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Measuring Degree of Departure from Extended Quasi-Symmetry for Square Contingency Tables

Medición del grado alejamiento del modelo extendido cuasi simétrico para tablas de contingencia cuadradas

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Abstract

For square contingency tables with ordered categories, the present paper proposes a measure to represent the degree of departure from the extended quasi-symmetry (EQS) model. It is expressed by using the Cressie-Read power-divergence or Patil-Taillie diversity index. The present paper also defines the maximum departure from EQS which indicates the maximum departure from the uniformity of ratios of symmetric odds-ratios. The measure lies between 0 and 1, and it is useful for not only seeing the degree of departure from EQS in a table but also comparing it in several tables.

***Key words*:** Contingency table, Kullback-Leibler information, Quasi-symmetry, Shannon entropy.

Resumen

El presente artículo propone una medida para representar el grado de alejamiento del modelo extendido cuasisimétrico (EQS, por su sigla en inglés) para tablas de contingencia con categorías ordenadas. Esta medida se expresa mediante el uso de la divergencia de potencia de Cressie-Read o el índice de diversidad Patil-Taillie. Nuestro trabajo también define el máximo alejamiento de EQS, el cual indica el alejamiento máximo de la uniformidad de razones de odds-ratios simétricos. La medida cae entre 0 y 1 y es útil no solo para determinar el grado de alejamiento de EQS en una tabla, sino también para comparar este grado de alejamiento en varias tablas.

***Palabras clave*:** cuasi-simetría, entropía de Shannon, información de Kullback-Leibler, tablas de contingencia.

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1. Introduction

Consider an $R \times R$ square contingency table with same row and column classifications. Let p_{ij} denote the probability that an observation will fall in the ith row and the jth column of the table ($i = 1, \dots, R; j = 1, \dots, R$). Bowker (1948) considered the symmetry (S) model defined by

$$p_{ij} = \phi_{ij} \quad \text{for} \quad i = 1, \dots, R; j = 1, \dots, R$$

where $\phi_{ij} = \phi_{ji}$ (Bishop, Fienberg & Holland 1975, p. 282). Caussinus (1965) considered the quasi-symmetry (QS) model defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad \text{for} \quad i = 1, \dots, R; j = 1, \dots, R$$

where $\psi_{ij} = \psi_{ji}$. A special case of this model obtained by putting $\{\alpha_i = \beta_i\}$ is the S model. For square tables with ordered categories, Tomizawa (1984) proposed the extended quasi-symmetry (EQS) model defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad \text{for} \quad i = 1, \dots, R; j = 1, \dots, R$$

where $\psi_{ij} = \gamma \psi_{ji}$ $(i < j)$. A special case of this model obtained by putting $\gamma = 1$ is the QS model. This is also expressed as, using the odds-ratios including the cell probabilities on the main diagonal,

$$\theta_{(i<j;j<k)} = \gamma \theta_{(j<k;i<j)} \quad \text{for} \quad i < j < k$$

where

$$\theta_{(i<j;j<k)} = \frac{p_{ij} p_{jk}}{p_{jj} p_{ik}}, \quad \theta_{(j<k;i<j)} = \frac{p_{ji} p_{kj}}{p_{ki} p_{jj}}$$

This indicates that the ratios of odds-ratios with respect to the main diagonal of the table are uniform for all $i < j < k$. The EQS model may be expressed as

$$D_{ijk} = \gamma D_{kji} \quad \text{for} \quad i < j < k,$$

where

$$D_{ijk} = p_{ij} p_{jk} p_{ki}, \quad D_{kji} = p_{kj} p_{ji} p_{ik}$$

For the analysis of square contingency tables, when a model does not hold, one may be interested in measuring how far the degree of departure from the model is. Thus some measures of various symmetry have been proposed. For example, Tomizawa (1994) and Tomizawa, Seo & Yamamoto (1998) proposed the measures to represent the degree of departure from the S model for square tables with *nominal* categories. Tomizawa, Miyamoto & Hatanaka (2001) proposed the measure for the S model for square tables with *ordered* categories. Tahata, Miyamoto & Tomizawa (2004) proposed the measure to represent the degree of departure from the QS model for square tables with *nominal* categories.

Generally, when the EQS model does not hold, we may apply a model which is extension of EQS model. Such models have been discussed by, e.g., Yamaguchi (1990), Tomizawa (1990) and Lawal (2004). On the other hand, we are also interested in measuring the degree of departure from the EQS model as described above. However a measure, which represents the degree of departure from the EQS model, does not exist. Therefore, we are interested in proposing a measure to represent the degree of departure from the EQS model, for square tables with *ordered* categories.

Table 1: Cross-classification of father and son social classes; taken from Hashimoto (2003, p. 142).

(a) Examined in 1955

	Son's class					
Father's class	(1)	(2)	(3)	(4)	(5)	Total
(1)	39	39	39	57	23	197
(2)	12	78	23	23	37	173
(3)	6	16	78	23	20	143
(4)	18	80	79	126	31	334
(5)	28	106	136	122	628	1020
Total	103	319	355	351	739	1867

(b) Examined in 1975

	Son's class					
Father's class	(1)	(2)	(3)	(4)	(5)	Total
(1)	29	43	25	31	4	132
(2)	23	159	89	38	14	323
(3)	11	69	184	34	10	308
(4)	42	147	148	184	17	538
(5)	42	176	377	114	298	1007
Total	147	594	823	401	343	2308

(c) Examined in 1995

	Son's class					
Father's class	(1)	(2)	(3)	(4)	(5)	Total
(1)	68	48	36	23	1	176
(2)	33	191	102	33	3	362
(3)	25	147	229	34	2	437
(4)	48	119	146	129	5	447
(5)	40	126	192	82	88	528
Total	214	631	705	301	99	1950

Consider the data in Table 1, taken from Hashimoto (2003, p. 142). These data describe the cross-classification of father and son social classes in Japan, which were examined in 1955, 1975, and 1995. Note that status (1) is Capitalist; (2) New-middle; (3) Working; (4) Self-employed; and (5) Farming. For social mobility data, one may be interested in considering the structure of symmetry instead of independence between row and column variables. Thus, for example the S, QS and EQS models would be useful for analyzing the data. For these data in Table 1, "$i \rightarrow j$" denotes the move to the son's class j from his father's class i. Thus $\{p_{ij}\}$ could be interpreted as transition probabilities. The EQS model indicates that for a given order $i < j < k$, the product of transition probabilities that connects a cyclic sequence of paths $i \to j \to k \to i$ (we shall call the probability for *right cyclic sequence of paths* $i \to j \to k \to i$ for convenience), which includes two upward moves $i \to j$ and $j \to k$ and one downward move $k \to i$, is γ times higher than the product of transition probabilities that represents a reverse cyclic sequence of paths $i \to k \to j \to i$ (we shall call the probability for *left cyclic sequence of paths* $i \to k \to j \to i$), which includes one upward move $i \to k$ and two downward moves $k \to j$ and $j \to i$.

The EQS model can also be expressed as

$$D_{ijk}^{(1)} = D_{ijk}^{(2)} \quad \text{for} \quad i < j < k, \tag{1}$$

where

$$D_{ijk}^{(1)} = \frac{D_{ijk}}{\sum_{s<t<u} D_{stu}}, \quad D_{ijk}^{(2)} = \frac{D_{kji}}{\sum_{s<t<u} D_{uts}}$$

For the data in Tables 1a, 1b and 1c, $D_{ijk}^{(1)}$ is conditional probability that for any three father-son pairs father's class and his son's class are (i, j), (j, k) and (k, i), on condition that there is right cyclic sequence of paths. Similarly, $D_{ijk}^{(2)}$ is conditional probability that for any three father-son pairs father's class and his son's class are (j, i), (k, j) and (i, k), on condition that there is left cyclic sequence of paths. In a similar manner to Tomizawa et al. (1998), we shall consider a measure which represents the degree of departure from EQS because the equation (1) states that there is a structure of symmetry between $\{D_{ijk}^{(1)}\}$ and $\{D_{ijk}^{(2)}\}$ for $i < j < k$.

Section 2 proposes the measure to represent the degree of departure from the EQS model. Section 3 gives the approximate confidence interval for the measure. Section 4 shows an example.

2. Measure of Extended Quasi-Symmetry

Assume that $\sum_{s<t<u} D_{stu} \neq 0$, $\sum_{s<t<u} D_{uts} \neq 0$ and $D_{ijk} + D_{kji} > 0$ for $i < j < k$. Let

$$E_{ijk}^{(1)} = \frac{D_{ijk}^{(1)}}{D_{ijk}^{(1)} + D_{ijk}^{(2)}}, \quad E_{ijk}^{(2)} = \frac{D_{ijk}^{(2)}}{D_{ijk}^{(1)} + D_{ijk}^{(2)}} \quad \text{for} \quad i < j < k$$

For the data in Tables 1a, 1b and 1c, $E_{ijk}^{(1)}$ is the proportion of the conditional probability $D_{ijk}^{(1)}$ to the sum of the conditional probabilities $D_{ijk}^{(1)} + D_{ijk}^{(2)}$. Similarly, $E_{ijk}^{(2)}$ is the proportion of $D_{ijk}^{(2)}$ to $D_{ijk}^{(1)} + D_{ijk}^{(2)}$. The EQS model can be expressed as

$$E_{ijk}^{(1)} = E_{ijk}^{(2)} = \frac{1}{2} \quad \text{for} \quad i < j < k$$

Consider the measure defined by

$$\Phi^{(\lambda)} = \frac{\lambda(\lambda+1)}{2(2^{\lambda}-1)} \sum_{i<j<k} \left(D_{ijk}^{(1)} + D_{ijk}^{(2)}\right) I_{ijk}^{(\lambda)} \quad \text{for} \quad \lambda > -1$$

where

$$I_{ijk}^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \left[E_{ijk}^{(1)} \left\{ \left(\frac{E_{ijk}^{(1)}}{1/2} \right)^{\lambda} - 1 \right\} + E_{ijk}^{(2)} \left\{ \left(\frac{E_{ijk}^{(2)}}{1/2} \right)^{\lambda} - 1 \right\} \right]$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Thus,

$$\Phi^{(0)} = \frac{1}{2(\log 2)} \sum_{i<j<k} \left(D_{ijk}^{(1)} + D_{ijk}^{(2)}\right) I_{ijk}^{(0)}$$

where

$$I_{ijk}^{(0)} = E_{ijk}^{(1)} \log \left(\frac{E_{ijk}^{(1)}}{1/2} \right) + E_{ijk}^{(2)} \log \left(\frac{E_{ijk}^{(2)}}{1/2} \right)$$

Note that a real value λ is chosen by the user. The $I_{ijk}^{(\lambda)}$ is the modified power-divergence and especially $I_{ijk}^{(0)}$ is the Kullback-Leibler information. For more details of the power-divergence, see Cressie & Read (1984). The measure $\Phi^{(\lambda)}$ would represent, essentially, the weighted sum of the power-divergence $I_{ijk}^{(\lambda)}$.

The measure may be expressed as

$$\Phi^{(\lambda)} = 1 - \frac{\lambda 2^{\lambda-1}}{2^{\lambda}-1} \sum_{i<j<k} \left(D_{ijk}^{(1)} + D_{ijk}^{(2)}\right) H_{ijk}^{(\lambda)} \quad \text{for} \quad \lambda > -1$$

where

$$H_{ijk}^{(\lambda)} = \frac{1}{\lambda} \left[1 - \left(E_{ijk}^{(1)}\right)^{\lambda+1} - \left(E_{ijk}^{(2)}\right)^{\lambda+1} \right]$$

with

$$\Phi^{(0)} = 1 - \frac{1}{2(\log 2)} \sum_{i<j<k} \left(D_{ijk}^{(1)} + D_{ijk}^{(2)}\right) H_{ijk}^{(0)}$$

where

$$H_{ijk}^{(0)} = -E_{ijk}^{(1)} \log E_{ijk}^{(1)} - E_{ijk}^{(2)} \log E_{ijk}^{(2)}$$

Note that $H_{ijk}^{(\lambda)}$ is the Patil & Taillie (1982) diversity index, which includes the Shannon entropy when $\lambda = 0$. Therefore, $\Phi^{(\lambda)}$ would represent one minus the weighted sum of the diversity index $H_{ijk}^{(\lambda)}$.

For each λ, the minimum value of $H_{ijk}^{(\lambda)}$ is 0 when $E_{ijk}^{(1)} = 0$ (then $E_{ijk}^{(2)} = 1$) or $E_{ijk}^{(2)} = 0$ (then $E_{ijk}^{(1)} = 1$), and the maximum value is $(2^{\lambda} - 1)/\lambda 2^{\lambda}$ (if $\lambda \neq 0$), $\log 2$ (if $\lambda = 0$), when $E_{ijk}^{(1)} = E_{ijk}^{(2)}$. Thus we see that $\Phi^{(\lambda)}$ lies between 0 and 1. Also for each λ, (i) there is a structure of EQS in the table (i.e., $E_{ijk}^{(1)} = E_{ijk}^{(2)} = 1/2$, (thus $D_{ijk}^{(1)} = D_{ijk}^{(2)}$) for any $i < j < k$) if and only if $\Phi^{(\lambda)} = 0$; and (ii) the degree of departure from EQS is the largest, in the sense that $E_{ijk}^{(1)} = 0$ (then $E_{ijk}^{(2)} = 1$) or $E_{ijk}^{(2)} = 0$ (then $E_{ijk}^{(1)} = 1$) (i.e., $D_{ijk}^{(1)} = 0$ (then $D_{ijk}^{(2)} > 0$) or $D_{ijk}^{(2)} = 0$ (then $D_{ijk}^{(1)} > 0$)) for any $i < j < k$, if and only if $\Phi^{(\lambda)} = 1$. Note that $\Phi^{(\lambda)} = 1$ indicates that $D_{ijk}^{(1)}/D_{ijk}^{(2)} = \infty$ for some $i < j < k$ and $D_{ijk}^{(1)}/D_{ijk}^{(2)} = 0$ for the other $i < j < k$, and therefore it seems appropriate to consider that then the degree of departure from EQS (i.e., from $D_{ijk}^{(1)}/D_{ijk}^{(2)} = 1$ for $i < j < k$) is largest.

According to the weighted sum of power-divergence or the weighted sum of Patil-Taillie diversity index, $\Phi^{(\lambda)}$ represents the degree of departure from EQS, and the degree increases as the value of $\Phi^{(\lambda)}$ increases.

3. Approximate Confidence Interval for Measure

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1, \ldots, R; j = 1, \ldots, R$) with $n = \sum\sum n_{ij}$. Assume that $\{n_{ij}\}$ have a multinomial distribution. We shall consider an approximate standard error and large-sample confidence interval for the measure $\Phi^{(\lambda)}$ using the delta method as described by Bishop et al. (1975, Section 14.6). The sample version of $\Phi^{(\lambda)}$, i.e., $\widehat{\Phi}^{(\lambda)}$, is given by $\Phi^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$, where $\widehat{p}_{ij} = n_{ij}/n$. Using the delta method, $\sqrt{n}(\widehat{\Phi}^{(\lambda)} - \Phi^{(\lambda)})$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance

$$\sigma^2 = \sum_{a=1}^{R-1} \sum_{b=a+1}^{R} \left\{ \frac{1}{p_{ab}} \left(A_{ab}^{(\lambda)} \right)^2 + \frac{1}{p_{ba}} \left(B_{ab}^{(\lambda)} \right)^2 \right\} - \left\{ \sum_{a=1}^{R-1} \sum_{b=a+1}^{R} \left(A_{ab}^{(\lambda)} + B_{ab}^{(\lambda)} \right) \right\}^2$$

where for $\lambda > -1$ and $\lambda \neq 0$,

$$\begin{aligned} A_{ab}^{(\lambda)} = \frac{2^{\lambda-1}}{2^\lambda - 1} \sum_{i<j<k} \Big[& (E_{ijk}^{(1)})^\lambda D_{ijk}^{(1)} \Big\{ I_{(a=i,b=j)} + I_{(a=j,b=k)} \\ & \qquad - \sum_{s<t<u} D_{stu}^{(1)} (I_{(a=s,b=t)} + I_{(a=t,b=u)}) \Big\} \\ & + (E_{ijk}^{(2)})^\lambda D_{ijk}^{(2)} \Big\{ I_{(a=i,b=k)} - \sum_{s<t<u} D_{stu}^{(2)} I_{(a=s,b=u)} \Big\} \\ + \lambda \Big(D_{ijk}^{(2)} (E_{ijk}^{(1)})^{\lambda+1} - D_{ijk}^{(1)} (E_{ijk}^{(2)})^{\lambda+1} \Big) & \Big\{ \Big(I_{(a=i,b=j)} + I_{(a=j,b=k)} - I_{(a=i,b=k)} \\ & - \sum_{s<t<u} (D_{stu}^{(1)} I_{(a=s,b=t)} + D_{stu}^{(1)} I_{(a=t,b=u)} - D_{stu}^{(2)} I_{(a=s,b=u)}) \Big) \Big\} \Big] \end{aligned}$$

and

$$A_{ab}^{(0)} = \frac{1}{2\log 2} \sum_{i<j<k} \Big[D_{ijk}^{(1)} (\log E_{ijk}^{(1)}) \Big\{ I_{(a=i,b=j)} + I_{(a=j,b=k)} - \sum_{s<t<u} D_{stu}^{(1)} (I_{(a=s,b=t)} + I_{(a=t,b=u)}) \Big\} + D_{ijk}^{(2)} (\log E_{ijk}^{(2)}) \Big\{ I_{(a=i,b=k)} - \sum_{s<t<u} D_{stu}^{(2)} I_{(a=s,b=u)} \Big\} \Big]$$

with

$$I_{(a=i,b=j)} = \begin{cases} 1 & (a=i \quad \text{and} \quad b=j) \\ 0 & (\text{otherwise}) \end{cases}$$

and where $B_{ab}^{(\lambda)}$ for $\lambda > -1$ is defined as $A_{ab}^{(\lambda)}$ obtained by interchanging $D_{ijk}^{(1)}$ and $D_{ijk}^{(2)}$ and by interchanging $E_{ijk}^{(1)}$ and $E_{ijk}^{(2)}$.

Although the detail is omitted, (i) when $\Phi^{(\lambda)} = 0$, we can get $\sigma^2 = 0$ by noting $D_{ijk}^{(1)} = D_{ijk}^{(2)}$ and $E_{ijk}^{(1)} = E_{ijk}^{(2)} = 1/2$ for $i < j < k$, and (ii) when $\Phi^{(\lambda)} = 1$, we can get $\sigma^2 = 0$ by noting $D_{ijk}^{(1)} = 0$ (then $E_{ijk}^{(1)} = 0$ and $E_{ijk}^{(2)} = 1$) for some $i < j < k$ and $D_{ijk}^{(2)} = 0$ (then $E_{ijk}^{(1)} = 1$ and $E_{ijk}^{(2)} = 0$) for the other $i < j < k$. Thus we note that the asymptotic distribution of $\widehat{\Phi}^{(\lambda)}$ is not applicable when $\Phi^{(\lambda)} = 0$ and $\Phi^{(\lambda)} = 1$. Let $\widehat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$. Then $\widehat{\sigma}/\sqrt{n}$ is an estimated approximate standard error for $\widehat{\Phi}^{(\lambda)}$.

4. An Example

Consider the data in Table 1 again. Then, the *maximum* departure from the EQS model indicates that for some $i < j < k$, the product of transition probabilities that connects $i \to j \to k \to i$ is zero, (and then the product of transition probabilities that represents $i \to k \to j \to i$ is not zero) and for the others the product of transition probabilities that connects $i \to j \to k \to i$ is not zero (and then the product of transition probabilities that represents $i \to k \to j \to i$ is zero); namely, the *stochastic circular* social mobility arises among any three father-son pairs.

Now we consider comparing the degree of departure from the EQS model for the data in Tables 1a, 1b and 1c. We choose $\lambda = 0$ because $\Phi^{(0)}$ is expressed as well known Kullback-Leibler information. Thus we apply the measure $\Phi^{(0)}$ for these data. Table 2 shows the estimated measure $\widehat{\Phi}^{(0)}$, estimated approximate standard error for $\widehat{\Phi}^{(0)}$, and approximate 95% confidence interval for $\Phi^{(0)}$. When the degrees of departure from the EQS model in Tables 1a, 1b and 1c are compared using the estimated measure $\widehat{\Phi}^{(0)}$, (i) the value of $\widehat{\Phi}^{(0)}$ is greater for Table 1a than for Tables 1b and 1c, and (ii) the value of $\widehat{\Phi}^{(0)}$ is greater for Table 1b than for Table 1c. Namely, the degree of departure from the EQS model for Table 1a is the largest, that for Table 1b is the second largest, and that for Table 1c is the smallest. Thus, the data in Table 1a rather than in Tables 1b and 1c are estimated to be close to the *maximum* departure from the EQS model.

Table 2: Estimated measure $\widehat{\Phi}^{(0)}$, estimated approximate standard error for $\widehat{\Phi}^{(0)}$, and approximate 95% confidence interval for $\Phi^{(0)}$, applied to Tables 1a, 1b, and 1c.

Table	Estimated measure	Standard error	Confidence interval
1a	0.076	0.039	(−0.001, 0.153)
1b	0.036	0.034	(−0.031, 0.102)
1c	0.011	0.018	(−0.024, 0.046)

5. Discussions and Conclusion

The measure $\Phi^{(\lambda)}$ always ranges between 0 and 1 independently of the dimension R and sample size n. But the likelihood-ratio statistic for testing goodness-of-fit of the EQS model depends on sample size n. For example, consider two $R \times R$ contingency tables, say, A and B, where the observed frequency in each cell for Table A has ten times that in the corresponding cell for table B. Then the value of likelihood-ratio statistic for testing goodness-of-fit of the EQS model for table A is ten times that for table B. However, when the ratios of odds-ratios, $\widehat{\theta}_{(i<j;j<k)}/\widehat{\theta}_{(j<k;i<j)}$, $i < j < k$, for table A is equal to that for table B, the value of measure $\widehat{\Phi}^{(\lambda)}$ for table A is equal to that for table B. Therefore, $\widehat{\Phi}^{(\lambda)}$ would be useful for comparing the degree of departure from EQS in several tables, even if several tables have different sample sizes.

As described in Section 2, the proposed measure would be useful when we want to see with single summary measure how degree the departure from EQS is toward the maximum degree of departure from EQS. We have defined the maximum degree of departure from EQS, namely, $D_{ijk}^{(1)}/D_{ijk}^{(2)} = \infty$ for some $i < j < k$ and $D_{ijk}^{(1)}/D_{ijk}^{(2)} = 0$ for the other $i < j < k$. This seems natural as the definition of the maximum departure from EQS that indicates $D_{ijk}^{(1)}/D_{ijk}^{(2)} = 1$ for $i < j < k$.

Table 3: Values of power-divergence test statistic $W^{(\lambda)}$ (with 5 degrees of freedom), applied to Tables 1a, 1b, and 1c.

λ	For Table 1a	For Table 1b	For Table 1c
−0.4	13.70	4.63	1.62
0.0	13.59	4.66	1.60
0.6	13.48	4.73	1.56
1.0	13.43	4.79	1.55
1.4	13.40	4.86	1.53

TABLE 4: Artificial data (n is sample size).

(a) $n = 700$

30	81	79	120
10	39	83	16
13	20	38	31
7	35	77	21

(b) $n = 668$

30	29	60	10
110	39	33	36
21	42	38	61
15	61	62	21

TABLE 5: Values of $\widehat{\Phi}^{(\lambda)}$, the test statistic $W^{(\lambda)}$ and $W^{(\lambda)}/n$ applied to Tables 4a and 4b.

(a) Values of $\widehat{\Phi}^{(\lambda)}$

λ	For Table 4a	For Table 4b
-0.4	0.268	0.225
0.0	0.363	0.304
0.6	0.436	0.364
1.0	0.456	0.381
1.4	0.463	0.387

(b) Values of $W^{(\lambda)}$

λ	For Table 4a	For Table 4b
-0.4	27.76	52.90
0.0	28.33	51.95
0.6	30.13	51.03
1.0	32.12	50.72
1.4	34.92	50.64

(c) Values of $W^{(\lambda)}/n$

λ	For Table 4a	For Table 4b
-0.4	0.040	0.079
0.0	0.040	0.078
0.6	0.043	0.076
1.0	0.046	0.076
1.4	0.050	0.076

Consider the data in Table 1, again. Cressie & Read (1984) proposed the power-divergence test statistic for testing goodness-of-fit of a model. Denote the power-divergence statistic for testing goodness-of-fit of the EQS model with $R(R-3)/2$

degrees of freedom by $W^{(\lambda)}$. Table 3 gives the values of $W^{(\lambda)}$ applied to the data in Tables 1a, 1b and 1c. The EQS model fits the data in Table 1a poorly; however, fits the data in Tables 1b and 1c well. This is similar to the results described in Section 4. Then, it may seem to many readers that $W^{(\lambda)}/n$ (for a given λ) is also a reasonable measure for representing the degree of departure from EQS. However, we point out that $W^{(\lambda)}$ can not measure the degree of departure from EQS toward the maximum degree of departure from EQS that is defined in Section 2, although $W^{(\lambda)}$ can test the goodness-of-fit of the EQS model. For example, consider the artificial data in Tables 4a and 4b. From Table 5, the value of $W^{(\lambda)}/n$ ($W^{(\lambda)}$) is less for Table 4a than for Table 4b; however, the value of $\widehat{\Phi}^{(\lambda)}$ is greater for Table 4a than for Table 4b. When we want to measure the degree of departure from EQS toward the maximum departure from the uniformity of ratios of symmetric odds-ratios (i.e., the maximum departure from EQS), the measure $\Phi^{(\lambda)}$ rather than $W^{(\lambda)}$ may be appropriate. Also, $W^{(\lambda)}$ rather than $\Phi^{(\lambda)}$ would be appropriate to test the goodness-of-fit of the EQS model.

As described in Section 1, Lawal (2004), Tomizawa (1990) and Yamaguchi (1990) considered the extension of EQS model. For testing goodness-of-fit of the EQS model under the assumption that the extension of EQS model holds true, the difference between the likelihood ratio statistic for the EQS and extension of EQS models has an asymptotic chi-squared distribution with degrees of freedom equal to the difference between degrees of freedom for two models. This statistic, which is useful for comparing pairs of models, is well known. So, the readers may consider that this statistic is also a reasonable measure for representing the degree of departure from EQS. However, since this statistic can not measure the degree of departure from EQS toward the maximum departure from EQS, $\Phi^{(\lambda)}$ rather than it would be preferable when we want to measure the degree of departure from EQS toward the maximum degree of departure from EQS.

We observe that the EQS model and the measure $\Phi^{(\lambda)}$ should be applied to square tables with *ordered* categories because it is not invariant under the arbitrary similar permutations of row and column categories.

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