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AN ORLIK–SOLOMON TYPE ALGEBRA FOR MATROIDS WITH A FIXED LINEAR CLASS OF CIRCUITS

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Abstract: A family $\mathcal{C}_L$ of circuits of a matroid M is a linear class if, given a modular pair of circuits in $\mathcal{C}_L$, any circuit contained in the union of the pair is also in $\mathcal{C}_L$. The pair $(M, \mathcal{C}_L)$ can be seen as a matroidal generalization of a biased graph. We introduce and study an Orlik–Solomon type algebra determined by $(M, \mathcal{C}_L)$. If $\mathcal{C}_L$ is the set of all circuits of M this algebra is the Orlik–Solomon algebra of M.

1 – Introduction

Let $\mathcal{A}_{\mathbb{C}} = \{H_1, \dots, H_n\}$ be a central and essential arrangement of hyperplanes in $\mathbb{C}^d$ (i.e, such that $\bigcap_{H_i \in \mathcal{A}_{\mathbb{C}}} H_i = \{0\}$). The manifold $\mathfrak{M} = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}_{\mathbb{C}}} H$ plays an important role in the Aomoto–Gelfand multivariable theory of hypergeometric functions (see [9] for a recent introduction from the point of view of arrangement theory). There is a rank d matroid $M := M(\mathcal{A}_{\mathbb{C}})$ on the ground set $[n]$ canonically determined by $\mathcal{A}_{\mathbb{C}}$: a subset $D \subseteq [n]$ is a *dependent set* of M if and only if there are scalars $\zeta_i \in \mathbb{C}$, $i \in D$, not all nulls, such that $\sum_{i \in D} \zeta_i \, \theta_{H_i} = 0$, where $\theta_{H_i} \in (\mathbb{C}^d)^*$ denotes a linear form such that $\mathrm{Ker}(\theta_{H_i}) = H_i$.

Let M be a matroid and $M^\star$ be its dual. In the following, we suppose that the ground set of M is $[n] := \{1, 2, \dots, n\}$ and its rank function is denoted by r_M.

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The subscript M in r_M will often be omitted. Let $\mathcal{C} = \mathcal{C}(M)$ be the family of circuits of M. Let $\mathbf{K}$ be a field and $E = \{e_1, \ldots, e_n\}$ be a finite set of order n. Let $\bigoplus_{e\in E} \mathbf{K}e$ be the vector space over $\mathbf{K}$ of basis E and $\mathcal{E}$ be the graded exterior algebra $\bigwedge\big(\bigoplus_{e\in E} \mathbf{K}e\big)$, i.e.,

$$\mathcal{E} \;:=\; \sum_{i=0} \mathcal{E}_i \;=\; \mathcal{E}_0(=\mathbf{K}) \oplus \mathcal{E}_1\Big(=\bigoplus_{e\in E} \mathbf{K}e\Big) \oplus \cdots \oplus \mathcal{E}_i\Big(=\bigwedge^{i}\Big(\bigoplus_{e\in E} \mathbf{K}e\Big)\Big) \oplus \cdots \;.$$

For every linearly ordered subset $X = \{i_1, \ldots, i_m\} \subseteq [n]$, $i_1 < \cdots < i_m$, let e_X be the monomial $e_X := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$. By definition set $e_\emptyset = 1 \in \mathbf{K}$. Consider the map $\partial\colon \mathcal{E} \to \mathcal{E}$, extended by linearity from the "differentials", $\partial e_i = 1$ for every $e_i \in E$, $\partial e_\emptyset = 0$ and

$$\partial e_X \;=\; \partial(e_{i_1} \wedge \cdots \wedge e_{i_m}) \;=\; \sum (-1)^j\, e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_m} \;.$$

The (graded) *Orlik–Solomon* $\mathbf{K}$*-algebra* $\mathrm{OS}(M)$ of the matroid M is the quotient $\mathcal{E}/\mathfrak{I}$ where $\mathfrak{I}$ denotes the (homogeneous) two-sided ideal of $\mathcal{E}$ generated by the set

$$\Big\{\partial e_C\colon\ C \in \mathcal{C}(M),\ |C| > 1\Big\} \cup \Big\{e_C\colon\ C \in \mathcal{C}(M),\ |C| = 1\Big\}$$

or equivalently by the set

$$\Big\{\partial e_C\colon\ C \in \mathcal{C}(M),\ |C| > 1\Big\} \cup \Big\{e_C\colon\ C \in \mathcal{C}(M)\Big\} \;.$$

The de Rham cohomology algebra $H^\bullet\big(\mathfrak{M}(\mathcal{A}_\mathbb{C}); \mathbf{K}\big)$ is shown to be isomorphic to the Orlik–Solomon $\mathbf{K}$-algebra of the matroid $M(\mathcal{A}_\mathbb{C})$, see [6, 7]. We refer to [5] for a recent discussion on the role of matroid theory in the study of Orlik–Solomon algebras.

2 – Linear class of circuits

Given a family $\mathcal{C}$ of circuits of a matroid M set

$$\mathcal{H}(\mathcal{C}) \;:=\; \Big\{H(C) = [n]\backslash C\colon\ C \in \mathcal{C}_L\Big\}$$

be the associated family of hyperplanes of $M^\star$. We recall that a pair $\{X, Y\}$ of subsets of the ground set $[n]$ is a *modular pair* of $M([n])$ if

$$\mathrm{r}(X) + \mathrm{r}(Y) \;=\; \mathrm{r}(X \cup Y) + \mathrm{r}(X \cap Y) \;.$$

Proposition 2.1. *Let $\{C_1, C_2\}$ be a pair of circuits of M and $\{H(C_1), H(C_2)\}$ be the associated hyperplanes of $M^\star$. The following four conditions are equivalent:*

- $\{C_1, C_2\}$ *is a modular pair of circuits of* M,
- $\{H(C_1), H(C_2)\}$ *is a modular pair of hyperplanes of* $M^\star$,
- $\mathrm{r}_M(C_1 \cup C_2) = |C_1 \cup C_2| - 2$,
- $\mathrm{r}_{M^\star}\big(H(C_1) \cap H(C_2)\big) = \mathrm{r}(M^\star) - 2\,(= n{-}r{-}2)$. ∎

Definition 2.2 ([10]). We say that the family of circuits $\mathcal{C}'$, $\mathcal{C}' \subseteq \mathcal{C}(M)$, is a *linear class of circuits* if, given a modular pair of circuits in $\mathcal{C}'$, all the circuits contained in the union of the modular pair are also in $\mathcal{C}'$. □

In the following we will always denote by $\mathcal{C}_L$ a linear class of circuits of the matroid M.

Definition 2.3. We say that the family $\mathcal{H}$ of hyperplanes of M is a *linear class of hyperplanes of M* if, given a modular pair of hyperplanes in $\mathcal{H}$, all the hyperplanes of M containing the intersection of the pair are also in $\mathcal{H}$. □

The following corollary is a direct consequence of Proposition 2.1 and Definitions 2.2 and 2.3.

Corollary 2.4. *The following two assertions are equivalent:*

- *The family $\mathcal{C}'$ is a linear class of circuits of M;*
- *The set $\mathcal{H}(\mathcal{C}')$ is a linear class of hyperplanes of $M^\star$.* ∎

Remark 2.5. The linear class of hyperplanes $\mathcal{H}(\mathcal{C}_L)$ of $M^\star$ determines a single-element extension

$$M^\star([n]) \overset{\mathcal{H}(\mathcal{C}_L)}{\hookrightarrow} N^\star\big([n{+}1]\big)\,,$$

where $\{n+1\}$ is in the closure in $N^\star([n+1])$ of a hyperplane H of $M^\star([n])$, if and only if $H \in \mathcal{H}(\mathcal{C}_L)$. Two special cases occur:

- If $\mathcal{C}_L = \mathcal{C}(M)$ the element $n+1$ is a coloop of $N([n+1])$.
- If $\mathcal{C}_L = \emptyset = \mathcal{H}(\mathcal{C}_L)$ the element $n{+}1$ is a is in general position in $N^\star([n{+}1])$.

In the literature $N([n+1])$ is called the *extended lift of $M([n])$ (determined by the linear class of circuits $\mathcal{C}_L$).* □

Lemma 2.6. *Let $N = N([n+1])$ be the extended lift of $M([n])$ determined by the linear class of circuits $\mathcal{C}_L$, $\mathcal{C}_L \neq \emptyset, \mathcal{C}(M)$. Then N has the family of circuits:*

$$\mathcal{C}(N) = \begin{cases} \mathcal{C}_L \cup \mathcal{C}_1 & \text{if } \left|\bigcup_{C\in\mathcal{C}_L} C\right| - \mathrm{r}_M\left(\bigcup_{C\in\mathcal{C}_L} C\right) = n-r-1 \;; \\ \mathcal{C}_L \cup \mathcal{C}_1 \cup \mathcal{C}_2 & \text{otherwise}, \end{cases}$$

where

$$\mathcal{C}_1 := \Big\{ C \cup \{n+1\} \colon\; C \in \mathcal{C}(M) \backslash \mathcal{C}_L \Big\} \;,$$

$$\mathcal{C}_2 := \Big\{ C' \cup C'' \colon\; C', C'' \text{ is a modular pair of } \mathcal{C}(M) \backslash \mathcal{C}_L \Big\} \;.$$

Proof: The matroid $N^\star([n+1])$ has the family of hyperplanes:

$$\mathcal{H}(N^\star) = \begin{cases} \mathcal{H}_0 \cup \mathcal{H}_1 & \text{if } \ \mathrm{r}_{M^\star}\left(\bigcap_{C\in\mathcal{C}_L} H(C)\right) = 1 \;; \\ \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2 & \text{otherwise}, \end{cases}$$

where

$$\mathcal{H}_0 := \Big\{ H \cup \{n+1\} \colon\; H \in \mathcal{H}(\mathcal{C}_L) \Big\} \;,$$

$$\mathcal{H}_1 := \Big\{ H(C') \colon\; C' \in \mathcal{C}(M) \backslash \mathcal{C}_L \Big\} \;,$$

$$\mathcal{H}_2 := \Big\{ H' \cap H'' \cup \{n+1\} \colon\; H', H'' \text{ is a modular pair of } \mathcal{H}\big(\mathcal{C}(M) \backslash \mathcal{C}_L\big) \Big\} \;. \blacksquare$$

3 – A bias algebra

The pair $(M, \mathcal{C}_L)$ can be seen as a matroidal generalization of the pair $(G, \mathcal{C}_L)$ (defining a biased graph) where G is a graph and $\mathcal{C}_L$ a set of balanced circuits of G. A biased graph is a graph together with a (linear) class of circuits which are called balanced. It is a generalisation of signed and gain graphs which are related to some special class of hyperplane arrangements. In the classical graphic hyperplane arrangements, a hyperplane has equation of the form $x_i = x_j$. In the "signed graphic" arrangements, the equations can be of the form $x_i = \pm x_j$. In the "gain graphic" arrangements, the equations can be of the form $x_i = g x_j$ (in the biased case) or of the form $x_i = x_j + g$ (in the lift case). All these definitions due to T. Zaslavsky are very natural and produce a nice theory [12, 13] in connection with graphs, matroids and arrangements. The following bias algebra is close related to the biased graphs (and its matroidal generalizations).

Definition 3.1. Let $\mathcal{C}_L$ be a linear class of circuits of the matroid $M([n])$ and $N = N([n+1])$ be the extended lift of $M([n])$ determined by $\mathcal{C}_L$. Let $\mathrm{OS}(N)$ be the Orlik–Solomon **K**-algebra of the matroid N. The *bias* **K**-*algebra* of the pair $(M, \mathcal{C}_L)$, denoted $\mathrm{Z}(M, \mathcal{C}_L)$, is the graded quotient of the Orlik–Solomon algebra $\mathrm{OS}(N)$ by the two-sided ideal generated by e_{n+1}, i.e.,

$$\mathrm{Z}(M, \mathcal{C}_L) := \mathrm{OS}(N)/\langle e_{n+1}\rangle \; . \; \Box$$

Remark 3.2 ([11])**.** This algebra is also known as the Orlik–Solomon algebra of the pointed matroid N, with basepoint $n+1$, see [5, Definition 3.2]. If N may be realized by a complex hyperplane arrangement, then $Z(M, \mathcal{C}_L)$ is isomorphic to the cohomology ring of the complement of the decone of this arrangement with respect to the $(n+1)^{\mathrm{st}}$ hyperplane, [7, Corollary 3.57]. Two special cases occur when M itself is realizable and $\mathcal{C}_L$ is either all of $\mathcal{C}(M)$ or the empty set. Indeed, suppose that M is the matroid associated to a complex hyperplane arrangement $\mathcal{A}$. Then $Z(M, \mathcal{C}(M))$ is isomorphic to the cohomology of the complement of $\mathcal{A}$ (i.e., the Orlik–Solomon algebra of M), and $Z(M, \emptyset)$ is isomorphic to the cohomology of the complement of the affine arrangement attained by translating each of the hyperplanes of $\mathcal{A}$ some distance away from the origin, so that every dependent set will have empty intersection. □

Theorem 3.3. *The bias* **K**-*algebra* $\mathrm{Z}(M, \mathcal{C}_L)$ *is independent of the order of the elements of* $M([n])$*, i.e., it is an invariant of the pair* $(M, \mathcal{C}_L)$*. For every linear class* $\mathcal{C}_L$*, the algebra* $\mathrm{Z}(M, \mathcal{C}_L)$ *is isomorphic to the quotient of the exterior* **K**-*algebra*

$$\mathcal{E} := \bigwedge\Big(\bigoplus_{i=1}^{n} \mathbf{K} e_i\Big) \tag{3.1}$$

by the two-sided ideal $\langle \Im(\mathcal{C}_L)\rangle$ *generated by the set*

$$\Im(\mathcal{C}_L) := \Big\{\partial e_C \colon\, C \in \mathcal{C}_L,\ |C| > 1\Big\} \cup \Big\{e_C \colon\, C \in \mathcal{C}(M)\Big\} \, .$$

Proof: Since the Orlik–Solomon **K**-algebra $\mathrm{OS}(N)$ does not depend of the ordering of the ground set the first part of the theorem follows. The second assertion is a straightforward consequence of Lemma 2.6. ∎

As the element e_{n+1} does not appear in the algebra $\mathrm{Z}(M, \mathcal{C}_L)$ we will omit it. We remark that the monomial e_X, $X \subseteq [n]$, in $\mathrm{Z}(M, \mathcal{C}_L)$ is different from zero if and only if X is an independent set of M.

Corollary 3.4. *The bias* $\mathbf{K}$*-algebra* $\mathrm{Z}(M,\mathcal{C}(M))$ *is the Orlik–Solomon* $\mathbf{K}$*-algebra of* $\mathrm{OS}(M)$*. Furthermore the bias* $\mathbf{K}$*-algebra* $\mathrm{Z}(M,\emptyset)$ *is isomorphic to the quotient of the exterior algebra* (3.1) *by the two-sided ideal generated by the set* $\{e_C\colon C\in\mathcal{C}(M)\}$*.* ∎

Definition 3.5. Given an independent set I, a non-loop element $x\in \mathrm{cl}(I)\setminus I$ is said to be $\mathcal{C}_L$*-active in* I if $C(x,I)$ (i.e., the unique circuit contained in $I\cup x$) is a circuit of the family $\mathcal{C}_L$ and x is the smallest element of $C(x,I)$. An independent set with at least one $\mathcal{C}_L$-active element is said to be $\mathcal{C}_L$*-active*, and $\mathcal{C}_L$*-inactive* otherwise. We denote by $a(I)$ the smallest $\mathcal{C}_L$-active element in an active independent set I. □

Definition 3.6. We say that a subset $U\subseteq[n]$ is a $\mathcal{C}_L$*-unidependent* (set of M) if it contains a unique circuit $C(U)$ of M, $C(U)\in\mathcal{C}_L$ and $|C(U)|>1$.

We say that a $\mathcal{C}_L$-unidependent set U is $\mathcal{C}_L$*-inactive* if the minimal element of $C(U)$, $\min C(U)$, is the the smallest $\mathcal{C}_L$-active element of the independent set $U\setminus \min C(U)$. Otherwise the set U is said $\mathcal{C}_L$*-active.* □

Definition 3.7. For every circuit $C\in\mathcal{C}_L$, $|C|>1$, the set $C\setminus\min(C)$, is said to be a $\mathcal{C}_L$*-broken circuit*. The family of $\mathcal{C}_L$-inactive independents, denoted $\mathrm{NBC}_{\mathcal{C}_L}$, is the family of independent sets of M not containing a $\mathcal{C}_L$-broken circuit. □

Set

$$\mathbf{nbc}_{\mathcal{C}_L} := \Big\{e_I\colon I\in \mathrm{NBC}_{\mathcal{C}_L}\Big\}\ ,$$

$$\mathbf{b}_{\Im(\mathcal{C}_L)} := \Big\{\partial e_U\colon U \text{ is } \mathcal{C}_L\text{-inactive unidependent}\Big\}\cup\Big\{e_D\colon D \text{ is dependent}\Big\}\ .$$

Theorem 3.8. *The sets* $\mathbf{nbc}_{\mathcal{C}_L}$ *and* $\mathbf{b}_{\Im(\mathcal{C}_L)}$ *are bases, respectively of the bias* $\mathbf{K}$*-algebra* $\mathrm{Z}(M,\mathcal{C}_L)$ *and of the ideal* $\langle\Im(\mathcal{C}_L)\rangle$*.*

Proof: We will show the two statements at the same time by proving that both sets are spanning and that they have the correct size. Let I be an independent set of M. If I is $\mathcal{C}_L$-active then we have

$$e_I \;=\; \sum_{x\in C(a(I),I)\setminus a(I)} \zeta_x\, e_{I\cup a(I)\setminus x}\ ,$$

where $\zeta(x)\in\{-1,1\}$. This is an expression for e_I whit respect to lexicographically smaller e_X where X is an independent of M and $|X|=|I|$. By induction, we get that the set $\mathbf{nbc}_{\mathcal{C}_L}$ is a generator of the graded algebra $\mathrm{Z}(M,\mathcal{C}_L)$.

Let U be a $\mathcal{C}_L$-unidependent set of M. Suppose that U is $\mathcal{C}_L$-active and let $a = \min C(U)$ and set $I := C(U) \backslash a$. Note that $\{C(U), C(a(I), I)\}$ is a modular pair of circuits of $\mathcal{C}_L$, so every circuit contained in the cycle $C(U) \cup C(a(I), I)$ is in $\mathcal{C}_L$. From the definition of the map ∂ we know that

$$\partial e_U \; = \sum_{x \in C(U) \backslash a} \epsilon_x \, \partial e_{U \cup a(I) \backslash x} \; ,$$

where $\epsilon_x \in \{-1, 1\}$. This is an expression for ∂e_U with respect to lexicographically smaller ∂e_X, where X is a $\mathcal{C}_L$-unidependent and $|U| = |X|$. By induction, we get that the set $\mathbf{b}_{\Im(\mathcal{C}_L)}$ is a generator of $\langle \Im(\mathcal{C}_L) \rangle$. By the definition of $\mathrm{Z}(M, \mathcal{C}_L)$, we know that

$$\dim\big(\mathrm{Z}(M, \mathcal{C}_L)\big) + \dim\big(\langle \Im(\mathcal{C}_L) \rangle\big) \; = \; \dim(\mathcal{E}) \; = \; 2^n \; .$$

Given a subset X of $[n]$, it is either dependent or independent $\mathcal{C}_L$-active or independent $\mathcal{C}_L$-inactive. To every independent $\mathcal{C}_L$-active independent set I corresponds uniquely the unidependent $\mathcal{C}_L$-inactive $I \cup a(I)$. We have then that

$$\big|\mathbf{nbc}_{\mathcal{C}_L}(M)\big| + \big|\mathbf{b}_{\Im(\mathcal{C}_L)}\big| \; = \; 2^n \; . \; \blacksquare$$

We define the deletion and contraction operation for an arbitrary subset of circuits $\mathcal{C}' \subseteq \mathcal{C}(M)$ setting:

$$\mathcal{C}' \backslash x \; := \; \Big\{ C \in \mathcal{C}' : \; x \notin C \Big\}$$

and

$$\mathcal{C}'/x \; := \; \begin{cases} \mathcal{C}' \backslash x & \text{if } x \text{ is a loop of } M \, , \\ \Big\{ C \backslash x : \; x \in C \in \mathcal{C}' \Big\} \uplus \Big\{ C \in \mathcal{C}' : \; x \notin \mathrm{cl}_M(C) \Big\} & \text{otherwise} \; . \end{cases}$$

From the preceding definition, we can see that given a circuit C of $\mathcal{C}'/x$, where x is a non-loop of M, there exists a unique circuit $\widehat{C} \in \mathcal{C}'$ such that

$$\widehat{C} \; := \; \begin{cases} C \cup x & \text{if } x \in \mathrm{cl}_M(C) \; , \\ C & \text{otherwise} \; . \end{cases}$$

Proposition 3.9. *Let M be a matroid and $\mathcal{C}_L$ be a linear class of circuits of M. For an element x of the matroid, the circuit sets $\mathcal{C}_L \backslash x$ and $\mathcal{C}_L / x$ are linear classes of $M \backslash x$ and M/x, respectively.*

Proof: The statement for the deletion is clear. If x is a loop the result is also clear for the contraction. Suppose that x is a non-loop of M. If $Y \subseteq X$ are sets such that $\mathrm{r}_M(X) = \mathrm{r}_M(Y) + 1$ then we have

$$\mathrm{r}_{M/x}(X\backslash x) \;=\; \mathrm{r}_{M/x}(Y\backslash x) + \epsilon\;, \qquad \epsilon \in \{0,1\}\;. \tag{3.2}$$

So, if $\{C_1, C_2\}$ is a modular pair of circuits of $\mathcal{C}_L/x$, $\{\widehat{C_1}, \widehat{C_2}\}$ is also a modular pair of circuits of $\mathcal{C}_L$. We see also from Equation 3.2 that if $C \subseteq C_1 \cup C_2$ is a circuit of M/x then $\widehat{C} \subseteq \widehat{C_1} \cup \widehat{C_2}$, so $\widehat{C} \in \mathcal{C}_L$ and necessarily $C \in \mathcal{C}_L/x$. ∎

Definition 3.10. For a pair $(M, \mathcal{C}_L)$ and an element x of M, we define the deletion and the contraction of the pair $(M, \mathcal{C}_L)$ by:

$$(M, \mathcal{C}_L)\backslash x \;:=\; (M\backslash x,\, \mathcal{C}_L\backslash x)$$

and

$$(M, \mathcal{C}_L)/x \;:=\; (M/x,\, \mathcal{C}_L/x)\;. \;\square$$

As a corollary of Theorem 3.3 we have:

Proposition 3.11. *For every element x of M, there is a unique monomorphism of vector spaces,*

$$\mathfrak{i}_x\colon\; \mathrm{Z}(M, \mathcal{C}_L)\backslash x \to \mathrm{Z}(M, \mathcal{C}_L)\;,$$

such that, for every independent set I of $M\backslash x$, we have $\mathfrak{i}_x(e_I) = e_I$. ∎

Proposition 3.12. *For every non-loop element x of M, there is a unique epimorphism of vector spaces, $\mathfrak{p}_x\colon\; \mathrm{Z}(M, \mathcal{C}_L) \to \mathrm{Z}(M, \mathcal{C}_L)/x$, such that, for every subset $I = \{i_1, \ldots, i_\ell\} \subseteq [n]$,*

$$\mathfrak{p}_x e_I \;:=\; \begin{cases} e_{I\backslash x} & \text{if } x \in I\;, \\ \pm\, e_{I\backslash y} & \text{if } \exists\, y \in I \text{ such that } \{x,y\} \in \mathcal{C}_L\;, \\ 0 & \text{otherwise}\;. \end{cases} \tag{3.3}$$

More precisely the value of the coefficient ± 1 in the second case is the sign of the permutation obtained by replacing y by x in I.

Proof: From Theorem 3.3, it is enough to prove that the map $\mathfrak{p}_x$ is well determined, i.e., for all $\mathcal{C}_L$-unidependent $U = (i_1, \ldots, i_m)$ set of M, we have

$$\mathfrak{p}_x\, \partial e_U = 0 \;\in\; \Im(\mathcal{C}_L/x)\;.$$

We can also suppose that x is the last element n. Note that if $n \in U$ then $U\backslash n$ is a $\mathcal{C}_L/n$-unidependent set of M/n. If $n \notin U$ but there is $y \in U$ and $\{n,y\} \in \mathcal{C}_L$, we know that $e_U = \pm\, e_{U\backslash y \cup n}$ in $\mathrm{Z}(M,\mathcal{C}_L)$. Suppose that $n \notin U$ and that there does not exist $y \in U$ such that $\{n,y\} \in \mathcal{C}_L$. Then it is clear that $\mathfrak{p}_n\,\partial e_U = 0$. Suppose that $n \in U$. It is easy to see that

$$\pm\, \mathfrak{p}_n\,\partial e_U \;=\; \sum_{j=1}^{m-1} e_{U\backslash\{j,n\}} \;=\; 0\ .$$

Finally, if an independent set I of M contains an element y such that $\{x,y\}$ is a circuit in $\mathcal{C}_L$, we know that there is a scalar $\chi(I;x,y) \in \{-1,1\}$ such that $e_I = \chi(I;x,y)\, e_{I\backslash y \cup x}$. More precisely the value of $\chi(I;x,y) \in \{-1,1\}$ is the sign of the permutation obtained by replacing y by x in I. ∎

Theorem 3.13. *Let M be a loop free matroid and $\mathcal{C}_L$ be a linear class of circuits of M. For every element x of M, there is a splitting short exact sequence of vector spaces*

$$0 \;\to\; \mathrm{Z}(M,\mathcal{C}_L)\backslash x \;\xrightarrow{\mathfrak{i}_x}\; \mathrm{Z}(M,\mathcal{C}_L) \;\xrightarrow{\mathfrak{p}_x}\; \mathrm{Z}(M,\mathcal{C}_L)/x \;\to\; 0\ . \tag{3.4}$$

Proof: From the definitions we know that $\mathfrak{p}_x \circ \mathfrak{i}_x$, is the null map so $\mathrm{Im}(\mathfrak{i}_x) \subseteq \mathrm{Ker}(\mathfrak{p}_x)$. We will prove the equality $\dim(\mathrm{Ker}(\mathfrak{p}_n)) = \dim(\mathrm{Im}(\mathfrak{i}_n))$. By a reordering of the elements of $[n]$ we can suppose that $x = n$. The minimal $\mathcal{C}_L/n$-broken circuits of M are the minimal sets X such that either X or $X \cup \{n\}$ is a $\mathcal{C}_L$-broken circuit of M (see [1, Proposition 3.2.e]). Then

$$\mathrm{NBC}_{\mathcal{C}_L/n} = \Big\{X\colon\ X \subseteq [n{-}1]\ \text{ and }\ X{\cup}\{n\} \in \mathrm{NBC}_{\mathcal{C}_L}\Big\}$$

and we have

$$\mathrm{NBC}_{\mathcal{C}_L} \;=\; \mathrm{NBC}_{\mathcal{C}_L\backslash n} \uplus \Big\{I \cup n\colon\ I \in \mathrm{NBC}_{\mathcal{C}_L/n}\Big\}\ . \tag{3.5}$$

So $\dim(\mathrm{Ker}(\mathfrak{p}_n)) = \dim(\mathrm{Im}(\mathfrak{i}_n))$. There is a morphism of vector spaces

$$\mathfrak{p}_n^{-1}\colon\ \mathrm{Z}(M,\mathcal{C}_L)/n \to \mathrm{Z}(M,\mathcal{C}_L)\ ,$$

where, for every $I \in \mathrm{NBC}_{\mathcal{C}_L/n}$, we have $\mathfrak{p}_n^{-1} e_I := e_{I\cup n}$. It is clear that $\mathfrak{p}_n \circ \mathfrak{p}_n^{-1}$ is the identity map. From Equation (3.5) we conclude that the exact sequence (3.4) splits. ∎

Remark 3.14. A large class of algebras, the so called χ-algebras (see [4] for more details), contain the Orlik–Solomon, Orlik–Terao [8] (associated to vectorial matroids) and Cordovil algebras [3] (associated to oriented matroids). Following the same ideas it is possible to generalize the definition of the bias algebras and obtain a class of bias χ-algebras, determined by a pair $(M, \mathcal{C}_L)$, and that contain all the mentioned algebras. □

Similarly to [4], we now construct, making use of iterated contractions, the dual basis $\boldsymbol{nbc}^*_{\mathcal{C}_L}$ of the standard basis $\boldsymbol{nbc}_{\mathcal{C}_L}$. Let $\mathrm{Z}(M, \mathcal{C}_L)_h$ be the subspace of $\mathrm{Z}(M, \mathcal{C}_L)$ generated by the set

$$\Big\{ e_X \colon\ X \text{ is an independent set of } M \text{ and } |X| = h \Big\} .$$

We associate to the (linearly ordered) independent set $I = (i_1, \ldots, i_h)$ of M the linear form on $\mathrm{Z}(M, \mathcal{C}_L)_h$, $\mathfrak{p}_I \colon \mathrm{Z}(M, \mathcal{C}_L)_h \to \mathbf{K}$,

$$\mathfrak{p}_I := \mathfrak{p}_{e_{i_1}} \circ \mathfrak{p}_{e_{i_2}} \circ \cdots \circ \mathfrak{p}_{e_{i_h}} . \tag{3.6}$$

We also associate to the linearly ordered independent $I = (i_1, \ldots, i_j)$ the flag of its final independent subsets, defined by

$$\Big\{ I_t \colon\ I_t = (i_t, \ldots, i_j),\ 1 \le t \le j \Big\} .$$

Proposition 3.15. *Let $I = (i_1, \ldots, i_h)$ and $J = (j_1, \ldots, j_h)$ be two linearly ordered independents of M, then we have $\mathfrak{p}_I(e_J) \neq 0$ if and only if there is a permutation $\tau \in \mathfrak{S}_h$ such that for every $1 \le t \le h$, $j_{\tau(t)} \in \mathrm{cl}(I_t)$ and $C(j_{\tau(t)}, I_t) \in \mathcal{C}_L$. When the permutation τ exists, it is unique and we have $\mathfrak{p}_I(e_J) = \mathrm{sgn}(\tau)$. In particular we have $\mathfrak{p}_I(e_I) = 1$ for any independent set I.*

Proof: The first equivalence is very easy to prove in both directions. To obtain the expression of $\mathfrak{p}_I(e_J)$ we just need to iterate h times the formula of contraction of Proposition 3.11. With the definition of the permutation τ we know that $\mathfrak{p}_I(e_{\tau(1)} \wedge \cdots \wedge e_{\tau(h)}) = 1$. By the antisymmetric of the wedge product we also have that $e_J = \mathrm{sgn}(\tau) \times e_{\tau(1)} \wedge \cdots \wedge e_{\tau(h)}$. And finally the last result comes from the fact that if $I = J$ then clearly $\tau = \mathrm{id}$. ■

Theorem 3.16. *The set $\{\mathfrak{p}_I \colon I \in \mathrm{NBC}_{\mathcal{C}_L}\}$ is the dual basis of the standard basis $\mathbf{nbc}_{\mathcal{C}_L}$ of $\mathrm{Z}(M, \mathcal{C}_L)$.*

Proof: Pick two elements e_I and e_J in $\mathbf{nbc}_{\mathcal{C}_L}$, $|I| = |J| = h$. We just need to prove that $\mathfrak{p}_I(e_J) = \delta_{IJ}$ (the Kronecker delta). From the preceding proposition we already have that $\mathfrak{p}_I(e_I) = 1$. Suppose for a contradiction that there exists a permutation τ such that $j_{\tau(t)} \in \mathrm{cl}(I_t)$ and $C(j_{\tau(t)}, I_t) \in \mathcal{C}_L$ for every $1 \le t \le h$. Suppose that $j_{\tau(m+1)} = i_{m+1}, \ldots, j_{\tau(h)} = i_h$ and $i_m \neq j_{\tau(m)}$. Then there is a circuit $C \in \mathcal{C}_L$ such that

$$i_m, j_{\tau(m)} \in C \subseteq \left\{i_m, j_{\tau(m)}, i_{m+1}, i_{m+2}, \ldots, i_h\right\} .$$

If $j_{\tau(m)} < i_m$ $\left[\text{resp. } i_m < j_{\tau(m)}\right]$ we conclude that $I \notin \mathrm{NBC}_{\mathcal{C}_L}$ $\left[\text{resp. } J \notin \mathrm{NBC}_{\mathcal{C}_L}\right]$, a contradiction. ∎

The following corollary is an extension of results of [2], [3] and [4].

Corollary 3.17. *Let $J = \{j_1, \ldots, j_\ell\}$ be an independent set of M such that the expansion of e_J in $\mathbf{nbc}_{\mathcal{C}_L}$ is $e_J = \sum_{I \in \mathbf{nbc}_{\mathcal{C}_L}} \xi(I, J) e_I$. Then the following are equivalent:*

- $\xi(I, J) \neq 0$,
- *there exists a permutation τ such that $e_{\tau(t)} \in \mathrm{cl}(I_t)$ and $C(j_{\tau(t)}, I_t) \in \mathcal{C}_L$ for every $1 \le t \le h$. Moreover, in the case where $\xi(I, J) \neq 0$ we have $\xi(I, J) = \mathrm{sgn}(\tau)$.* ∎

REFERENCES

[**1**] Brylawski, T. – The broken-circuit complex, *Trans. Amer. Math. Soc.,* 234(2) (1977), 417–433.

[**2**] Cordovil, R. and Etienne, G. – A note on the Orlik–Solomon algebra, *European J. of Combin.,* 22 (2001), 165–170.

[**3**] Cordovil, R. – A commutative algebra for oriented matroids, *Discrete and Computational Geometry,* 27 (2002), 73–84.

[**4**] Cordovil, R. and Forge, D. – Diagonal bases in Orlik–Solomon type algebras, *Annals of Combinatorics,* 7 (2003), 247–257.

[**5**] Falk, Michael J. – Combinatorial and algebraic structure in Orlik–Solomon algebras, *European J. Combin.,* 22(5) (2001), 687–698.

[**6**] Orlik, Peter; Solomon, Louis – Combinatorics and topology of complements of hyperplanes, *Invent. Math.,* 56(2) (1980), 167–189.

[**7**] Orlik, Peter; Terao, Hiroaki – *Arrangements of Hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 300, Springer-Verlag, Berlin, 1992.

[**8**] Orlik, Peter; Terao, Hiroaki – Commutative algebras for arrangements, *Nagoya Math. J.,* 134, (1994), 65–73.

[**9**] Orlik, Peter; Terao, Hiroaki – *Arrangements and hypergeometric integrals*, MSJ Memoirs **9**, Mathematical Society of Japan, Tokyo, 2001.

[**10**] Tutte, W.T. – Lectures on matroids, *J. Res. Nat. Bur. Standards Sect. B,* 69B (1965), 1–47.

[**11**] Proudfoot, Nicholas – Private communication.

[**12**] Zaslavsky, T. – Biased graphs. I. Bias, balance, and gains, *J. Combin. Theory Ser. B,* 47 (1989), 32–52.

[**13**] Zaslavsky, T. – Biased graphs. II. The three matroids, *J. Combin. Theory Ser. B,* 51 (1991), 46–72.

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