiplication by $\theta \neq 0$ and accounting for (6) and finally making $\beta =$ const. (this is equivalent to discarding β in the resulting equation and loosing the connection of $\mathbf{S}$ and $\mathbf{Q}$ with β) we arrive at the "*heat-propagation" equation* in the form

$$\text{(24)}\qquad \left.\frac{\partial\,\,(S\theta)}{\partial\,t}\right|_{\mathbf{X}} + \nabla_R.\mathbf{Q} = S\dot{\theta} + A.\dot{\alpha}\, \equiv \bar{\Phi}^{th} + \Phi^{intr}.$$

Then working in reverse, in this approximation one recovers the equations (compare to [14])

$$\theta \frac{\partial S}{\partial t} + \nabla_R . \mathbf{Q} = \Phi^{intr},$$
$$\frac{\partial S}{\partial t} + \nabla_R . \mathbf{S} = \sigma^{th} + \sigma^{intr},$$

where $\sigma^{th} = -S \nabla_R (\ln \theta)$ is the *thermal entropy source* and

$$\sigma^{intr} = \theta^{-1} A . \dot{\alpha},$$
$$\Phi^{intr} = \theta \, \sigma^{intr}$$

are the *intrinsic* entropy source and the *intrinsic* dissipation , respectively. In the present classical limit, $bf S$ and $bf S = bf S\theta$ are now given by a constitutive equation obtained by invoking the noncontradiction with the second law of thermodynamics which here locally reads

$$\sigma = \sigma^{th} + \sigma^{intr} \geq 0.$$

We have recovered all equations or constraints of the "classical theory" by applying the scheme proposed in eqn. (21).

5. Canonical four-dimensional space-time formulation

Equations (23) and (24) present an obvious space-time symmetry (see the two right-hand sides). This obviously suggest considering these two equations as space and time-like components of a unique *four-dimensional equation* in the appropriate space and the canonical momentum $\mathbf{P}^{mech}$ and the quantity θS (an energy which is the difference between internal and free energies) as dual space-time quantities, i.e., they together form a four-dimensional canonical momentum

$$\mathbf{P}_{(4)} = \left(\mathbf{P}^{mech}, P_4 = \theta \, S \right).$$

We let the reader check that eqns. (23) and (24) can in fact be rewritten in the following pure 4-dimensional or 4×4 formalism in an Euclidean 4-dim space (compare to World-invariant kinematics in [20])

(25) $$\frac{\partial}{\partial X^\beta} B^\beta_{.\alpha} = f_\alpha \equiv \bar{A} . \frac{\partial}{\partial X^\alpha} \mu - \left. \frac{\partial W}{\partial X^\alpha} \right|_{\exp l} = \left. \frac{\partial L}{\partial X^\alpha} \right|_{(\mathbf{F}, \mathbf{v} \; fixed)}$$

$$\bar{A} = (A, S) \quad \mu = (\alpha, \theta),$$
$$X^\alpha (\alpha = 1, 2, 3, 4) = \left\{ X^K \; (K = 1, 2, 3) \, , \; X^4 = t \right\}$$

$$B^\beta_{.\alpha} = \left\{ \begin{matrix} B^K_{.L} = -b^K_L & B^4_{.L} = P^{mech}_L \\ B^K_{.4} = Q^K & B^4_{.4} = \theta \, S \end{matrix} \right\}, \; P_{(4)} = \left(B^4_{.L} \, , \; B^4_{.4} \right)$$

or, introducing intrinsically four-dimensional gradients and divergence in E^4 for eqn. (25),

(26) $$div_{E^4} \, \mathbf{B}^{mech} = \nabla_{E^4} L \, |_{mech}$$

where the right-hand side means the gradient computed keeping the "mechanical" fields (**F**,**v**) *fixed.* Equation (26) represents the canonical form of the balance of canonical momentum and the heat-propagation equation for anelastic, anisotropic, finitely deformable solid heat conductors. The 4-dimensional formalism introduced is somewhat different from that used by Maugin ([17]) or Herrmann and Kienzler ([7]). However, in the absence of intrinsic dissipative processes and for isothermal processes, eqns. (23) and (24) - or eqn. (26) reduce to those of Kijowski and Magli ([8]) in isothermal thermoelasticity, the second equation reducing obviously to the simple equation

$$\partial(\theta S)/\partial t = 0.$$

This shows the closedness of the present approach with the general relativistic Hamiltonian scheme.

6. Conclusion

The procedure used in this paper is essentially that of a *gauge theory* as practiced in modern physics. We have artificially enlarged the state space of the theory by adding one coordinate (the material gradient of the "potential" γ) to this space and then projected the resulting equations back onto the original state space. The latter could not accommodate dissipative processes, but the enlarged one does. Recurring to the classical dissipative formulation then requires this projection or "return to reality". In the mean time, a variational formulation has indeed been proposed.

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