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REGULAR TRIANGLES AND ISOCLINIC TRIANGLES IN THE GRASSMANN MANIFOLDS $G_2(\mathbb{R}^N)$*

Abstract. We give a complete set of orthogonal invariants for triangles in $G_2(\mathbb{R}^n)$. As a consequence we characterize regular triangles and isoclinic triangles and we exhibit the existence regions of these objects in comparison with the angular invariants associated to them.

1. Introduction

By trigonometry in a given Riemannian space we mean the study of triples of points in that space; more precisely, one wants to find a complete system of isometrical invariants which permits to determine uniquely the isometry class of the triple of points.

Trigonometry plays a fundamental role in geometry: indeed, the study of the geometric properties of a given space is necessarily linked to the study of the most simple geometric objects in that space, namely the triangles.

In classical trigonometry, i.e. trigonometry in Euclidean spaces, spheres and hyperbolic spaces, we know that a triangle depends on three essential parameters (for example two sides and the enclosed angle, provided triangular inequalities are verified). These spaces are rank-one symmetric spaces with constant curvature. The situation in the other rank-one symmetric spaces (i.e. projective spaces and hyperbolic spaces, which are the corresponding non-compact duals) is more complicated. Trigonometry in these spaces has been revealed by Brehm in [3] after partial results of Blaschke and Terheggen, Coolidge, Hsiang (see [2, 4, 8]). Brehm shows that a triangle depends on four invariants; he introduces the "shape invariant" σ which, in addition to the three side lengths, permits to determine uniquely the isometry class of a triangle (these four invariants must, of course, satisfy some inequalities). A geometrical interpretation of σ can be found in [7].

For what concerns symmetric spaces of higher rank, we only know the trigonometry in the Lie group $SU(3)$ which is a rank-two symmetric space. These results are due to Aslaksen [1]. Using an algebraic approach, Aslaksen shows, thanks to invariant theory, that the isometry class of a triangle depends on eight essential parameters.

In this paper we examine trigonometry in another rank-two symmetric space, namely the real Grassmann manifold $G_2(\mathbb{R}^n)$. This survey has been started up by Hangan in [6]; moreover, some results have been discovered by Fruchard in [5] using a different approach. General laws of trigonometry in the symmetric spaces of non-compact type have been settled by Leuzinger in [9].

A first obvious application of trigonometry consists in studying some particular triangles such as regular triangles and isoclinic triangles. In a forthcoming paper ([12]) we will apply

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these results to the 4-tuples, to the regular 4-tuples and finally to n-tuples. Complete details can be found in [11].

2. Invariants of triangles in $G_2(\mathbb{R}^n)$

Let $\mathbb{R}^n$ be the Euclidean space endowed with the usual scalar product $\langle\cdot,\cdot\rangle$. The Grassmann manifold $G_2(\mathbb{R}^n)$ is the set of non oriented 2-planes in $\mathbb{R}^n$. Let a 2-plane X spanned by an orthonormal basis $\{u,v\}$; we can represent X as an irreducible bivector $X = u \wedge v$ (up to the sign), i.e. as an element of the exterior algebra $\Lambda^2(\mathbb{R}^n)$. To the 2-plane X, we can also associate the orthogonal projector denoted with P_X and defined as:

$$P_X(x) = \langle x,u\rangle u + \langle x,v\rangle v\,.$$

Conversely, we can associate to a 2-dimensional projector P_X the 2-plane $Im(P_X)$. With respect to a fixed orthonormal basis in $\mathbb{R}^n$, P_X will be represented by a symmetric, idempotent matrix with trace 2, which does not depend on the basis defining X. If we change the basis in $\mathbb{R}^n$, the matrix will be altered by conjugation with an orthogonal matrix. In other words, to X we associate a conjugation class of symmetric, idempotent matrices with trace 2. Let us take now $\{X,Y\} \in G_2(\mathbb{R}^n)$ and consider the angle between $v \in X$ and its orthogonal projection $P_Y(v)$; we denote α_1, α_2 respectively the minimum and maximum angle as v varies in X (with $v \neq 0$). These angles are called critical angles and they permit to introduce a distance in $G_2(\mathbb{R}^n)$, defined as:

$$d(X,Y) = \sqrt{\alpha_1^2 + \alpha_2^2}\,.$$

In comparison with this distance, the orthogonal group $O(n,\mathbb{R})$ acts as an isometry group and the Grassmann manifold can be considered as the homogeneous rank-two symmetric manifold

$$G_2(\mathbb{R}^n) = \frac{O(n)}{O(2)\times O(n-2)}\,.$$

Consider now $\{X,Y,Z\} \in G_2(\mathbb{R}^n)$. The orthogonal projections in X of the unit circles of Y and Z respectively are two ellipses. The angle between the great axes of these two ellipses, denoted with ω_X, is called inner angle and represents the rotation angle between the critical directions of $\{X,Y\}$ and $\{X,Z\}$ (see [5, 6]). So, to a triangle $\{X,Y,Z\}$ we can associate nine angular invariants: six critical angles (two for each pair of planes) and three inner angles ω_X, ω_Y, ω_Z.

Let $\{A,B,C\} \in G_2(\mathbb{R}^n)$, we can find an orthonormal basis $\{e_1,\dots,e_6\}$ in $\mathbb{R}^6$ with respect to which the triangle $\{A,B,C\}$ takes the following form (see [5, 6, 11] for details):

$$\begin{cases} A = e_1 \wedge e_2 \\ B = \epsilon_1 \wedge \epsilon_2 = (\cos c_1 e_1 + \sin c_1 e_3) \wedge (\cos c_2 e_2 + \sin c_2 e_4) \\ C = \bar\epsilon_1 \wedge \bar\epsilon_2 = (\cos b_1 \bar e_1 + \sin b_1 u) \wedge (\cos b_2 \bar e_2 + \sin b_2 v) \end{cases} \tag{1}$$

where $\{b_1,b_2\}$, $\{c_1,c_2\}$ are the critical angles of $\{A,C\}$ and $\{A,B\}$ respectively and $\{u,v\}$ is an orthonormal system in $A^\perp$, i.e. $u = \sum_{i=3}^6 u_i e_i$ and $v = \sum_{i=3}^6 v_i e_i$ with

$$\begin{cases} \|u\| = \|v\| = 1 \\ \langle u,v\rangle = 0\,. \end{cases} \tag{2}$$

(1) is called *canonical form* of the triangle.

We assume that $0 < b_1 < b_2 < \frac{\pi}{2}$ and $0 < c_1 < c_2 < \frac{\pi}{2}$. Moreover, we can choose the critical directions such that

$$\begin{cases} \bar{e}_1 = \cos\omega_A e_1 + \sin\omega_A e_2 \\ \bar{e}_2 = -\sin\omega_A e_1 + \cos\omega_A e_2 \end{cases} \tag{3}$$

with $0 < \omega_A < \frac{\pi}{2}$. Such a triangle will be called *generic*. Some special triangles will be studied separately hereafter.

REMARK 1. Thanks to the action of the orthogonal group on $(A + B)^{\perp}$, we can impose that $v_6 = 0$, $v_5 > 0$ and $u_6 > 0$.

The parameters u_5, u_6, v_5 can be uniquely deduced from u_3, u_4, v_3, v_4; indeed, the conditions (2) lead to:

$$\begin{aligned} v_5 &= \sqrt{1 - v_3^2 - v_4^2} \\ u_5 &= -\frac{u_3v_3 + u_4v_4}{v_5} \\ u_6 &= \sqrt{1 - u_3^2 - u_4^2 - u_5^2} \end{aligned}$$

so, we must impose the following existence conditions

$$f = v_3^2 + v_4^2 - 1 \leq 0 \tag{C1}$$

$$g = u_3^2 + u_4^2 + u_5^2 - 1 \leq 0 \tag{C2}$$

which is equivalent to:

$$g = u_3^2 + u_4^2 + v_3^2 + v_4^2 - (u_3v_4 - u_4v_3)^2 - 1 \leq 0\,.$$

Hence, we deduce that the canonical form contains nine independent parameters.

DEFINITION 1. *Two triangles $\{A, B, C\}$ and $\{\bar{A}, \bar{B}, \bar{C}\}$ are isometric if there exists $\phi \in O(n)$ such that $\phi(A) = \bar{A}$, $\phi(B) = \bar{B}$, $\phi(C) = \bar{C}$.*

In [11], we establish the following lemma:

LEMMA 1. *Two triangles in $G_2(\mathbb{R}^6)$ are isometric if and only if they have the same canonical form.*

From this lemma, we deduce that the isometry class of a triangle is determined by a set of invariants which enables us to determine uniquely the parameters in the canonical form. Recall that to each plane X we associate a conjugation class of matrices representing the orthogonal projector P_X. We can denote with the same letter the plane and the matrix associated to the projector; indeed, the isometry group is the orthogonal group $O(n)$ which acts on matrices by conjugation. Consequently, the geometric problem of finding the isometry class of the planes $\{A, B.C\}$ turns into the algebraic problem of finding a complete set of orthogonal invariants for the symmetric matrices $\{A, B, C\}$. According to Procesi [13], such a set is composed of traces (and determinants) of opportune combinations of these matrices. Such a set of invariants can be found in [10]. Another more symmetric set will be given hereafter.

From now on, when considering any combination between A, B and C, we shall consider the restriction to the starting plane; for example, $A.B.C$ will mean $P_A \circ P_B \circ P_C \circ P_A$.

The invariant $\det(A.B.C)$ has a nice topological interpretation; let

$$\Pi : G_2(\mathbb{R}^6) \longrightarrow \mathbb{RP}^{14}$$

be the Plücker embedding and $\sigma = \langle x, y\rangle.\langle y, z\rangle.\langle z, x\rangle$ be the shape invariant (see [3]) for the triangle $([x], [y], [z])$ in the real projective space. We have:

PROPOSITION 1. $\det(A.B.C) = \sigma(\Pi(A), \Pi(B), \Pi(C))$.

See [11] for a proof.

In the real projective space, $\sigma > 0$ if and only if the geodesic triangle is null-homotopic; $\sigma < 0$ if and only if the geodesic triangle is non null-homotopic (see [3]).

A fundamental problem is the following: as the algebraic dimension of the orbit space

$$\frac{G_2(\mathbb{R}^6) \times G_2(\mathbb{R}^6) \times G_2(\mathbb{R}^6)}{O(6, \mathbb{R})}$$

representing the isometry class of triangles in $G_2(\mathbb{R}^6)$ is nine, it is natural to ask to what extent the nine angular invariants define the isometry class of the triangle. We have the following

THEOREM 1. *There exist at most sixteen non isometric generic triangles having prescribed critical angles and inner angles.*

Proof. We must determine the parameters in the canonical form. The parameters b_1, b_2, c_1, c_2 and ω_A are already known. However, they can be determined thanks to the following invariants:

$$(4)\qquad \left\{\begin{array}{rcl} \mathrm{tr}(A.B) & = & \cos^2 c_1 + \cos^2 c_2 \\ \det(A.B) & = & \cos^2 c_1 \cos^2 c_2 \\ \mathrm{tr}(A.C) & = & \cos^2 b_1 + \cos^2 b_2 \\ \det(A.C) & = & \cos^2 b_1 \cos^2 b_2 \\ \mathrm{tr}(A.B.A.C) & = & \left(\cos^2 b_1 \cos^2 c_1 + \cos^2 b_2 \cos^2 c_2\right) \cos^2 \omega_A \\ & & +\left(\cos^2 b_1 \cos^2 c_2 + \cos^2 b_2 \cos^2 c_1\right) \sin^2 \omega_A\,. \end{array}\right.$$

So, we only have to determine u_3, u_4, v_3, v_4 using the remaining invariants: a_1 and a_2, the critical angles of the pair $\{B, C\}$, and the inner angles ω_B and ω_C.

Let us perform the change of parameters:

$$(5)\qquad \left\{\begin{array}{l} x = \langle \epsilon_1, \bar{\epsilon}_1\rangle = \cos b_1 \cos c_1 \cos \omega_A + \sin b_1 \sin c_1 u_3 \\ y = \langle \epsilon_1, \bar{\epsilon}_2\rangle = -\cos b_2 \cos c_1 \sin \omega_A + \sin b_2 \sin c_1 v_3 \\ z = \langle \epsilon_2, \bar{\epsilon}_1\rangle = \cos b_1 \cos c_2 \sin \omega_A + \sin b_1 \sin c_2 u_4 \\ t = \langle \epsilon_2, \bar{\epsilon}_2\rangle = \cos b_2 \cos c_2 \cos \omega_A + \sin b_2 \sin c_2 v_4\,. \end{array}\right.$$

We deduce that determining u_3, u_4, v_3, v_4 is equivalent to determining x, y, z, t. Now, if we permute cyclically A, B and C in the expressions (4), we see that the invariants $\mathrm{tr}(B.C)$, $\det(B.C)$, $\mathrm{tr}(B.A.B.C)$, $\mathrm{tr}(C.A.C.B)$ are determined by the remaining critical angles a_1, a_2 and inner angles ω_B, ω_C.

On the other hand, they have an equivalent expression by calculating directly on the canonical form. Finally, we have the following quadratic system in the parameters x, y, z, t:

$$\text{(6)} \qquad \begin{cases} x^2 + y^2 + z^2 + t^2 = \operatorname{tr}(B.C) \\ (xt - yz)^2 = \det(B.C) \\ \cos^2 c_1\big(x^2 + y^2\big) + \cos^2 c_2\big(z^2 + t^2\big) = \operatorname{tr}(B.A.B.C) \\ \cos^2 b_1\big(x^2 + z^2\big) + \cos^2 b_2\big(y^2 + t^2\big) = \operatorname{tr}(C.A.C.B)\,. \end{cases}$$

If (x, y, z, t) is a solution of (6) then the following are also solutions of (6): $(-x, -y, -z, -t)$, $(-x, -y, z, t)$, $(x, y, -z, -t)$, $(-x, y, z, -t)$, $(x, -y, -z, t)$, $(-x, y, -z, t)$, $(x, -y, z, -t)$. Finally, we obtain two groups of eight solutions given by:

$$\text{(7)} \qquad \begin{cases} x = \cos a_1 \cos\omega_B \cos\omega_C \pm \cos a_2 \sin\omega_B \sin\omega_C \\ y = \mp\cos a_1 \cos\omega_B \sin\omega_C + \cos a_2 \sin\omega_B \cos\omega_C \\ z = \cos a_1 \sin\omega_B \cos\omega_C \mp \cos a_2 \cos\omega_B \sin\omega_C \\ t = \mp\cos a_1 \sin\omega_B \sin\omega_C - \cos a_2 \cos\omega_B \cos\omega_C\,. \end{cases}$$

This completes the proof □

REMARK 2. A. Fruchard found the same result using a different approach (see [5]). The sixteen solutions are reached if the conditions $(C1)$ and $(C2)$ are satisfied. A. Fruchard shows that all the solutions exist if the critical angles are greater than $\arccos\frac{1}{3}$.

We will consider additional algebraic invariants to distinguish the sixteen solutions. We consider at first:

$$\det(A.B.C) = \cos b_1 \cos b_2 \cos c_1 \cos c_2 (xt - yz)\,.$$

The factor $xt - yz$, when substituting in (7) takes only the values $\pm\cos a_1 \cos a_2$, so $\det(A.B.C)$ separates the sixteen solutions in two groups.

Finally, we consider the following four invariants, evaluated on the canonical form:

$$\begin{aligned}
\operatorname{tr}(A.B.C) &= \cos b_1 \cos c_1 \cos\omega_A x - \cos b_2 \cos c_1 \sin\omega_A y \\
&\quad + \cos b_1 \cos c_2 \sin\omega_A z + \cos b_2 \cos c_2 \cos\omega_A t \\
\operatorname{tr}(A.B.C.A.C) &= \cos^3 b_1 \cos c_1 \cos\omega_A x - \cos^3 b_2 \cos c_1 \sin\omega_A y \\
&\quad + \cos^3 b_1 \cos c_2 \sin\omega_A z + \cos^3 b_2 \cos c_2 \cos\omega_A t \\
\operatorname{tr}(A.B.C.A.B) &= \cos b_1 \cos^3 c_1 \cos\omega_A x - \cos b_2 \cos^3 c_1 \sin\omega_A y \\
&\quad + \cos b_1 \cos^3 c_2 \sin\omega_A z + \cos b_2 \cos^3 c_2 \cos\omega_A t \\
\operatorname{tr}(A.B.C.B.C) &= \big[\cos b_1 \cos c_1\big(\cos^2 a_1 + \cos^2 a_2\big)\cos\omega_A \\
&\quad - \cos b_2 \cos c_2 (xt - yz)\cos\omega_A\big]x \\
&\quad -\big[\cos b_2 \cos c_1\big(\cos^2 a_1 + \cos^2 a_2\big)\sin\omega_A \\
&\quad - \cos b_1 \cos c_2 (xt - yz)\sin\omega_A\big]y \\
&\quad +\big[\cos b_1 \cos c_2\big(\cos^2 a_1 + \cos^2 a_2\big)\sin\omega_A \\
&\quad - \cos b_2 \cos c_1 (xt - yz)\sin\omega_A\big]z \\
&\quad +\big[\cos b_2 \cos c_2\big(\cos^2 a_1 + \cos^2 a_2\big)\cos\omega_A \\
&\quad - \cos b_1 \cos c_1 (xt - yz)\cos\omega_A\big]t\,.
\end{aligned}$$

This gives us a linear system with four equations in the parameters x, y, z, t (indeed, $xt - yz = \pm\cos a_1 \cos a_2$). The determinant of the coefficients matrix is:

$$- \cos a_1 \cos a_2 \cos b_1 \cos b_2 \cos c_1 \cos c_2 \big(\cos^2 b_1 - \cos^2 b_2 \big)^2 \cdot \\ \cdot \big(\cos^2 c_1 - \cos^2 c_2 \big) \sin^2 \omega_A \cos^2 \omega_A$$

if $xt - yz > 0$, otherwise it is the opposite, and never vanishes (the case $b_1 = b_2$ and $c_1 = c_2$ will be studied separately in another section).

We conclude so that these invariants determine uniquely x, y, z, t (i.e. they separate the sixteen orbits). This completes the proof of the following theorem:

THEOREM 2. *The isometry class of a generic triangle* $\{A, B, C\}$ *in* $G_2(\mathbb{R}^6)$ *is uniquely determined by the following list of orthogonal invariants:* $L_{ABC} = [\mathrm{tr}(A.B),\ \det(A.B),\ \mathrm{tr}(A.C),\ \det(A.C),\ \mathrm{tr}(B.C),\ \det(B.C),\ \mathrm{tr}(A.B.C),\ \det(A.B.C),\ \mathrm{tr}(A.B.A.C),\ \mathrm{tr}(B.A.B.C),\ \mathrm{tr}(C.A.C.B),\ \mathrm{tr}(A.B.C.A.B),\ \mathrm{tr}(A.B.C.A.C),\ \mathrm{tr}(A.B.C.B.C)]$.

REMARK 3. As a triangle depends essentially on nine continuous parameters, we shall expect to find five syzygies between the fourteen invariants of the list L_{ABC}. According to the general theory (see [13]) the syzygies (functional relations between non independent invariants) are consequences of the Hamilton-Cayley theorem.

3. Regular triangles

DEFINITION 2. *A triangle* $\{A, B, C\}$ *will be called regular if it admits the symmetric group* S_3 *as isometry group.*

We want now to feature regular triangles; by virtue of Theorem 2, we must impose that each invariant of the list L_{ABC} does not vary under the action of each permutation of S_3. However, it is sufficient to impose the invariance under the action of the generators of S_3. As generators, we can consider

$$R : (A, B, C) \longrightarrow (B, C, A)$$
$$S : (A, B, C) \longrightarrow (A, C, B)\,.$$

By considering the action of R and S on the elements of L_{ABC}, we deduce immediately the following:

THEOREM 3. *A triangle* $\{A, B, C\}$ *in* $G_2(\mathbb{R}^6)$ *is regular if and only if*

(i) $\mathrm{tr}(A.B) = \mathrm{tr}(A.C) = \mathrm{tr}(B.C)$

(ii) $\det(A.B) = \det(A.C) = \det(B.C)$

(iii) $\mathrm{tr}(A.C.A.B) = \mathrm{tr}(B.A.B.C) = \mathrm{tr}(C.A.C.B)$

(iv) $\mathrm{tr}(A.B.C.A.B) = \mathrm{tr}(A.B.C.A.C) = \mathrm{tr}(A.B.C.B.C)$.

REMARK 4. The elements $\mathrm{tr}(A.B.C)$ and $\det(A.B.C)$ are always invariant under the influence of permutations of $\{A, B, C\}$ because these matrices are symmetric.

Let us deduce now some consequences from conditions $(i), \ldots, (iv)$.

From conditions (i) and (ii), we deduce:

$$a_1 = b_1 = c_1 =^{not} \alpha$$
$$a_2 = b_2 = c_2 =^{not} \beta$$

this means that the triangle is equilateral.
From condition (iii) we deduce that:

$$\omega_A = \omega_B = \omega_C =^{not} \omega\,.$$

So, a regular triangle possesses only three angular invariants, namely α, β and ω. We already know that there exist at most sixteen non isometric triangles having angular invariants α, β and ω (these triangles are called "semi-regular" by Fruchard). Which ones are regular? We show the following:

THEOREM 4. *There exist at most four non isometric regular triangles having prescribed critical angles and inner angles.*

Proof. Let us suppose α, β and ω are given. We must determine the parameters of the canonical form, such that conditions $(i), \ldots, (iv)$ are satisfied.

• $\operatorname{tr}(A.B) = \operatorname{tr}(A.C)$ and $\det(A.B) = \det(A.C)$ imply $b_1 = c_1 = \alpha$ and $b_2 = c_2 = \beta$.

• $\operatorname{tr}(A.B.C.A.B) - \operatorname{tr}(A.B.C.A.C) = \cos\alpha\cos\beta\big(\cos^2\alpha - \cos^2\beta\big)\sin\omega(y+z)$ with $y+z = \sin\alpha\sin\beta(u_4 + v_3)$ according to (5).

So, $\operatorname{tr}(A.B.C.A.B) = \operatorname{tr}(A.B.C.A.C)$ if and only if $u_4 = -v_3$.

• From (6), we get $\operatorname{tr}(B.A.B.C) = \operatorname{tr}(C.A.C.B)$ if and only if $y^2 = z^2$. This condition is already verified because $y = -z$.

• From $\operatorname{tr}(A.B) = \operatorname{tr}(B.C)$ and $\det(A.B) = \det(B.C)$ we deduce:

$$\text{(8)} \qquad \begin{cases} x^2 + 2y^2 + t^2 = \cos^2\alpha + \cos^2\beta \\ xt + y^2 = \pm\cos\alpha\cos\beta \end{cases}$$

which imply:

$$x - t = \pm(\cos\alpha - \cos\beta) \quad \text{if} \quad xt + y^2 > 0$$
$$x - t = \pm(\cos\alpha + \cos\beta) \quad \text{if} \quad xt + y^2 < 0\,.$$

• $\operatorname{tr}(A.B.A.C) = \operatorname{tr}(B.A.B.C)$ if and only if $\omega_A = \omega_B = \omega$ gives $x^2 + y^2 = \cos^2\beta + \big(\cos^2\alpha - \cos^2\beta\big)\cos^2\omega_A$.

When considering the following system:

$$\text{(9)} \qquad \begin{cases} x^2 + t^2 = \cos^2\alpha + \cos^2\beta - 2y^2 \\ xt + y^2 = \pm\cos\alpha\cos\beta \\ x^2 + y^2 = \cos^2\beta + \big(\cos^2\alpha - \cos^2\beta\big)\cos^2\omega_A \end{cases}$$

we deduce that:

$$y = \pm(\cos\alpha + \cos\beta)\sin\omega_A\cos\omega_A \quad \text{if} \quad xt - y^2 > 0$$
$$y = \pm(\cos\alpha - \cos\beta)\sin\omega_A\cos\omega_A \quad \text{if} \quad xt - y^2 < 0\,.$$

• Finally, by calculating on the canonical form, we get:

$$\operatorname{tr}(A.B.C.A.B) = \operatorname{tr}(A.B.C.B.C)$$

if and only if

$$x\cos\beta\cos\omega_A - t\cos\alpha\cos\omega_A + y(\cos\beta-\cos\alpha)\sin\omega_A = 0$$
$$\text{if}\quad xt+y^2>0$$

and

$$x\cos\omega_A\big(\cos\alpha\cos\beta+\cos^2\beta\big)+t\cos\omega_A\big(\cos\alpha\cos\beta+\cos^2\alpha\big)$$
$$-y\sin\omega_A(\cos\alpha+\cos\beta)^2=0$$
$$\text{if}\quad xt+y^2<0\,.$$

By separating the two cases $xt+y^2>0$ (<0), we find the following four solutions (where we denote $a=\cos\alpha$, $b=\cos\beta$, $\epsilon_1=\pm1$, $\epsilon_2=\pm1$):

$$(10)\qquad \begin{cases} x=\epsilon_1 a\cos^2\omega+\epsilon_2 b\sin^2\omega\\ y=-(\epsilon_2 a-\epsilon_1 b)\sin\omega\cos\omega\\ z=-y\\ t=-\epsilon_1 a\sin^2\omega-\epsilon_2 b\cos^2\omega\,. \end{cases}$$

□

These solutions can be deduced as particular solutions of (7); we just have to consider $a_1=b_1=c_1=\alpha$, $a_2=b_2=c_2=\beta$ and $\omega_A=\omega_B=\omega_C=\omega$.

The problem now is to establish when the four solutions exist effectively. The relations (5) reduce to:

$$(11)\qquad \begin{cases} x=\cos^2\alpha\cos\omega+\sin^2\alpha u_3\\ y=-\cos\alpha\cos\beta\sin\omega+\sin\alpha\sin\beta v_3\\ z=-y\\ t=\cos^2\beta\cos\omega+\sin^2\beta v_4\,. \end{cases}$$

If we combine now the (11) with the (10), we get the following (where we denote $w=\cos\omega$):

$$(12)\qquad \begin{cases} u_3=\dfrac{aw^2+b\big(1-w^2\big)-a^2w}{1-a^2}\\ v_3=\dfrac{(-a+b)w\sqrt{1-w^2}+ab\sqrt{1-w^2}}{\sqrt{\big(1-a^2\big)\big(1-b^2\big)}}\\ u_4=-v_3\\ v_4=\dfrac{-a\big(1-w^2\big)-bw^2-b^2w}{1-b^2}\,. \end{cases}$$

If we substitute these values in the conditions $(C1)$ and $(C2)$, we obtain the existence region for the first regular triangle, expressed in terms of the parameters a, b, w:

$$(13)\qquad \begin{cases} f(a,b,w)\le 0\\ g(a,b,w)\le 0\,. \end{cases}$$

These conditions can be nicely factorized, with the aid of *Mathematica*^{TM}. We get explicitly:

$$\begin{aligned}\big(a^2-1\big)\big(b^2-1\big)^2 f(a,b,w) &= \big[1-a^2+ab-b^2+(b-a)w+\big(b-a^2\big)w^2\big]\big[1-a^2\\ &\quad -ab-b^2+\big(a-b-2ab^2\big)w+\big(a^2-b^2\big)w^2\big]\\ \big(a^2-1\big)^2\big(b^2-1\big)^2 g(a,b,w) &= [(w-1)a-(w+1)b+1][(w-1)b-(w+1)a\\ &\quad -1]\big[1-a^2+ab-b^2+(b-a)w+(b-a)^2w^2\big]^2\,.\end{aligned}$$

Figure 1

The three other cases have analogous expressions, we just have to transform a into $-a$, b into $-b$ and a, b into $-a$, $-b$. When we fix ω, the existence region expressed in terms of a and b is the interior of a domain bordered by two lines and an ellipse arc (for each triangle). We must of course consider the region under the bisectrix $a = b$. An example taking $\omega = \frac{\pi}{3}$ is shown in Figure 1.

For existence regions in the semi-regular case, see also [5].

4. Isoclinic triangles

Another interesting class of triangles is given by the isoclinic triangles.

DEFINITION 3. *Two planes A, B in $G_2(\mathbb{R}^6)$ are called isoclinic if the angle between any nonzero vector in A and its orthogonal projection in B is constant; in other words, the two critical angles coincide and all the directions are critical directions. A triangle $\{A, B, C\}$ in $G_2(\mathbb{R}^6)$ will be called isoclinic if each sub-pair of planes is isoclinic.*

It is straightforward that if $\{A, B\}$ is isoclinic, then $(P_A)|_B$ and $(P_B)|_A$ are similarities.

The first step now is to deduce a canonical form for $\{A, B, C\}$. Let us write:

$$\begin{cases} A = e_1 \wedge e_2 \\ B = \epsilon_1 \wedge \epsilon_2 = (\cos c e_1 + \sin c e_3) \wedge (\cos c e_2 + \sin c e_4) \\ C = \bar{\epsilon}_1 \wedge \bar{\epsilon}_2 = (\cos b e_1 + \sin b u) \wedge (\cos b e_2 + \sin b v) \,. \end{cases} \tag{14}$$

In (14), we imposed $b_1 = b_2 = b$, $c_1 = c_2 = c$. We can set the inner angle $\omega_A = 0$ by choosing judiciously the critical directions; the same can be done for ω_B but the last inner angle $\omega_C =^{not} \omega$ does not vanish in general. Let us put:

$$\begin{cases} x = \cos b \cos c + \sin b \sin c u_3 \\ y = \sin b \sin c v_3 \\ z = \sin b \sin c u_4 \\ t = \cos b \cos c + \sin b \sin c v_4 \,. \end{cases} \tag{15}$$

The last condition we have to impose is $a_1 = a_2$, which leads to:

$$\begin{aligned} \mathrm{tr}(B.C) &= x^2 + y^2 + z^2 + t^2 = 2\cos^2 a \,, \\ \det(B.C) &= (xt - yz)^2 = \cos^4 a \,. \end{aligned}$$

This gives $\left(x^2 + y^2 + z^2 + t^2\right)^2 = 4(xt - yz)^2$ if and only if $\left[(x-t)^2 + (y+z)^2\right]\left[(x+t)^2 + (y-z)^2\right] = 0$.
Two cases can occur:

1. $x = t$ and $y = -z$ corresponds to positive triangles
2. $x = -t$ and $y = z$ corresponds to negative triangles.

Using (15), we deduce then:

1. $u_3 = v_4$ and $u_4 = -v_3$
2. $u_3 + v_4 = \cot b \cot c$ and $u_4 = v_3$.

The isoclinic triangles depend so on the four invariants a, b, c and ω. Let us see to what extent these parameters determine the triangle. We introduce first the following

DEFINITION 4. *A triangle $\{A, B, C\}$ in $G_2(\mathbb{R}^6)$ is positive (negative respectively) if* $\det(A.B.C) > 0$ *(*< 0 *respectively).*

We have now:

THEOREM 5. *1. There are at most two positive isoclinic triangles in $G_2(\mathbb{R}^6)$ having prescribed critical angles a, b, c and inner angle ω.*

2. There are at most two negative isoclinic triangles in $G_2(\mathbb{R}^6)$ having prescribed critical angles a, b, c.

Proof. We just have to determine the parameters x and y.

Consider the linear application $P_A \circ P_B \circ P_C \circ P_A$. We study the two cases 1 and 2 separately.
1. Calculus gives:

$$\begin{aligned} \mathrm{tr}(A.B.C) &= 2\cos b \cos c x \\ \det(A.B.C) &= \cos^2 a \cos^2 b \cos^2 c \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(B.C) &= 2\big(x^2+y^2\big) = 2\cos^2 a \\ \det(B.C) &= \big(x^2+y^2\big)^2 = \cos^4 a \end{aligned}$$

which imply that $x^2+y^2=\cos^2 a$. The characteristic polynomial of $A.B.C$ is:

$$P_\lambda = \lambda^2 - 2\cos b\cos c x\lambda + \cos^2 a\cos^2 b\cos^2 c = 0$$

so, the eigenvalues are $\cos b\cos c(x\pm iy)$. On the other hand, we know that $A.B.C$ is a similarity, as the result of the composition of an homothetic transformation with magnification factor $\rho = \cos a\cos b\cos c$ and a rotation with angle ω.

Such a similarity admits the following eigenvalues:

$$\cos a\cos b\cos c e^{\pm i\omega} = \cos a\cos b\cos c(\cos\omega \pm i\,\sin\omega)\,.$$

Comparing now the two sets of eigenvalues gives:

$$\text{(16)}\qquad \begin{cases} \cos b\cos c x = \cos a\cos b\cos c\cos\omega \\ \pm\cos b\cos c y = \cos a\cos b\cos c\sin\omega \end{cases}$$

which imply:

$$\text{(17)}\qquad \begin{cases} x = \cos a\cos\omega \\ y = \pm\cos a\sin\omega\,. \end{cases}$$

Substituting then in (15) gives the two solutions

$$\text{(18)}\qquad \begin{cases} u_3 = \dfrac{\cos a\cos\omega - \cos b\cos c}{\sin b\sin c} \\ u_4 = -v_3 \\ v_3 = \pm\dfrac{\cos a\sin\omega}{\sin b\sin c} \\ v_4 = u_3\,. \end{cases}$$

2. Calculus gives:

$$\begin{aligned} \mathrm{tr}(A.B.C) &= 0 \\ \det(A.B.C) &= -\cos^2 a\cos^2 b\cos^2 c\,. \end{aligned}$$

The eigenvalues of $A.B.C$ are $\lambda = \pm\cos a\cos b\cos c$ and the similarity reduces to an homothetic transformation. This means we can choose the critical directions such that ω vanishes. In this case, according to (7), y also vanishes. Finally, it remains $x^2 = \cos^2 a$ and we get the two solutions:

$$\text{(19)}\qquad \begin{cases} u_3 = \dfrac{\pm\cos a - \cos b\cos c}{\sin b\sin c} \\ u_4 = v_3 = 0 \\ v_4 = \dfrac{\pm\cos a - \cos b\cos c}{\sin b\sin c}\,. \end{cases}$$

□

The solutions (18) and (19) exist effectively if the conditions $(C1)$ and $(C2)$ are satisfied. In the semi-regular case (for $a = b = c$), these conditions take the form:

in the case (18):

$$\begin{aligned} f(a,\omega) &= \frac{-1 + 3\cos^2 a - 2\cos^3 a\cos\omega}{\sin^4 a} \le 0 \\ g(a,\omega) &= -\frac{\left(1 - 3\cos^2 a + 2\cos^3 a\cos\omega\right)^2}{\sin^8 a} \le 0\,. \end{aligned}$$

For the second triangle, we just have to transform $\cos a$ into $-\cos a$;

in the case (19):

$$\begin{aligned} f(a) &= \frac{2\cos a - 1}{(\cos a - 1)^2} \le 0 \\ g(a) &= \frac{(2\cos a + 1)(2\cos a - 1)}{\sin^4 a} \le 0\,. \end{aligned}$$

For the second triangle, the same remark as for (18) holds.

Furthermore, the positive isoclinic triangles in $G_2(\mathbb{R}^6)$ behave as triangles in $\mathbb{CP}^2$. We have:

PROPOSITION 2. *1. There is a one-to-one correspondence between positive isoclinic triangles in $G_2(\mathbb{R}^6)$ and generic triangles in the complex projective plane $\mathbb{CP}^2$.*

2. The inequalities satisfied by the shape invariant of the projective triangle are equivalent to the conditions $(C1)$ and $(C2)$ for the Grassmannian triangle.

3. If a, b, c denote the length of the sides of the projective triangle, the link between the shape invariant σ and the inner angle ω is:

$$\sigma = \cos a\cos b\cos c\cos\omega\,.$$

Proof. 1. To an element $\bar{X} = [x] \in \mathbb{CP}^2$, we can associate the plane

$$X = x \wedge ix \in G_2(\mathbb{C}^3) \subset G_2(\mathbb{R}^6)$$

(X does not depend on the representing vector x).

Let us consider the canonical form of a triangle $\{\bar{A}, \bar{B}, \bar{C}\}$ in $\mathbb{CP}^2$ (see [3]):

$$\text{(20)} \qquad \begin{cases} \bar{A} = e_1 \\ \bar{B} = \cos c e_1 + \sin c e_2 \\ \bar{C} = \cos b e_2 + (z_2 + i\tilde{z}_2)e_2 + z_3 e_3 \end{cases}$$

with $z_2, \tilde{z}_2, z_3 \in \mathbb{R}$; $\tilde{z}_2, z_3 \ge 0$, $z_2^2 + \tilde{z}_2^2 + z_3^2 = \sin^2 b$.

If we set $ie_1 = e_4$, $ie_2 = e_5$, $ie_3 = e_6$, we get the following triangle in $G_2(\mathbb{R}^6)$:

$$\text{(21)} \qquad \begin{cases} \bar{A} = e_1 \wedge e_4 \\ \bar{B} = (\cos c e_1 + \sin c e_2) \wedge (\cos c e_4 + \sin c e_5) \\ \bar{C} = (\cos b e_1 + z_2 e_2 + z_3 e_3 + \tilde{z}_2 e_5) \wedge \left(-\tilde{z}_2 e_2 + \cos b e_4 + z_2 e_5 + z_3 e_6\right). \end{cases}$$

The calculus of $\mathrm{tr}(A.B)$, $\det(A.B)$, $\mathrm{tr}(A.C)$, $\det(A.C)$, $\mathrm{tr}(B.C)$, $\det(B.C)$ shows immediately that the triangle is isoclinic. Moreover:

$$\left\{\begin{array}{l} x = \cos b \cos c + z_2 \sin c \\ y = -\tilde{z}_2 \sin c \\ z = \tilde{z}_2 \sin c \\ t = \cos b \cos c + z_2 \sin c \end{array}\right. \tag{22}$$

shows that $\det(A.B.C) > 0$.

Conversely, to the triangle (14), we can associate the triangle in $\mathbb{CP}^2$ given by:

$$\left\{\begin{array}{l} \bar{A} = e_1 \\ \bar{B} = \cos c e_1 + \sin c e_3 \\ \bar{C} = \cos b e_1 + \sin b(-iu_3 + u_4)e_4 + z_3 e_5 + z_4 e_6 \end{array}\right. \tag{23}$$

(not in canonical form).

2. Let a, b, c denote the side lengths of $\{\bar{A}, \bar{B}, \bar{C}\}$ and

$$d([x],[y]) = \arccos \frac{|\langle x, y\rangle|}{\|x\| \cdot \|y\|}$$

the distance function. We have:

$$\begin{aligned} \cos d\left(\bar{A}, \bar{B}\right) &= \cos c \\ \cos d\left(\bar{A}, \bar{C}\right) &= \cos b \\ \cos^2 d\left(\bar{B}, \bar{C}\right) &= (\cos b \cos c + z_2 \sin c)^2 + \tilde{z}_2^2 \sin^2 c \\ &= x^2 + y^2 = \cos^2 a \,. \end{aligned}$$

Moreover, $\sigma = \cos b \cos c(\cos b \cos c + z_2 \sin c) = \cos b \cos c x$ (see [3]) but $x = \cos a \cos \omega$, which proves 3.

The existence conditions are

$$g = v_3^2 + v_4^2 - 1 \leq 0 \tag{C2}$$

and substituting into (15), we get:

$$\frac{y^2 + x^2 + \cos^2 b \cos^2 c - 2x \cos b \cos c - \sin^2 b \sin^2 c}{\sin^2 b \sin^2 c} \leq 0$$

$$f = u_3^2 + u_4^2 + v_3^2 + v_4^2 - (u_3 v_4 - u_4 v_3)^2 - 1 \leq 0 \tag{C1}$$

gives:

$$f = 2\left(v_3^2 + v_4^2\right) - \left(v_3^2 + v_4^2\right)^2 - 1 \leq 0 \,.$$

If $(C2)$ is satisfied, it implies automatically that $(C1)$ is also satisfied.

Now, the inequalities involving the shape invariant σ are:

$$\frac{1}{2}\left(\cos^2 a + \cos^2 b + \cos^2 c - 1\right) \leq \sigma \leq |\sigma| \leq \cos a \cos b \cos c \,.$$

Substituting $\sigma = \cos b \cos cx$ and $x^2 + y^2 = \cos^2 a$, we obtain:

$$\frac{1}{2}\left(x^2 + y^2 + \cos^2 b + \cos^2 c - 1\right) \le \cos b \cos cx \le \cos b \cos c|x| \le \cos b \cos c\sqrt{x^2 + y^2}\,.$$

The first inequality is equivalent to $(C2)$ because $\cos^2 b \cos^2 c - \sin^2 b \sin^2 c = \cos^2 b + \cos^2 c - 1$; the second inequality is always true. □

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References

[1] ASLAKSEN H., *Laws of trigonometry on $SU(3)$*, Trans. A.M.S. **317** (1990) (1), 127–142.

[2] BLASCHKE W., TERHEGGEN H., *Trigonometria Hermitiana*, Rend. Sem. Mat. Univ. Roma **3** (1939), 153–161.

[3] BREHM U., *The shape invariant of triangles and trigonometry in 2-point homogeneous spaces*, Geom. Dedicata **33** (1990) (1), 59–76.

[4] COOLIDGE J.L., *Hermitian metrics*, Annals of Math. **22** (1921), 11–28.

[5] FRUCHARD A., *Les triplets de la variété de Grassmann $G_2(\mathbb{R}^6)$*, Preprint 1995.

[6] HANGAN TH., *Formules de trigonométrie sur la variété de Grassmann*, Rend. Sem. Mat. Pol. Torino **50** (1992) (4), 367–380.

[7] HANGAN TH., MASALA G., *A geometrical interpretation of the shape invariant for geodesics triangles in complex projective spaces*, Geom. Dedicata **49** (1994), 129–134.

[8] HSIANG W.-Y., *On the laws of trigonometries of 2-point homogeneous spaces*, Ann. Global Anal. Geom. **7** (1989) (1), 29–45.

[9] LEUZINGER E., *On the trigonometry of symmetric spaces*, Comment. Math. Helv. **67** (1992), 252–292.

[10] MASALA G., *Théorèmes de congruence pour triplets de points dans la variété de Grassmann $G_2(\mathbb{R}^n)$*, Rend. Circ. Mat. Palermo **45** (1996) (3) - Serie 2, 351–376.

[11] MASALA G., *Trigonométrie et polyèdres dans les variétés de Grassmann $G_2(\mathbb{R}^n)$*, Doctoral Dissertation, U.H.A.-Mulhouse, 1996.

[12] MASALA G., *Congruence theorem for 4-tuples in the Grassmann manifold $G_2(\mathbb{R}^8)$*, Preprint.

[13] PROCESI C., *The invariant theory of $n \times n$ matrices*, Adv. in Math. **19** (1976), 306–381.

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