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INVERSION OF MATRIX CONVOLUTION TYPE OPERATORS WITH SYMMETRY

L.P. Castro° and F.-O. Speck•

Recommended by A. Ferreira dos Santos

Abstract: We consider matrix convolution type operators that carry a certain symmetry due to the presence of even or odd extensions. The study is motivated by mathematical physics applications where this kind of operators appears. In connection with this interest, a class of Hölder continuous Fourier symbols is taken into consideration. The main result consists of sufficient conditions for the invertibility of such operators including a presentation of the corresponding inverse operator in terms of an asymmetric factorization of the symbol matrix. Moreover the asymptotic behavior of the factors is analyzed.

1 – Introduction

We consider *matrix convolution type operators with symmetry*, acting between Bessel potential spaces, which have the form

$$T \;=\; r_+ A\ell^c \;=\; r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}\ell^c:\; \times_{j=1}^m H^{r_j}(\mathbb{R}_+) \;\to\; \times_{j=1}^m H^{s_j}(\mathbb{R}_+)\;, \tag{1.1}$$

where ℓ^c is the even ℓ^e or odd ℓ^o continuous extension operator from $\times_{j=1}^m H^{r_j}(\mathbb{R}_+)$

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into $\times_{j=1}^{m} H^{r_j}(\mathbb{R})$ in the case where all $r_j \in (-1/2, 3/2)$ or all $r_j \in (-3/2, 1/2)$, respectively, $A : \times_{j=1}^{m} H^{r_j}(\mathbb{R}) \to \times_{j=1}^{m} H^{s_j}(\mathbb{R})$, is a bounded translation invariant operator and r_+ denotes the operator of restriction to the corresponding Bessel potential space on the positive half-line.

It is well-known that A can be represented by $A = \mathcal{F}^{-1}\phi \cdot \mathcal{F}$ where $\mathcal{F}$ denotes the Fourier transformation and $\phi \in [L^\infty_{\mathrm{loc}}(\mathbb{R})]^{m\times m}$. Moreover we shall assume that the matrix Fourier symbols ϕ are from the following class of Hölder continuous matrix functions:

$$\phi \in \left\{ \phi \in \mathcal{G}[L^\infty_{\mathrm{loc}}(\mathbb{R})]^{m\times m} \ : \ \phi_0 = \lambda_-^s \, \phi \, \lambda^{-r} \in [C^\alpha(\ddot{\mathbb{R}})]^{m\times m} \right\}, \tag{1.2}$$

where $C^\alpha(\ddot{\mathbb{R}})$ is the class of Hölder continuous functions, with exponent $\alpha \in (0,1]$, on $\ddot{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, admitting two possible different finite limits at infinity. Later on, we will also make use of the subclass $C^\alpha(\dot{\mathbb{R}})$ of Hölder continuous functions, with exponent $\alpha \in (0,1]$, on $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, admitting the same finite limit at (plus and minus) infinity. We let $\mathcal{G}X$ denote the subclass of invertible elements of a unital algebra X. For complex numbers or complex valued functions d and multi-indices $\eta = (\eta_1, ..., \eta_m)$, we use the notation

$$d^\eta = \mathrm{diag}[d^{\eta_1}, ..., d^{\eta_m}] = \mathrm{diag}[d^{\eta_j}] \ , \tag{1.3}$$

$$\lambda_-^\eta(\xi) = \mathrm{diag}\Big[(\xi - i)^{\eta_j}\Big] \ , \tag{1.4}$$

$$\lambda^\eta(\xi) = \mathrm{diag}\Big[(\xi^2 + 1)^{\eta_j/2}\Big] \ , \tag{1.5}$$

$$\lambda_+^\eta(\xi) = \mathrm{diag}\Big[(\xi + i)^{\eta_j}\Big], \qquad \xi \in \mathbb{R} \ . \tag{1.6}$$

Let us recall that the Bessel potential spaces $H^s(\mathbb{R}_+)$, $s \in \mathbb{R}$, are the spaces of generalized functions on $\mathbb{R}_+$ which have extensions into $\mathbb{R}$ that belong to $H^s(\mathbb{R})$; i.e., they belong to the space of tempered distributions φ such that

$$\|\varphi\|_{H^s(\mathbb{R})} = \Big\| \mathcal{F}^{-1}(\xi^2+1)^{s/2} \cdot \mathcal{F}\varphi \Big\|_{L^2(\mathbb{R})} < \infty \ . \tag{1.7}$$

For this framework, our operator T in (1.1) is well-defined and bounded; for the scalar case cf. [7] (in particular Lemma 2.1 and Lemma 2.2).

Operators with the form of T appear in several mathematical physics problems. In particular, they occur in (stationary as well as non-stationary) wave diffraction problems involving rectangularly wedged obstacles [6, 13]. Therefore, a representation of the inverse operator or invertibility conditions for such operators are very useful for those applications.

In the present work, in Section 2, we present a lifting procedure for T and therefore obtain new operators equivalent to T, but defined in the framework of Lebesgue spaces. For this setting, in Section 3, certain factorization concepts are proposed for the Fourier symbols of the operators in study. Connections between different kinds of the presented factorizations are also exposed. In Section 4, sufficient conditions and inverse formulas are provided for T depending on the existence of the factorizations in the previous section. Assuming certain Hölder continuous behavior of the symbols, detailed information about the factors is obtained. This leads also to corresponding information about a representation of the inverse of T, which may be viewed as the main result of this work (Corollary 4.6). In Section 5, weaker conditions are imposed such that conclusions about the generalized invertibility of T are obtained. The last two sections contain also asymptotic representations of the factors that can be used to describe the asymptotic behavior of the solution of diffraction problems [14].

2 – Relation with convolution type operators with symmetry in Lebesgue spaces

We will perform a lifting of the initially presented operator T to the L^2 spaces, taking for that (in the odd or even extension ℓ^c case) the largest possible range of indices $r = (r_1, ..., r_m)$ in the domain space of (1.1) where the operator is bounded.

Proposition 2.1. *Let $r \in (-1/2, 3/2)$ and $\ell^c = \ell^e$ (or $r \in (-3/2, 1/2)$ and $\ell^c = \ell^o$). Then the operator* (1.1) *can be lifted into $[L^2(\mathbb{R}_+)]^m$, i.e., there are linear homeomorphisms E, F such that*

$$
\begin{aligned}
&T = E\, T_0\, F \\
&T_0 = r_+ \mathcal{F}^{-1} \phi_0 \cdot \mathcal{F} \ell^c \;:\; [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R}_+)]^m
\end{aligned}
\tag{2.1}
$$

where $\phi_0 \in [C^\alpha(\ddot{\mathbb{R}})]^{m\times m}$. More precisely, we have

$$\mathcal{F}^{-1}\phi_0 \cdot \mathcal{F} = \left(\mathcal{F}^{-1}\lambda_-^s \cdot \mathcal{F}\right)\left(\mathcal{F}^{-1}\phi \cdot \mathcal{F}\right)\left(\mathcal{F}^{-1}\lambda^{-r} \cdot \mathcal{F}\right) \tag{2.2}$$

$$E = r_+ \mathcal{F}^{-1} \lambda_-^{-s} \cdot \mathcal{F} \ell \tag{2.3}$$

$$F = r_+ \mathcal{F}^{-1} \lambda^r \cdot \mathcal{F} \ell^c \,, \tag{2.4}$$

where $\ell : [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R})]^m$ denotes an arbitrary extension, i.e., E is independent of that choice.

Proof: We will use the notations $L^{2,e}(\mathbb{R})$ and $L^{2,o}(\mathbb{R})$, for the $L^2(\mathbb{R})$ elements that are even or odd functions, respectively.

Consider the first case with $\ell^c = \ell^e$. For the corresponding values of r and s we can write

$$(2.5)\qquad \begin{aligned} F^{-1}F \;&=\; \Big(r_+\mathcal{F}^{-1}\lambda^{-r}\cdot\mathcal{F}\Big)\,(\ell^e r_+)\,\Big(\mathcal{F}^{-1}\lambda^{r}\cdot\mathcal{F}\ell^e\Big) \\ &: \times_{j=1}^m H^{r_j}(\mathbb{R}_+) \to \big[L^{2,e}(\mathbb{R})\big]^m \to \big[L^{2,e}(\mathbb{R})\big]^m \to \times_{j=1}^m H^{r_j}(\mathbb{R}_+)\end{aligned}$$

and $\mathcal{F}^{-1}\lambda^r\cdot\mathcal{F}$ preserves the "even function property", since its symbol is even. So we may drop the middle term $\ell^e r_+$ and obtain $F^{-1}F = I$ in $\times_{j=1}^m H^{r_j}(\mathbb{R}_+)$. By analogy we have $FF^{-1} = I$ in $[L^2(\mathbb{R}_+)]^m$. On the other hand it is known [10, Lemma 4.6] that

$$(2.6)\qquad E^{-1}E \;=\; r_+\mathcal{F}^{-1}\lambda_-^{s}\cdot\mathcal{F}\ell r_+\mathcal{F}^{-1}\lambda_-^{-s}\cdot\mathcal{F}\ell \;=I \quad \text{in} \;\; \big[L^2(\mathbb{R}_+)\big]^m$$

and $EE^{-1} = I$ in $\times_{j=1}^m H^{s_j}(\mathbb{R}_+)$ for any $s = (s_1, ..., s_m) \in \mathbb{R}^m$.

The rest of the formulas is obvious and the proof for ℓ^o runs the same way, if we replace ℓ^e by ℓ^o and $\big[L^{2,e}(\mathbb{R})\big]^m$ by $\big[L^{2,o}(\mathbb{R})\big]^m$ in (2.5). ∎

We note that the identity in (2.1) between T and T_0 is an operator equivalence relation. This simple but useful observation allows the initial consideration of a factorization procedure in L^2 spaces that later on will be transposed to the initial context of Bessel potential spaces.

The operator T_0 in (2.1), can be regarded as a Wiener–Hopf–Hankel operator but we find the present approach more convenient (see [7] for the scalar case and [6] for applications where it was used with great efficiency).

3 – Asymmetric and anti-symmetric generalized matrix factorizations

The present section is concerned with new kinds of factorizations of matrix functions that later on will play a central role in the characterization of the invertibility of our convolution type operator T. These definitions generalize corresponding notions presented in [7]. The roots for such factorizations can be found in the well-known notion of *generalized factorization* in the theory of singular integral, Toeplitz, and classical Wiener–Hopf operators [15], in the theory of general Wiener–Hopf operators [18], and in the recent work on Toeplitz plus Hankel operators [1, 2, 8].

Let $[L^2_\pm(\mathbb{R})]^{m\times m}$ be the image of $[L^2(\mathbb{R})]^{m\times m}$ under the action of the projector

$$P_\pm = \frac{1}{2}(I \pm S_\mathbb{R})$$

associated with the Hilbert transformation $S_\mathbb{R}$. For our subspaces $[X(\mathbb{R})]^{m\times m}$ of $[L^2(\mathbb{R})]^{m\times m}$, we will denote by $[X(\mathbb{R},\rho)]^{m\times m}$ the corresponding weighted spaces whose elements φ fulfill $\rho\varphi \in [X(\mathbb{R})]^{m\times m}$, for some weight function ρ.

Definition 3.1. A matrix function $\phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$ admits an *asymmetric generalized factorization with respect to* L^2 *and* ℓ^e, written as

$$\phi \;=\; \phi_- \;\mathrm{diag}[\zeta^{\kappa_j}]\,\phi_e \tag{3.1}$$

if $\kappa_1,\dots,\kappa_m \in \mathbb{Z}$, $\zeta(\xi) = (\xi - i)/(\xi + i)$ for $\xi \in \mathbb{R}$, $\phi_- \in [L^2_-(\mathbb{R},\lambda_-^{-2})]^{m\times m}$, $\phi_-^{-1} \in [L^2_-(\mathbb{R},\lambda_-^{-1})]^{m\times m}$, $\phi_e \in [L^{2,e}(\mathbb{R},\lambda^{-1})]^{m\times m}$, $\phi_e^{-1} \in [L^{2,e}(\mathbb{R},\lambda^{-2})]^{m\times m}$ and if

$$V_e = A_e^{-1}\ell^e r_+ A_-^{-1} \tag{3.2}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^m$ possessing a bounded extension to $[L^2(\mathbb{R})]^m$, with

$$A_e \;=\; \mathcal{F}^{-1}\phi_e\cdot\mathcal{F}\;, \tag{3.3}$$

$$A_- = \mathcal{F}^{-1}\phi_-\cdot\mathcal{F}\;. \;\square \tag{3.4}$$

As usual the factor spaces, where the factors of ϕ can be found, are the closures of the spaces of bounded rational functions without poles in the closed lower half-plane $\overline{\mathbb{C}_-} = \{\xi \in \mathbb{C} : \mathrm{Im}\, m(\xi) \le 0\}$ or of those which are even, respectively, due to the weighted L^2 norm.

When all k_j in (3.1) are zero, we will denote the factorization as a *canonical asymmetric generalized factorization with respect to* L^2 *and* ℓ^e and so we shall use the word *canonical* in other kinds of factorizations.

Note that the weights λ_-^{-1}, λ^{-1} have the common decrease at infinity that is used for generalized factorization in $[L^2(\mathbb{R})]^{m\times m}$ [15] whilst the weights λ_-^{-2}, λ^{-2} admit an increase of the factors ϕ_- and ϕ_e^{-1}, respectively, that is one order higher than usual. That choice of the spaces was firstly proposed in [7] for the scalar case where it turned out to be most appropriate for constructive factorization of $\mathcal{G}C^\alpha(\ddot{\mathbb{R}})$ symbols.

Definition 3.2. Furthermore, we speak of an *asymmetric generalized factorization with respect to L^2 and ℓ^o*, if a matrix function $\phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$ admits the form of (3.1), with $\kappa_1, ..., \kappa_m \in \mathbb{Z}$, $\phi_- \in [L^2(\mathbb{R}, \lambda_-^{-1})]^{m\times m}$, $\phi_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{m\times m}$, $\phi_e \in [L^{2,e}(\mathbb{R}, \lambda^{-2})]^{m\times m}$, $\phi_e^{-1} \in [L^{2,e}(\mathbb{R}, \lambda^{-1})]^{m\times m}$ and if

$$V_o = A_e^{-1} \ell^o r_+ A_-^{-1} \tag{3.5}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^m$ possessing a bounded extension to $[L^2(\mathbb{R})]^m$, and with A_e and A_- as in (3.3) and (3.4), respectively. ▫

Note that here the increase orders of the factors are exchanged compared with those in Definition 3.1.

Given a matrix-valued function φ, on the real line, we will abbreviate by $\widetilde{\varphi}$ that one defined by

$$\widetilde{\varphi}(\xi) = \varphi(-\xi)\,, \qquad \xi \in \mathbb{R}\ . \tag{3.6}$$

Definition 3.3. A matrix function $\psi \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$ admits an *anti-symmetric generalized factorization with respect to L^2 and ℓ^e*, written as

$$\psi = \psi_- \ \mathrm{diag}[\zeta^{2\kappa_j}]\ \widetilde{\psi}_-^{-1} \tag{3.7}$$

if $\kappa_1, ..., \kappa_m \in \mathbb{Z}$, $\psi_- \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{m\times m}$, $\psi_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-1})]^{m\times m}$, and if

$$U_e = \widetilde{A}_- \ell^e r_+ A_-^{-1} \tag{3.8}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^m$ possessing a bounded extension to $[L^2(\mathbb{R})]^m$, with $\widetilde{A}_- = \mathcal{F}^{-1}\widetilde{\psi}_- \cdot \mathcal{F}$ and $A_- = \mathcal{F}^{-1}\psi_- \cdot \mathcal{F}$. ▫

Definition 3.4. We will say that a matrix function $\psi \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$ admits an *anti-symmetric generalized factorization with respect to L^2 and ℓ^o*, if ψ can be written as in (3.7), with $\kappa_1, ..., \kappa_m \in \mathbb{Z}$, $\psi_- \in [L^2_-(\mathbb{R}, \lambda_-^{-1})]^{m\times m}$, $\psi_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{m\times m}$, and if

$$U_o = \widetilde{A}_- \ell^o r_+ A_-^{-1}\ , \tag{3.9}$$

where $\widetilde{A}_- = \mathcal{F}^{-1}\widetilde{\psi}_- \cdot \mathcal{F}$ and $A_- = \mathcal{F}^{-1}\psi_- \cdot \mathcal{F}$, is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^m$ possessing a bounded extension to $[L^2(\mathbb{R})]^m$. ▫

In view of the study of our convolution type operator T (and T_0), we will take profit of the properties of the following auxiliary Toeplitz operator (also having in mind the last defined anti-symmetric generalized factorization)

$$S_0 = P_+\psi_{0|[L^2_+(\mathbb{R})]^m} : \; [L^2_+(\mathbb{R})]^m \to [L^2_+(\mathbb{R})]^m \;, \tag{3.10}$$

where the *symbol* ψ_0 of S_0 is given by

$$\psi_0 = \phi_0\, \widetilde{\phi}_0^{-1} \;. \tag{3.11}$$

In a sense, the role of the even or odd extension in the operator T_0 is here incorporated in the symmetry of the symbol of S_0.

Theorem 3.5. *Let $\phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$ and consider $\psi = \phi\, \widetilde{\phi}^{-1}$.*

(i) *If ϕ admits an asymmetric generalized factorization with respect to L^2 and ℓ^c,*

$$\phi = \phi_- \operatorname{diag}[\zeta^{\kappa_j}]\, \phi_e \;, \tag{3.12}$$

then ψ admits an anti-symmetric generalized factorization with respect to L^2 and ℓ^c in the form

$$\psi = \phi_- \operatorname{diag}[\zeta^{2\kappa_j}]\, \widetilde{\phi}_-^{-1} \;. \tag{3.13}$$

(ii) *If ψ admits an anti-symmetric generalized factorization with respect to L^2 and ℓ^c,*

$$\psi = \psi_- \operatorname{diag}[\zeta^{2\kappa_j}]\, \widetilde{\psi}_-^{-1} \;, \tag{3.14}$$

then ϕ admits an asymmetric generalized factorization with respect to L^2 and ℓ^c in the form

$$\phi \;=\; \psi_- \operatorname{diag}[\zeta^{\kappa_j}] \left(\operatorname{diag}[\zeta^{-\kappa_j}]\psi_-^{-1}\phi\right) \;, \tag{3.15}$$

where $\operatorname{diag}[\zeta^{-\kappa_j}]\psi_-^{-1}\phi$ is the even factor.

Proof: We will present the proof for $\ell^c = \ell^e$. The case $\ell^c = \ell^o$ runs analogously, with obvious changes.

(**i**) Assuming an asymmetric generalized factorization with respect to L^2 and ℓ^e for ϕ,

$$\phi = \phi_- \operatorname{diag}[\zeta^{\kappa_j}]\, \phi_e \ , \tag{3.16}$$

with $\kappa_j \in \mathbb{Z}$, $j = 1, ..., m$, $\phi_- \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{m\times m}$, $\phi_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-1})]^{m\times m}$, $\phi_e \in [L^{2,e}(\mathbb{R}, \lambda^{-1})]^{m\times m}$, $\phi_e^{-1} \in [L^{2,e}(\mathbb{R}, \lambda^{-2})]^{m\times m}$ and where

$$V_e = \mathcal{F}^{-1}\phi_e^{-1} \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\phi_-^{-1} \cdot \mathcal{F} \tag{3.17}$$

is an operator defined on a dense subspace of $[L^2(\mathbb{R})]^m$ possessing a bounded extension to $[L^2(\mathbb{R})]^m$, we start by choosing the same "minus" factor ϕ_- for the factorization of ψ and observe in addition that

$$\widetilde{\phi}^{-1} = \phi_e^{-1} \operatorname{diag}[\zeta^{\kappa_j}]\, \widetilde{\phi}_-^{-1} \tag{3.18}$$

holds due to the even property of ϕ_e. Therefore,

$$\begin{aligned} \psi = \phi\, \widetilde{\phi}^{-1} &= \Big(\phi_- \operatorname{diag}[\zeta^{\kappa_j}]\, \phi_e\Big) \Big(\phi_e^{-1} \operatorname{diag}[\zeta^{\kappa_j}]\, \widetilde{\phi}_-^{-1}\Big) \\ &= \phi_- \operatorname{diag}[\zeta^{2\kappa_j}]\, \widetilde{\phi}_-^{-1} \ , \end{aligned} \tag{3.19}$$

with

$$\phi_- \in [L^2_-(\mathbb{R}, \lambda_-^{-2})]^{m\times m}, \qquad \phi_-^{-1} \in [L^2_-(\mathbb{R}, \lambda_-^{-1})]^{m\times m} \ , \tag{3.20}$$

or equivalently

$$\widetilde{\phi}_- \in [L^2_+(\mathbb{R}, \lambda_+^{-2})]^{m\times m}, \qquad \widetilde{\phi}_-^{-1} \in [L^2_+(\mathbb{R}, \lambda_+^{-1})]^{m\times m} \ . \tag{3.21}$$

The supposition of having an asymmetric generalized factorization includes that

$$V = \mathcal{F}^{-1}\phi_e^{-1} \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\phi_-^{-1} \cdot \mathcal{F} \tag{3.22}$$

is a bounded operator (densely defined) in $[L^2(\mathbb{R})]^m$. As in the theory of generalized factorizations [12, Section 9], this last condition (3.22) can be equivalently replaced by others. In particular, together with (3.18) we obtain that

$$U_e = \mathcal{F}^{-1}\widetilde{\phi}_- \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\phi_-^{-1} \cdot \mathcal{F} \tag{3.23}$$

is a bounded operator also (densely defined) in $[L^2(\mathbb{R})]^m$.

(ii) If ψ admits an anti-symmetric generalized factorization with respect to L^2 and ℓ^e,

$$\psi = \phi\,\widetilde{\phi}^{-1} = \psi_-\ \mathrm{diag}[\zeta^{2\kappa_j}]\,\widetilde{\psi}_-^{-1}\ , \tag{3.24}$$

then choosing

$$\phi_e = \mathrm{diag}[\zeta^{-\kappa_j}]\,\psi_-^{-1}\,\phi \tag{3.25}$$

$$\phi_- = \psi_- \tag{3.26}$$

it directly follows that

$$\phi = \phi_-\ \mathrm{diag}[\zeta^{\kappa_j}]\,\phi_e\ . \tag{3.27}$$

In addition, due to (3.24), we have

$$\psi_-^{-1}\,\phi\,\widetilde{\phi}^{-1} = \mathrm{diag}[\zeta^{2k_j}]\,\widetilde{\psi}_-^{-1} \tag{3.28}$$

$$\widetilde{\psi}_-^{-1}\,\widetilde{\phi} = \mathrm{diag}[\zeta^{-2k_j}]\,\psi_-^{-1}\,\phi\ , \tag{3.29}$$

and therefore (please remember (3.25), as well as the first identity in (3.24))

$$\widetilde{\phi_e} = \mathrm{diag}[\zeta^{\kappa_j}]\,\widetilde{\psi}_-^{-1}\,\widetilde{\phi} = \mathrm{diag}[\zeta^{-\kappa_j}]\,\psi_-^{-1}\phi = \phi_e\ , \tag{3.30}$$

which in particular shows that ϕ_e is an even function.

Now, due to the anti-symmetric generalized factorization of ψ, we already know that

$$\phi_- = \psi_- \in [L^2_-(\mathbb{R},\lambda_-^{-2})]^{m\times m}\ , \qquad \phi_-^{-1} = \psi_-^{-1} \in [L^2_-(\mathbb{R},\lambda_-^{-1})]^{m\times m} \tag{3.31}$$

which together with the fact that $\phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$, and the form of the even function ϕ_e in (3.25) leads to

$$\phi_e \in [L^{2,e}(\mathbb{R},\lambda^{-1})]^{m\times m}\ , \qquad \phi_e^{-1} \in [L^2(\mathbb{R},\lambda^{-2})]^{m\times m}\ . \tag{3.32}$$

Finally, similarly as in part (i), we obtain that

$$V_e = \mathcal{F}^{-1}\phi_e^{-1}\cdot\mathcal{F}\ell^e r_+\mathcal{F}^{-1}\phi_-^{-1}\cdot\mathcal{F} \tag{3.33}$$

is bounded in $[L^2(\mathbb{R})]^m$ (as an extended operator from a dense subspace). ∎

4 – Sufficient conditions for invertibility and representation of inverses

In this section, based on the factorizations presented above, we start with a sufficient condition for the invertibility of our convolution type operator in study. At the end of the section, we will be in the position to present the main result of this work: a representation of the inverse of T under certain conditions on the Hölder continuous Fourier symbol ϕ_0 of T_0, in terms of a canonical asymmetric generalized factorization of ϕ_0, which allows an asymptotic analysis.

Theorem 4.1. *Let us turn to our first operator* (1.1) *assuming* (1.2). *If the matrix function* ϕ_0 *admits a canonical asymmetric generalized factorization with respect to* L^2 *and* ℓ^c *(see* (3.1)*),*

$$\phi_0 \,=\, \phi_- \, \phi_e \ , \tag{4.1}$$

then T *is an invertible operator with inverse given by*

$$T^{-1} = F^{-1}\, r_+ A_e^{-1} \ell^c r_+ A_-^{-1} \ell \, E^{-1} \ , \tag{4.2}$$

where E *and* F *are defined in* (2.3)–(2.4), $A_e = \mathcal{F}^{-1}\phi_e \cdot \mathcal{F}$, $A_- = \mathcal{F}^{-1}\phi_- \cdot \mathcal{F}$, *and* $\ell : [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R})]^m$ *is an arbitrary extension (which particular choice is indifferent for the definition of* T^{-1}*).*

Proof: Let $\ell^c = \ell^e$. The statement can be achieved through a direct computation:

$$\begin{aligned}
T\,T^{-1} &= (E\, r_+ A_- A_e \ell^e\, F)\ (F^{-1}\, r_+ A_e^{-1} \ell^e r_+ A_-^{-1} \ell\, E^{-1}) \\
&= E\, r_+ A_- A_e \ell^e r_+ A_e^{-1} \ell^e r_+ A_-^{-1} \ell\, E^{-1} \\
&= E\, r_+ A_- \ell^e r_+ A_-^{-1} \ell\, E^{-1} \\
&= E\, E^{-1} \\
&= I_{\times_{j=1}^m H^{s_j}(\mathbb{R}_+)} \ ,
\end{aligned}$$

where we omitted the first term $\ell^e r_+$ of the second line due to the factor (invariance) property of A_e^{-1} that yields $A_e \ell^e r_+ A_e^{-1} \ell^e r_+ = \ell^e r_+$. Similarly we dropped the term $\ell^e r_+$ in $A_- \ell^e r_+ A_-^{-1} \ell$ due to a factor property of A_-.

For $T^{-1}\, T$ we have an analogous computation:

$$\begin{aligned}
T^{-1} T &= (F^{-1}\, r_+ A_e^{-1} \ell^e r_+ A_-^{-1} \ell\, E^{-1})\ (E\, r_+ A_- A_e \ell^e\, F) \\
&= F^{-1}\, r_+ A_e^{-1} \ell^e r_+ A_-^{-1} \ell\, r_+ A_- A_e \ell^e\, F
\end{aligned}$$

$$
\begin{aligned}
&= F^{-1} r_+ A_e^{-1} \ell^e r_+ A_e \ell^e F \\
&= F^{-1} F \\
&= I_{\times_{j=1}^m H^{r_j}(\mathbb{R}_+)} \ ,
\end{aligned}
$$

where we may omit the term ℓr_+ in the second line since A_-^{-1} is "minus type" and $\ell^e r_+$ can be dropped subsequently due to the factor (invariance) property of A_e that yields $A_e^{-1} \ell^e r_+ A_e \ell^e = \ell^e$.

The case $\ell^c = \ell^o$ is proved by analogy. ∎

Theorem 4.2. *If* $\psi_0 = \phi_0 \widetilde{\phi}_0^{-1}$, *see* (3.11), *admits a canonical anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^c,

$$
\psi_0 = \phi_- \, \widetilde{\phi}_-^{-1} \ , \tag{4.3}
$$

then the Toeplitz operator S_0 *presented in* (3.10) *is an invertible operator with inverse given by*

$$
S_0^{-1} = P_+ \widetilde{\phi}_- P_+ \phi_-^{-1}{}_{|[L^2_+(\mathbb{R})]^m} : \ [L^2_+(\mathbb{R})]^m \to [L^2_+(\mathbb{R})]^m \ . \tag{4.4}
$$

Proof: Having the canonical anti-symmetric generalized factorization of ψ_0, the result is a consequence of the "minus" and "plus" factor properties of ϕ_- and $\widetilde{\phi}_-^{-1}$, respectively. Therefore, a direct computation shows the statement:

$$
\begin{aligned}
S_0^{-1} S_0 &= P_+ \widetilde{\phi}_- P_+ \phi_-^{-1} \, P_+ \phi_- \widetilde{\phi}_-^{-1}{}_{|[L^2_+(\mathbb{R})]^m} \\
&= P_+ \widetilde{\phi}_- P_+ \widetilde{\phi}_-^{-1}{}_{|[L^2_+(\mathbb{R})]^m} \\
&= I_{[L^2_+(\mathbb{R})]^m} \ ,
\end{aligned} \tag{4.5}
$$

and

$$
\begin{aligned}
S_0 S_0^{-1} &= P_+ \phi_- \widetilde{\phi}_-^{-1} \, P_+ \widetilde{\phi}_- P_+ \phi_-^{-1}{}_{|[L^2_+(\mathbb{R})]^m} \\
&= P_+ \phi_- P_+ \phi_-^{-1}{}_{|[L^2_+(\mathbb{R})]^m} \\
&= I_{[L^2_+(\mathbb{R})]^m} \ . ∎
\end{aligned} \tag{4.6}
$$

The following proposition gives us an idea about the possible structure of the *intermediate space* [4] in factorizations of T due to corresponding asymmetric generalized factorizations of its lifted Fourier symbol.

Proposition 4.3. *Let an asymmetric generalized factorization of a matrix* $\phi_0 \in [L^\infty(\mathbb{R})]^{m\times m}$, *with respect to* L^2 *and* ℓ^c, *be given in the form*

$$\phi_0 = \phi_- \operatorname{diag}[\zeta^{\kappa_j}]\, \phi_e\ . \tag{4.7}$$

Then the following assertions are equivalent:

(i) *there are real numbers*

$$\delta_j \in \begin{cases} (-1/2,\, 3/2)\,, & \text{if } \ell^c = \ell^e \\ (-3/2,\, 1/2)\,, & \text{if } \ell^c = \ell^o\ , \end{cases} \tag{4.8}$$

such that

$$r_+\mathcal{F}^{-1}\phi_e \cdot \mathcal{F}\ell^c :\ [L^2(\mathbb{R}_+)]^m \to \times_{j=1}^m H^{\delta_j}(\mathbb{R}_+) \tag{4.9}$$

$$r_+\mathcal{F}^{-1}\phi_- \cdot \mathcal{F}\ell :\ \times_{j=1}^m H^{\delta_j}(\mathbb{R}_+) \to [L^2(\mathbb{R}_+)]^m \tag{4.10}$$

are bijections;

(ii) *the matrices* ϕ_e *and* ϕ_- *have the properties*

$$\operatorname{diag}[\lambda^{\delta_j}]\, \phi_e \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m} \tag{4.11}$$

$$\phi_- \operatorname{diag}[\lambda_-^{-\delta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m} \tag{4.12}$$

where δ_j *are the same as before.*

Proof: For $\nu \in (-1/2,\, 3/2)$ and $\ell^c = \ell^e$ (or $\nu \in (-3/2,\, 1/2)$ and $\ell^c = \ell^o$), $s \in \mathbb{R}$, the following operators are bijective

$$r_+\mathcal{F}^{-1}\lambda^\nu \cdot \mathcal{F}\ell^c :\ H^\nu(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \tag{4.13}$$

$$r_+\mathcal{F}^{-1}\lambda_-^s \cdot \mathcal{F}\ell :\ H^\nu(\mathbb{R}_+) \to H^{\nu-s}(\mathbb{R}_+) \tag{4.14}$$

where ℓ denotes any extension into $H^\nu(\mathbb{R})$, cf. [7], Lemma 2.1 and Lemma 2.2, as well as [10], Theorem 4.4 and Lemma 4.6. Herein, the indicated values of ν are exactly those for which the operator in (4.13) is boundedly invertible [7] whilst that one in (4.14) is a bijection for any $\nu, s \in \mathbb{R}$ [10]. Hence we have that the invertibility of the operators defined by (4.9) and (4.10) is equivalent to the invertibility of the operators

$$r_+\mathcal{F}^{-1}\operatorname{diag}[\lambda^{\delta_j}] \cdot \mathcal{F}\ell^c r_+\mathcal{F}^{-1}\phi_e \cdot \mathcal{F}\ell^c :\ [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R}_+)]^m \tag{4.15}$$

$$r_+\mathcal{F}^{-1}\phi_- \cdot \mathcal{F}\ell r_+\mathcal{F}^{-1}\operatorname{diag}[\lambda_-^{-\delta_j}] \cdot \mathcal{F}\ell :\ [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R}_+)]^m\ . \tag{4.16}$$

But, by use of the even factor property of ϕ_e and also the holomorphic extendibility of λ_- and ϕ_-, we may drop $l^c r_+$ and $l r_+$, in (4.15) and (4.16), and then the conditions (4.11) and (4.12) are necessary and sufficient for the invertibility of the operators given by (4.15) and (4.16). ∎

The following result is essential for asymptotic considerations and has its roots in the work of N.P. Vekua [19], F. Penzel [16], and the second author [17].

Theorem 4.4. *Suppose that ψ_0 (see (3.11)) has the following properties:*

- **(i)** $\psi_0 \in \mathcal{G}[C^\alpha(\ddot{\mathbb{R}})]^{m\times m}$ *for some* $\alpha \in (0,1]$;
- **(ii)** *there are m complex numbers $\eta_1, ..., \eta_m$ and an invertible constant matrix U such that*

$$-\frac{1}{2} < \operatorname{Re}\eta_j < \frac{1}{2}, \qquad \lim_{\xi\to\pm\infty} U \operatorname{diag}\left[\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\eta_j}\right] U^{-1} = \psi_0(\pm\infty) \; ; \tag{4.17}$$

- **(iii)** $\alpha > \frac{1}{2} + \max_j \operatorname{Re}\eta_j$;
- **(iv)** *the matrix ψ_0 admits a canonical anti-symmetric generalized factorization with respect to L^2 and ℓ^c.*

Then there are

$$M_- \in [L^2_-(\mathbb{R})]^{m\times m}, \qquad M_+ \in [L^2_+(\mathbb{R})]^{m\times m} \tag{4.18}$$

such that ψ_0 has the form

$$\psi_0 = \psi_- \, \widetilde{\psi}_-^{-1} \tag{4.19}$$

as a canonical anti-symmetric generalized factorization with respect to L^2 and ℓ^c, with

$$\psi_- = \Big(U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big) \operatorname{diag}[\lambda_-^{\eta_j}] \tag{4.20}$$

$$\widetilde{\psi}_-^{-1} = \operatorname{diag}[\lambda_+^{-\eta_j}]\Big(U + M_+ \operatorname{diag}[\lambda_+^{-\eta_j}]\Big)^{-1} \tag{4.21}$$

where $U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]$, $U + M_+ \operatorname{diag}[\lambda_+^{-\eta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$.

Proof: Let us take into consideration the even case (the odd case is analogous). From (iv), we have that

$$\psi_0 = \psi_- \, \widetilde{\psi}_-^{-1} \tag{4.22}$$

with $\lambda_-^{-2}\psi_- \in \left[L^2_-(\mathbb{R})\right]^{m\times m}$, $\lambda_+^{-1}\widetilde{\psi}_-^{-1} \in \left[L^2_+(\mathbb{R})\right]^{m\times m}$, $\lambda_-^{-1}\psi_-^{-1} \in \left[L^2_-(\mathbb{R})\right]^{m\times m}$ and $\lambda_+^{-2}\widetilde{\psi}_- \in \left[L^2_+(\mathbb{R})\right]^{m\times m}$. Due to assertions (i) and (ii) the matrix function ψ_0 has the representation

$$\psi_0 \;=\; U\,\mathrm{diag}\!\left[\left(\frac{\lambda_-}{\lambda_+}\right)^{\eta_j}\right] U^{-1} + \psi_{00}\;, \tag{4.23}$$

where $\psi_{00} \in [C^\alpha(\dot{\mathbb{R}})]^{m\times m}$, and $\psi_{00}(\pm\infty) = 0$. With the result of Theorem 4.2, we take profit of the existence of a matrix $M_+ \in \left[L^2_+(\mathbb{R})\right]^{m\times m}$ being the solution of the following system of singular integral equations (cf. (3.10))

$$S_0\,M_+ \;=\; -P_+\psi_{00}\,U\,\mathrm{diag}[\lambda_+^{\eta_j}]. \tag{4.24}$$

Please observe that the right-hand side of (4.24) belongs to $\left[L^2_+(\mathbb{R})\right]^{m\times m}$, due to (4.23), the action of P_+ and the assertions (i), (ii), and (iii) in the hypotheses. In addition, due to assertion (iv), Theorem 3.5 and Theorem 4.2, we point out that the matrix M_+ is uniquely defined by (4.24):

$$M_+ \;=\; -P_+\widetilde{\psi}_-P_+\psi_-^{-1}P_+\psi_{00}\,U\,\mathrm{diag}[\lambda_+^{\eta_j}]\;. \tag{4.25}$$

Then we also introduce a matrix $M_- \in \left[L^2_-(\mathbb{R})\right]^{m\times m}$ in the following way

$$M_- \;=\; \psi_0 M_+ + \psi_{00}\,U\,\mathrm{diag}[\lambda_+^{\eta_j}]\;. \tag{4.26}$$

In fact, from the construction of M_- in (4.26) and Equation (4.24), we obtain

$$P_+M_- \;=\; S_0\,M_+ + P_+\psi_{00}\,U\,\mathrm{diag}[\lambda_+^{\eta_j}] \;=\; 0\;, \tag{4.27}$$

which allows us to conclude that the matrix function M_- has a holomorphic extension into the lower half-plane.

From the representation (4.23) and the definition of M_-, introduced in (4.26), it follows that

$$\begin{aligned}\psi_0\left(U\,\mathrm{diag}[\lambda_+^{\eta_j}] + M_+\right) \;&=\; U\,\mathrm{diag}[\lambda_-^{\eta_j}] + \psi_{00}\,U\,\mathrm{diag}[\lambda_+^{\eta_j}] + \psi_0 M_+\\ &=\; U\,\mathrm{diag}[\lambda_-^{\eta_j}] + M_-\;.\end{aligned} \tag{4.28}$$

By (4.22) and (4.28) we get

$$\widetilde{\psi}_-^{-1}\left(U\,\mathrm{diag}[\lambda_+^{\eta_j}] + M_+\right) \;=\; \psi_-^{-1}\left(U\,\mathrm{diag}[\lambda_-^{\eta_j}] + M_-\right)\;. \tag{4.29}$$

The right-hand side of (4.29) is an analytic function in the half-plane $\operatorname{Im}\xi > 0$, continuous in the closed half-plane, while the left-hand side is analytic for $\operatorname{Im}\xi < 0$ and continuous in the closed half-plane where the increase at infinity is algebraic. Since both sides coincide on the real axis, it follows that they are restrictions of an analytic function in the whole complex plane to the half-planes $\operatorname{Im}\xi \geq 0$ and $\operatorname{Im}\xi \leq 0$, respectively. According to the possible increase of the factors in (4.22) and recognizing that the functions $\lambda_\pm^{\eta_j}(\xi)$ grow slower than $|\xi|$, Liouville's theorem applies and we obtain that both sides in (4.29) equal the same constant matrix.

This constant matrix is invertible because we already know that the matrix $\widetilde{\psi}_-^{-1}$ is invertible almost everywhere and, for sufficiently large $|\xi|$, the same holds for the matrix $U \operatorname{diag}[\lambda_+^{\eta_j}](\xi) + M_+(\xi)$. The latter fact can be seen by passing from the real line to the unit circle and then applying Banach's fixed point principle to the corresponding equation of (4.24). In this case, the regularity of such a solution can also be analyzed with the help of classical results from the theory of singular integral equations in Hölder spaces with weight (cf. the lemmata 4.5, 4.6 and 4.6 in [17]).

Therefore, we may use the following representation for the factors of ψ_0:

$$\psi_- = \Big(U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big) \operatorname{diag}[\lambda_-^{\eta_j}]\,, \tag{4.30}$$

$$\widetilde{\psi}_-^{-1} = \operatorname{diag}[\lambda_+^{-\eta_j}] \Big(U + M_+ \operatorname{diag}[\lambda_+^{-\eta_j}]\Big)^{-1}. \tag{4.31}$$

From (4.24) it also follows that $M_+ \operatorname{diag}[\lambda_+^{-\eta_j}]$ vanishes at infinity, so we have $U + M_+ \operatorname{diag}[\lambda_+^{-\eta_j}],\ U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$. ∎

Corollary 4.5. *Suppose that ϕ_0 has the following properties:*

(i) $\phi_0 \in \mathcal{G}\Big[C^\alpha(\ddot{\mathbb{R}})\Big]^{m\times m}$ *for some* $\alpha \in (0,1]$;

(ii) *there are m complex numbers $\eta_1, ..., \eta_m$ and an invertible constant matrix U such that*

$$-\frac{1}{2} < \operatorname{Re}\eta_j < \frac{1}{2}\,, \quad \lim_{\xi\to\pm\infty} U \operatorname{diag}\left[\left(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\right)^{\eta_j}\right] U^{-1} = \phi_0(\pm\infty)\,\phi_0^{-1}(\mp\infty)\,;$$

(iii) $\alpha > \frac{1}{2} + \max_j \operatorname{Re}\eta_j$;

(iv) *the matrix ϕ_0 admits a canonical asymmetric generalized factorization with respect to L^2 and ℓ^c.*

Then there is a matrix-valued function

$$M_- \in [L_-^2(\mathbb{R})]^{m\times m}\,, \tag{4.32}$$

such that we can present a canonical asymmetric generalized factorization of ϕ_0, *with respect to* L^2 *and* ℓ^c, *in the form*

$$\begin{aligned}
\phi_0 &= \phi_- \, \phi_e \ , \\
\phi_- &= \Big(U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big) \operatorname{diag}[\lambda_-^{\eta_j}] \\
\phi_e &= \operatorname{diag}[\lambda_-^{-\eta_j}] \Big(U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big)^{-1} \phi_0
\end{aligned} \tag{4.33}$$

where $U + M_- \operatorname{diag}[\lambda_-^{-\eta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$.

Proof: The result is a direct combination of Theorem 3.5 and Theorem 4.4. ∎

Corollary 4.6. *Under the conditions of the last result, the convolution type operator in* (1.1)–(1.2) *is invertible and its inverse has the form*

$$T^{-1} = F^{-1} \, r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell^c r_+ \mathcal{F}^{-1} \phi_-^{-1} \cdot \mathcal{F} \ell \, E^{-1} \ , \tag{4.34}$$

where E *and* F *are defined in* (2.3)–(2.4), $\ell : [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R})]^m$ *is an arbitrary continuous extension and* ϕ_- *and* ϕ_e *have the form of* (4.33).

Proof: This result follows from Theorem 4.1 and Corollary 4.5. ∎

5 – Sufficient conditions for generalized invertibility and representation of generalized inverses

In this section, we are concerned with the question of generalized invertibility of the convolution type operator T, presented in (1.1). Firstly, this will be done depending only on the existence of a convenient factorization of the (matrix) symbol (in both cases of our convolution type operator with symmetry, see Theorem 5.1, and of the Toeplitz operator S_0, see Theorem 5.2). Secondly, with additional assumptions (particularly on the behavior of the symbol jump at infinity, cf. (5.11) and (5.22)), more detailed information about the generalized invertibility is obtained at the end.

Theorem 5.1. *Let the operator* T *be given by* (1.1) *with symbol* ϕ *satisfying* (1.2). *If the matrix function* ϕ_0 *admits an asymmetric generalized factorization with respect to* L^2 *and* ℓ^c *(see* (3.1)*),*

$$\phi_0 = \phi_- \operatorname{diag}[\zeta^{\kappa_j}] \, \phi_e \ , \tag{5.1}$$

then T has a reflexive generalized inverse given by

$$T^- = F^{-1}\, r_+ A_e^{-1} \ell^c r_+ D^{-1} \ell^c r_+ A_-^{-1} \ell\, E^{-1} \ , \tag{5.2}$$

where E and F are defined in (2.3)–(2.4), *$A_e = \mathcal{F}^{-1}\phi_e \cdot \mathcal{F}$, $D = \mathcal{F}^{-1}\,\mathrm{diag}[\zeta^{\kappa_j}] \cdot \mathcal{F}$, $A_- = \mathcal{F}^{-1}\phi_- \cdot \mathcal{F}$, and $\ell : [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R})]^m$ is an arbitrary extension (which particular choice is indifferent for the definition of T^{-1}).*

Proof: First of all, we remark that we will use the decomposition

$$D = D_- D_+ \ , \tag{5.3}$$

where $D_\pm = \mathcal{F}^{-1}\,\mathrm{diag}[\zeta^{\beta_{j\pm}}] \cdot \mathcal{F}$, with

$$\beta_{j+} = \begin{cases} \kappa_j & \text{if} \ \ \kappa_j \geq 0 \\ 0 & \text{if} \ \ \kappa_j \leq 0 \end{cases} \tag{5.4}$$

and

$$\beta_{j-} = \begin{cases} 0 & \text{if} \ \ \kappa_j \geq 0 \\ \kappa_j & \text{if} \ \ \kappa_j \leq 0 \end{cases} \ . \tag{5.5}$$

Let us study the case $\ell^c = \ell^e$ and consider, in a suitable dense subspace,

$$\begin{aligned} T\,T^-\,T &= (E\,r_+ A_- D A_e \ell^e\, F)\ (F^{-1}\, r_+ A_e^{-1} \ell^e r_+ D^{-1} \ell^e r_+ A_-^{-1} \ell\, E^{-1}) \\ &\quad\ (E\,r_+ A_- D A_e \ell^e\, F) \\ &= E\,r_+ A_- D A_e \ell^e r_+ A_e^{-1} \ell^e r_+ D^{-1} \ell^e r_+ A_-^{-1} \ell r_+ A_- D A_e \ell^e\, F \qquad (5.6) \\ &= E\,r_+ A_- D_- D_+ \ell^e r_+ D_+^{-1} D_-^{-1} \ell^e r_+ D_- D_+ A_e \ell^e\, F \qquad (5.7) \\ &= E\,r_+ A_- D_- \ell^e r_+ D_+ A_e \ell^e\, F \\ &= T \ , \end{aligned}$$

where we omitted the first term $\ell^e r_+$ of (5.6) in (5.7) due to the factor (invariance) property of A_e^{-1} that yields $A_e \ell^e r_+ A_e^{-1} \ell^e r_+ = \ell^e r_+$. Similarly we dropped the term ℓr_+ in $\ell^e r_+ A_-^{-1} \ell r_+ A_-$ due to a factor property of A_-^{-1}. Analogous arguments apply to the "plus" and "minus" type factors D_-^{-1} and D_+^{-1}, respectively. More precisely: If one of the factors D_+ or D_- equals I (as in the scalar case), then $D_- \ell^e r_+ D_-^{-1} \ell^e r_+ D_- = D_- \ell^e r_+$ or $D_+ \ell^e r_+ D_+^{-1} \ell^e r_+ D_+ = \ell^e r_+ D_+$ holds, respectively. Here, in the diagonal matrix case, the situation is identical for each place in the diagonal which rectifies the simplification in the last but one step.

For $T^- T T^-$ we have an analogous computation:

$$\begin{aligned}
T^-TT^- &= (F^{-1}\, r_+A_e^{-1}\ell^e r_+D^{-1}\ell^e r_+A_-^{-1}\ell\, E^{-1})\ (E\, r_+A_-DA_e\ell^e\, F)\\
&\quad (F^{-1}\, r_+A_e^{-1}\ell^e r_+D^{-1}\ell^e r_+A_-^{-1}\ell\, E^{-1})\\
&= F^{-1}\, r_+A_e^{-1}\ell^e r_+D^{-1}\ell^e r_+A_-^{-1}\ell\, r_+A_-DA_e\ell^e\\
&\quad r_+A_e^{-1}\ell^e r_+D^{-1}\ell^e r_+A_-^{-1}\ell\, E^{-1} \qquad (5.8)\\
&= F^{-1}\, r_+A_e^{-1}\ell^e r_+D^{-1}\ell^e r_+D\ell^e r_+D^{-1}\ell^e r_+A_-^{-1}\ell\, E^{-1}\\
&= F^{-1}\, r_+A_e^{-1}\ell^e r_+D_+^{-1}D_-^{-1}\ell^e r_+D_-\\
&\quad D_+\ell^e r_+D_+^{-1}D_-^{-1}\ell^e r_+A_-^{-1}\ell\, E^{-1}\\
&= F^{-1}\, r_+A_e^{-1}\ell^e r_+D_+^{-1}\ell^e r_+D_-^{-1}\ell^e r_+A_-^{-1}\ell\, E^{-1}\\
&= T^-
\end{aligned}$$

where we may omit the term ℓr_+ in (5.8) since A_-^{-1} is "minus type" and the third $\ell^e r_+$ is unnecessary in (5.8) due to the factor (invariance) property of A_e that yields $A_e\ell^e r_+A_e^{-1}\ell^e r_+ = \ell^e r_+$. In addition, due to the "minus" factor property of D_- and D_+^{-1}, as well as, the "plus" factor property of D_+ and D_-^{-1}, one has $D_-^{-1}\ell^e r_+D_-D_+\ell^e r_+D_+^{-1} = \ell^e r_+$.

The case $\ell^c = \ell^o$ is proved by analogy. ∎

Theorem 5.2. *If* $\psi_0 = \phi_0\widetilde{\phi}_0^{-1}$, *see* (3.11), *admits an anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^c,

$$\psi_0 \;=\; \phi_-\ \mathrm{diag}[\zeta^{2\kappa_j}]\,\widetilde{\phi}_-^{-1}\ , \tag{5.9}$$

then the Toeplitz operator S_0 *presented in* (3.10) *is a generalized invertible operator and a generalized inverse of it is given by*

$$S_0^- \;=\; P_+\widetilde{\phi}_-P_+\,\mathrm{diag}[\zeta^{-2\kappa_j}]P_+\phi_-^{-1}{}_{|[L^2_+(\mathbb{R})]^m} :\ [L^2_+(\mathbb{R})]^m \to [L^2_+(\mathbb{R})]^m\ . \tag{5.10}$$

Proof: The result is derived from a direct computation as in the proof of Theorem 4.2, and therefore omitted here. ∎

Theorem 5.3. *Suppose that* ψ_0 *(see* (3.11)*) has the following properties:*

(i) $\psi_0 \in \mathcal{G}[C^\alpha(\ddot{\mathbb{R}})]^{m\times m}$ *for some* $\alpha \in (0,1]$;

(ii) *there is an invertible constant matrix* V *and* m *complex numbers* $\eta_1, ..., \eta_m$ *such that*

$$-\frac{1}{2} < \operatorname{Re}\eta_j < \frac{1}{2}\,, \qquad \psi_0^{-1}(+\infty)\,\psi_0(-\infty) = V\operatorname{diag}[e^{-2\pi i\eta_j}]V^{-1}\,; \tag{5.11}$$

(iii) $\alpha > \frac{1}{2} + \max_j \operatorname{Re}\eta_j$ *;*

(iv) *the matrix* ψ_0 *admits an anti-symmetric generalized factorization with respect to* L^2 *and* ℓ^c*.*

Then there is a matrix

$$M_- \in [L^2_-(\mathbb{R})]^{m\times m} \tag{5.12}$$

such that ψ_0 *has the form*

$$\psi_0 \;=\; \psi_-\,\operatorname{diag}[\zeta^{2\kappa_j}]\,\widetilde{\psi}_-^{-1} \tag{5.13}$$

of an anti-symmetric generalized factorization with respect to L^2 *and* ℓ^c*, with*

$$\psi_- = \Big(\psi_0(+\infty)V + M_-\operatorname{diag}[\lambda_-^{-\eta_j}]\Big)\,\operatorname{diag}[\lambda_-^{\eta_j-2\kappa_j}] \tag{5.14}$$

where $\psi_0(+\infty)V + M_-\operatorname{diag}[\lambda_-^{-\eta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$.

Proof: We will take into consideration only the even case $\ell^c = \ell^e$ (the odd case, $\ell^c = \ell^o$, is analogous). First of all, from (iv), we have that

$$\psi_0 \;=\; \psi_-\,\operatorname{diag}[\zeta^{2\kappa_j}]\,\widetilde{\psi}_-^{-1} \tag{5.15}$$

with $k_j \in \mathbb{Z}$, $\lambda_-^{-2}\psi_- \in [L^2_-(\mathbb{R})]^{m\times m}$, $\lambda_+^{-1}\widetilde{\psi}_-^{-1} \in [L^2_+(\mathbb{R})]^{m\times m}$, $\lambda_-^{-1}\psi_-^{-1} \in [L^2_-(\mathbb{R})]^{m\times m}$, and $\lambda_+^{-2}\widetilde{\psi}_- \in [L^2_+(\mathbb{R})]^{m\times m}$. On the other hand, due to assertions (i) and (ii), the matrix function ψ_0 has also the representation

$$\psi_0(\xi) \;=\; \psi_0(+\infty)V\operatorname{diag}\bigg[\Big(\frac{\lambda_-(\xi)}{\lambda_+(\xi)}\Big)^{\eta_j}\bigg]\,V^{-1} + \psi_{00}(\xi)\;, \tag{5.16}$$

where $\psi_{00} \in [C^\alpha(\dot{\mathbb{R}})]^{m\times m}$, with $\psi_{00}(\pm\infty) = 0$.

Since Theorem 5.2 and condition (iv) of the hypotheses imply that S_0 is a generalized invertible operator, we obtain projections P_1 and P_2 in $[L^2_+(\mathbb{R})]^m$ such that $\operatorname{im} P_1 = \ker S_0$ and $\operatorname{im} P_2$ is a complement of $\operatorname{im} S_0 = \ker P_2$.

Therefore, we have

$$S_0P_1 \,=\, 0 \,=\, P_2S_0 \tag{5.17}$$

and

$$\begin{array}{ccccc} & & & & \ker S_0 \\ & & & & \| \\ [L^2_+(\mathbb{R})]^m & = & \ker P_1 & \dot{+} & \operatorname{im} P_1 \\ & & \downarrow S_1 & & \\ [L^2_+(\mathbb{R})]^m & = & \ker P_2 & \dot{+} & \operatorname{im} P_2 \\ & & \| & & \\ & & \operatorname{im} S_0 & & \end{array}$$

where $S_1 = \operatorname{Rst} S_0 : \ker P_1 \to \ker P_2$ is bijective as a restriction of S_0 on $\ker P_1$ acting onto $\ker P_2$.

Because of these facts and (iii) we can introduce $M_+ \in \left[L^2_+(\mathbb{R})\right]^{m\times m}$ in the following way

$$M_+ = -P_+ S_1^{-1}(I-P_2)P_+\psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}] \tag{5.18}$$

and define

$$M_- \ = \ \psi_0 M_+ + \psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}]\ . \tag{5.19}$$

Then, according to the definition of S_1, P_1, P_2, as well as the definition of M_- and M_+, we obtain

$$\begin{aligned} P_+M_- \ &= \ P_+\psi_0 M_+ + P_+\psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}] \\ &= \ -S_0 S_1^{-1}(I-P_2)P_+\psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}] \ + \ P_+\psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}] \\ &= \ -(I-P_2)P_+\psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}] \ + \ P_+\psi_{00} V \operatorname{diag}[\lambda_+^{\eta_j}] \\ &= \ (P_2 P_+ \psi_{00}) V \operatorname{diag}[\lambda_+^{\eta_j}] \\ &= \ 0 \end{aligned}$$

which in particular shows that $M_- \in \left[L^2_-(\mathbb{R})\right]^{m\times m}$.

From the representation (5.16) and by the definition of M_- it follows that

$$\begin{aligned} \psi_0(\xi)\left(V \operatorname{diag}[\lambda_+^{\eta_j}] + M_+(\xi)\right) \ &= \ \psi_0(+\infty) V \operatorname{diag}[\lambda_-^{\eta_j}] \\ &\quad + \psi_{00}(\xi) V \operatorname{diag}[\lambda_+^{\eta_j}] \ + \ \psi_0(\xi)\, M_+(\xi) \\ &= \ \psi_0(+\infty) V \operatorname{diag}[\lambda_-^{\eta_j}] + M_-(\xi) \end{aligned}$$

thus by (5.15), we obtain

$$\begin{aligned} &\operatorname{diag}[\lambda_+^{-2\kappa_j}]\, \widetilde{\psi}_-^{-1}(\xi) \left(V \operatorname{diag}[\lambda_+^{\eta_j}] + M_+(\xi)\right) \ = \\ &\qquad \operatorname{diag}[\lambda_-^{-2\kappa_j}]\, \psi_-^{-1}(\xi) \left(\psi_0(+\infty) V \operatorname{diag}[\lambda_-^{\eta_j}] + M_-(\xi)\right) \end{aligned}$$

where the left and the right-hand side of this equation are equal to an invertible constant matrix function.

By the same method indicated in the proof of Theorem 4.4 it is possible to show the invertibility of $\psi_0(+\infty)V + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]$. Thus we can write

$$\psi_- = \Big(\psi_0(+\infty)V + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big) \operatorname{diag}[\lambda_-^{-2\kappa_j+\eta_j}] \,. \tag{5.20}$$

Finally, from (5.18)–(5.19) and $\psi_0 \in [L^\infty(\mathbb{R})]^{m\times m}$, it follows that $\psi_0(+\infty)V + M_- \operatorname{diag}[\lambda_-^{-\eta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$. ∎

Corollary 5.4. *Suppose that ϕ_0 has the following properties:*

(i) $\phi_0 \in \mathcal{G}\Big[C^\alpha(\ddot{\mathbb{R}})\Big]^{m\times m}$ *for some* $\alpha \in (0,1]$;

(ii) *there are m complex numbers $\eta_1, ..., \eta_m$ and an invertible constant matrix V such that*

$$-\frac{1}{2} < \operatorname{Re}\eta_j < \frac{1}{2} \,, \tag{5.21}$$

$$\phi_0(-\infty)\,\phi_0^{-1}(+\infty)\,\phi_0(-\infty)\,\phi_0^{-1}(+\infty) \;=\; V \operatorname{diag}[e^{-2\pi i\eta_j}]\,V^{-1} \,; \tag{5.22}$$

(iii) $\alpha > \frac{1}{2} + \max_j \operatorname{Re}\eta_j$;

(iv) *the matrix ϕ_0 admits an asymmetric generalized factorization with respect to L^2 and ℓ^c .*

Then there is a matrix-valued function

$$M_- \in [L^2_-(\mathbb{R})]^{m\times m} \,, \tag{5.23}$$

such that we can present an asymmetric generalized factorization of ϕ_0 with respect to L^2 and ℓ^c by

$$\phi_0 \;=\; \phi_- \operatorname{diag}[\zeta^{\kappa_j}]\,\phi_e \,, \tag{5.24}$$

where the factors have the form

$$\phi_- \;=\; \Big(\phi_0(+\infty)\,\phi_0^{-1}(-\infty)\,V + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big) \operatorname{diag}[\lambda_-^{\eta_j-2\kappa_j}] \tag{5.25}$$

$$\phi_e \;=\; \operatorname{diag}[\lambda_+^{\kappa_j}\,\lambda_-^{\kappa_j-\eta_j}] \Big(\phi_0(+\infty)\,\phi_0^{-1}(-\infty)\,V + M_- \operatorname{diag}[\lambda_-^{-\eta_j}]\Big)^{-1} \phi_0 \tag{5.26}$$

and $\phi_0(+\infty)\,\phi_0^{-1}(-\infty)\,V + M_- \operatorname{diag}[\lambda_-^{-\eta_j}] \in \mathcal{G}[L^\infty(\mathbb{R})]^{m\times m}$.

Proof: The result is a direct consequence of Theorem 3.5 and Theorem 5.3. ∎

Corollary 5.5. *Under the conditions of the last result, the convolution type operator T (presented in* (1.1)–(1.2)*) is generalized invertible and a generalized inverse of it has the form*

$$T^- = F^{-1}\, r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell^c \\ r_+ \mathcal{F}^{-1}\, \mathrm{diag}[\zeta^{-\kappa_j}] \cdot \mathcal{F} \ell^c r_+ \mathcal{F}^{-1} \phi_-^{-1} \cdot \mathcal{F} \ell\, E^{-1} \ , \tag{5.27}$$

where E and F are defined in (2.3)–(2.4)*, $\ell : [L^2(\mathbb{R}_+)]^m \to [L^2(\mathbb{R})]^m$ is an arbitrary extension and ϕ_- and ϕ_e have the form of* (5.25)–(5.26).

Proof: This result follows from Theorem 5.1 and Corollary 5.4. ∎

We end up with some final remarks.

1. The results of the last two sections are also useful for application to systems $Tf = g$ where additional conditions on the regularity of $g \in \times_{j=1}^m H^{s_j}(\mathbb{R}_+)$ are assumed from the beginning. This will lead to a refined description of the behavior of the solutions f of the corresponding systems. Particular results into this direction can be found in [16, 19]. The point is that the operator T in (1.1) does not directly allow the consideration in a scale of Bessel potential spaces according to the restrictions on r_j. However, for

$$H^{r,e}(\mathbb{R}) = \Big\{\varphi \in H^r(\mathbb{R}) : \ \varphi = J\varphi\Big\} \tag{5.28}$$

$$H^{r,o}(\mathbb{R}) = \Big\{\varphi \in H^r(\mathbb{R}) : \ \varphi = -J\varphi\Big\} \tag{5.29}$$

(where $J\varphi(x) = \varphi(-x)$ for $\varphi \in H^r(\mathbb{R})$, $r \geq 0$, and $J\varphi(\upsilon) = \varphi(J\upsilon)$ for test functions υ in the case of $r < 0$), a modified operator

$$\widetilde{T} = r_+ A : \ \times_{j=1}^m H^{r_j,c}(\mathbb{R}) \to \times_{j=1}^m H^{s_j}(\mathbb{R}_+) \tag{5.30}$$

(where c replaces e or o, respectively) admits the ideas that we know from [17]: (a) consideration of the family of restricted operators $\widetilde{T}_k$ where the orders r_j, s_j are simultaneously replaced by $r_j + k$, $s_j + k$, $k \in \mathbb{N}$; (b) generalized inversion of $\widetilde{T}_k$ in the sense of the last paragraph, simultaneously following from the factorization of ϕ_0; (c) representation of the preceding generalized inverse of T by a series in terms of generalized inverses of $\widetilde{T}_k$, cf. (3.2) in [17], that yields asymptotic expansions of solutions f for smooth data g.

2. For the generalization of the present results to Sobolev(-Slobodečkiĭ) spaces $W^{s,p}$, $1 < p < \infty$, it is necessary to work in Proposition 2.1 with orders $r \in (-1+1/p, 1+1/p)$ for $\ell^c = \ell^e$ or $r \in (-2+1/p, 1/p)$ for $\ell^c = \ell^o$, respectively, cf. [5]. The weights in the factorization theorems of Section 3 have to be modified correspondingly, cf. [15]. This leads to different parameter intervals for δ_j in (4.8) and sufficient conditions (4.17) etc. later on.

3. The question of constructive asymmetric matrix factorization is open. In some cases it can be reduced to the question of constructive generalized factorization, for instance if ϕ is triangular and so is ψ in Theorem 3.5. For matrix functions of Daniele–Khrapkov type and other, this step is already difficult in general, see the articles [3, 9, 11] for an overview on constructive generalized matrix factorization.

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Luís Castro,
Departamento de Matemática, Universidade de Aveiro,
Campus Universitário, 3810-193 Aveiro – PORTUGAL
E-mail: `lcastro@mat.ua.pt`

and

Frank-Olme Speck,
Departamento de Matemática, Instituto Superior Técnico, U.T.L.,
Avenida Rovisco Pais, 1049-001 Lisboa – PORTUGAL
E-mail: `fspeck@math.ist.utl.pt`