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KURAMOTO–SIVASHINSKY EQUATION IN DOMAINS WITH MOVING BOUNDARIES

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Abstract: In the non-cylindrical domain $Q = \{(x,t);\ \alpha_1(t) < x < \alpha_2(t),\ t \in (0,T)\}$ we consider the initial-boundary value problem for the one-dimensional Kuramoto–Sivashinsky equation

$$u_t + u\,u_x + \beta\,u_{xx} + \delta\,u_{xxxx} \,=\, 0 \;.$$

We prove the existence and uniqueness of global weak, strong and smooth solutions. The exponential decay of the solutions is also proved.

1 – Introduction

The Kuramoto–Sivashinsky (K-S) equation describes the thermo-diffusive instability in flame fronts and was derived independently by Sivashinsky [8] and Kuramoto [6]. The largest part of publications concerned with the K-S equation was devoted to its physical aspects. Recently appeared papers where some results on the existence and uniqueness of global solutions to the Cauchy problem were obtained, see Biagioni, Bona, Iorio and Scialom [2]. Controllability and stabilization results for the K-S equation with periodic boundary conditions were obtained by He, Glowinski, Gorman and Periaux [5].

The Cauchy problem for the multi-dimensional analogue of the K-S equation was discussed by Biagioni and Gramchev [3].

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In the paper of Tadmor [9] the well-posedness of the Cauchy problem was proved for the one-dimensional K-S equation. It was shown that the Cauchy problem admits a unique smooth solution continuously depending on initial data.

Concerning moving boundaries problems we address the reader to Limaco Ferrel and Medeiros [7] where the nonlinear Kirchhoff equation with moving ends is investigated.

Here we study the one-dimensional K-S equation in a bounded domain with moving boundaries. We prove the existence and uniqueness of global weak, strong and smooth solutions and prove that the weak solutions are smooth for $t > 0$. Finally, we prove the exponential decay of the solutions as $t \to +\infty$.

2 – Statement of the problem

Let

$$\alpha_1(t) < x < \alpha_2(t), \quad t \in [0,T], \qquad \gamma(t) = \alpha_2(t) - \alpha_1(t) \geq \delta_0 > 0 \ ;$$

and

$$\alpha_1, \alpha_2 \in \mathbf{C}^1[0,\infty) \quad \text{ with } \ |\alpha_1'(t)| + |\alpha_2'(t)| \leq \delta_1 < \infty \ .$$

We denote through

$$Q = \Big\{(x,t);\ \alpha_1(t) < x < \alpha_2(t),\ t \in (0,T)\Big\} \ .$$

In Q we consider the Kuramoto–Sivashinsky equation:

$$Lu = u_t + u\,u_x + \beta\,u_{xx} + \delta\,u_{xxxx} = 0 \quad \text{ in } \ Q \ , \tag{2.1}$$

where $\beta, \delta > 0$, with the initial data,

$$u(x,0) = u_0(x)\,, \qquad \alpha_1(0) < x < \alpha_2(0) \ . \tag{2.2}$$

On the moving boundaries the following conditions are specified

$$\begin{aligned} &u(\alpha_1(t),t) = u(\alpha_2(t),t) = 0 \ , \\ &u_{xx}(\alpha_1(t),t) = u_{xx}(\alpha_2(t),t) = 0\,, \qquad t \in [0,T] \ . \end{aligned} \tag{2.3}$$

Changing variables,

$$(x,t) \leftrightarrow (y,t)\,, \qquad u(x(y,t),t) = v(y,t) \ ,$$

where

$$y = \frac{x - \alpha_1(t)}{\gamma(t)} \ ,$$

we transform Q into the rectangle $\tilde{Q} = (0,1) \times (0,T)$, and (2.1)–(2.3) into the following problem,

$$Lv \;=\; v_t + \frac{1}{\gamma(t)}\, v\, v_y - \frac{y\,\gamma'(t) + \alpha_1'(t)}{\gamma(t)}\, v_y + \frac{\beta}{\gamma^2(t)}\, v_{yy} + \frac{\delta}{\gamma^4(t)}\, v_{yyyy} \;=\; 0 \ , \tag{2.4}$$

$$v(0,t) = v(1,t) = v_{yy}(0,t) = v_{yy}(1,t) = 0 \ , \tag{2.5}$$

$$v(y,0) = v_0(y) = u_0\Big(\alpha(0) + y\gamma(0)\Big) \ . \tag{2.6}$$

Because the transformation $(x,t) \leftrightarrow (y,t)$ is a diffeomorphism, then, solving (2.4)–(2.6), we solve also problem (2.1)–(2.3). To solve (2.4)–(2.6), we use the method of Faedo–Galerkin.

3 – Strong solutions

Let $y \in (0,1)$, $t \in (0,T)$ and $\tilde{Q} = (0,1) \times (0,T)$. We define $W_k(0,1)$ as the subspace of those functions g from $H^k(0,1)$ such that

$$\frac{\partial^{2j} g}{\partial y^{2j}}\bigg|_{y=0,1} = 0\,, \qquad j = 0, ..., \left[\frac{k}{2}\right] - 1 \ .$$

Theorem 3.1. *Let $v_0 \in W_2(0,1)$. Then there exists a function $v(y,t)$,*

$$v \;\in\; L^\infty(0,T; W_2(0,1)) \cap L^2(0,T; W_4(0,1))\,, \qquad v_t \in L^2(\tilde{Q})$$

which is a unique strong solution to (2.4)–(2.6).

Proof: Let $w_j(y)$ be the eigenfunctions of

$$\begin{cases} w_{jyy} + \lambda_j w_j \;=\; 0, \quad y \in (0,1) \ , \\ \qquad w_j|_{y=0,1} \;=\; 0 \ . \end{cases} \tag{3.1}$$

It is known that the $w_j(y)$ generate a basis in $W_k(0,1)$ which is orthonormal in $L^2(0,1)$. We seek the approximate solutions to (2.4)–(2.6) in the form,

$$v^N(y,t) = \sum_{j=1}^{N} g_j^N(t)\, w_j(y) \ ,$$

where $g_j^N(t)$ are solutions to the following Cauchy problem for the normal system of N ordinary differential equations,

$$\tag{3.2} \begin{cases} (Lv^N, w_j)(t) = 0\,, \quad (u,v)(t) = \displaystyle\int_0^1 u(y,t)\,v(y,t)\,dy\;, \\ \qquad\qquad g_j^N(0) = (v_0, w_j)\,, \quad j = 1,...,N\;. \end{cases}$$

Solutions to (3.2) exist on some interval $(0, T_N)$. To extend them to any interval $(0,T)$ and to pass to the limit as $N \to +\infty$, we need a priori estimates.

From now on C represents any positive constants and C_ε any positive constants depending on $\varepsilon > 0$.

Estimate 1: Substituting in (3.2) w_j for v^N, we obtain the following inequality,

$$\tag{3.3} \frac{1}{2}\frac{d}{dt}|v^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v^N_{yy}(t)|^2 \,\le\, \frac{\delta_1}{\delta_0}|v^N_y(t)|\,|v^N(t)| + \frac{\beta}{\delta_0^2}\,|v^N_{yy}(t)|\,|v^N(t)|\;.$$

Due to the Ehrling inequalities, (see Adams [1]), for any $\varepsilon > 0$,

$$|v^N_y(t)| \,\le\, \varepsilon|v^N_{yy}(t)| + C_\varepsilon|v^N(t)|$$

and

$$|v^N_y(t)|\,|v^N(t)| \,\le\, \varepsilon|v^N_{yy}(t)|^2 + C_\varepsilon|v^N(t)|^2\;.$$

Using the Young inequality, we rewrite (3.3) for any $\varepsilon > 0$ as follows,

$$\tag{3.4} \frac{1}{2}\frac{d}{dt}|v^N(t)|^2 + \frac{\delta}{\gamma^4(t)}|v^N_{yy}(t)|^2 \,\le\, \left[\frac{\delta_1}{2\,\delta_0}\,\varepsilon^2 + \frac{\beta}{\delta_0^2}\,\varepsilon\right]|v^N_{yy}(t)|^2 + C_\varepsilon|v^N(t)|^2\;.$$

Choosing $\varepsilon > 0$ such that

$$\frac{\delta}{\gamma^4(t)} - \left[\frac{\delta_1}{2\,\delta_0}\,\varepsilon^2 + \frac{\beta}{\delta_0^2}\,\varepsilon\right] \,\ge\, \frac{\delta}{2\,\gamma^4(t)}\;,$$

we obtain from (3.4)

$$\tag{3.5} \frac{d}{dt}|v^N(t)|^2 + |v^N_{yy}(t)|^2 \,\le\, C|v^N(t)|^2\;,$$

where $C > 0$ is a constant independent of N, v^N and t.

Integrating (3.5) over $[0,t]$, $t < T$, we have by the Gronwall lemma

$$\tag{3.6} |v^N(t)|^2 + \int_0^t |v^N_{yy}(\tau)|^2\,d\tau \,\le\, C(|v_0|^2)\;.$$

This estimate permits us to extend the local solution to the whole interval $[0,T]$. On the other hand, by Rolle's theorem,

$$v_y^N(y,t) \;=\int_\xi^y v_{ss}^N(s,t)\,ds$$

for some $\xi\in(0,1)$. Then

$$|v_y^N(t)|^2 \le |v_{yy}^N(t)|^2\ .$$

This and (3.6) imply

$$\int_0^t |v_y^N(\tau)|^2\,d\tau \;\le\; C\,|v_0|^2\ . \tag{3.7}$$

Estimate 2: To obtain higher estimates, we multiply Lv^N by $\lambda_j^2\,g_j^N(t)$, sum over $j=1,...,N$, and come to the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_{yy}^N(t)|^2 + \frac{\delta}{\gamma^4(t)}\,|v_{yyyy}^N(t)|^2 \;\le& \\ \le\; \frac{2\,\delta_1}{\delta_0}\,|v_y^N(t)|\,|v_{yyyy}^N(t)| \;+\; \frac{1}{\delta_0}|(v^N v_y^N, v_{yyyy}^N)(t)| \;+\; &\frac{\beta}{\delta_0^2(t)}\,|v_{yy}^N(t)|\,|v_{yyyy}^N(t)|\ .\end{aligned} \tag{3.8}$$

By the Ehrling inequalities,

$$|v_y^N(t)| \;\le\; \varepsilon\,|v_{yyyy}^N(t)| + C_\varepsilon |v^N(t)|\,, \qquad \varepsilon>0$$

and

$$|v_{yyy}^N(t)| \;\le\; \varepsilon\,|v_{yyyy}^N(t)| + C_\varepsilon |v^N(t)|\ .$$

Using this, the Gagliardo–Nirenberg inequalities and (3.6), the terms of (3.8) may be estimated as follows,

$$\begin{aligned}\frac{1}{\delta_0}|(v^N v_y^N, v_{yyyy}^N)(t)| \;&\le\; C\,|v^N(t)|\,|v_y^N(t)|^{\frac12}\,|v_{yy}^N(t)|^{\frac12}\,|v_{yyyy}^N(t)| \\ &\le\; C_\varepsilon(1+|v_{yy}^N(t)|^2) + \epsilon\,|v_{yyyy}^N(t)|^2\ .\end{aligned} \tag{3.9}$$

Choosing ε sufficiently small in (3.9) we come from (3.8) to the inequality,

$$\frac{d}{dt}|v_{yy}^N(t)|^2 + |v_{yyyy}^N(t)|^2 \;\le\; C(1+|v_{yy}^N(t)|^2)\ .$$

By the Gronwall lemma,

$$|v_{yy}^N(t)|^2 + \int_0^T |v_{yyyy}(\tau)|^2\,d\tau \;\le\; C(|v_0|^2_{H^2(0,1)})\ . \tag{3.10}$$

From estimates (3.6) and (3.10), we conclude that

$$v^N \text{ is bounded in } L^\infty(0,T;W_2(0,1)\cap L^2(0,T;W_4(0,1)\ . \tag{3.11}$$

On the other hand, from (3.2), we deduce

$$\begin{aligned} |v_t^N(t)|^2 \le\ & \frac{1}{\delta_0}\,|(v^N v_y^N, v_t^N)(t)| + \frac{2\,\delta_1}{\delta_0}\,|v_y^N(t)|\,|v_t^N(t)| \\ & + \frac{\beta}{\delta_0}\,|(v_{yy}^N, v_t^N)(t)| + \frac{\delta}{\delta_0^4}\,|v_{yyyy}^N(t)|\,|v_t^N(t)|\ . \end{aligned} \tag{3.12}$$

The first term in the right hand side of (3.12) is estimated as follows

$$\frac{1}{\delta_0}\,|(v^N v_y^N, v_t^N)(t)| \le C\,|v_y^N(t)|^{\frac12}\,|v_{yy}^N(t)|^{\frac12}\,|v^N(t)|\,|v_t^N(t)|\ . \tag{3.13}$$

Taking into account (3.6), (3.7) and (3.13), we get from (3.12)

$$\int_0^t |v_\tau^N(\tau)|^2\,d\tau \le \varepsilon\int_0^t |v_\tau^N(\tau)|^2\,d\tau + C_\varepsilon\,, \qquad \varepsilon>0\ .$$

Then, for $\varepsilon>0$ sufficiently small,

$$v_t^N \text{ is bounded in } L^2(0,T;L^2(0,1))$$

and, consequently, v^N is bounded in $\overline{\widetilde{Q}}$ uniformly in N.

Using (3.11) and compactness arguments, we can pass to the limit in (3.2) as $N\to\infty$, therewith to prove the existence result of Theorem 3.1. ∎

Uniqueness of strong solutions follows from uniqueness of weak solutions proved in Theorem 4.1.

4 – Weak solutions

In this section we prove that if $v_0\in L^2(0,1)$, that is $u_0\in L^2(\alpha_1(0),\alpha_2(0))$, then system (2.4)–(2.6) has a unique weak solution. This implies the uniqueness of a strong solution.

Theorem 4.1. *Let $v_0\in L^2(0,1)$. Then there exists a unique weak solution $v(y,t)$ to the problem*

$$Lv=0 \quad \text{in } \ L^2(0,T;H^{-2}(0,1)\ ,$$

$$v(0,t) = v(1,t) = v_{yy}(0,t) = v_{yy}(1,t) = 0\,, \qquad t \in (0,T)\ ,$$

$$v(y,0) = v_0(y)\,, \qquad y \in (0,1)$$

such that

$$v \in L^\infty(0,T,L^2(0,1)) \cap L^2(0,T;H^2(0,1))\,, \qquad v_t \in L^2(0,T;H^{-2}(0,1))\ .$$

Proof: Taking into account classical density results, we can find a sequence $\{v_0^\nu\}$ in $W_2(0,1)$ which converges to v_0 in $L^2(0,1)$.

From Theorem 3.1, for each ν we have a solution v^ν to the problem,

$$Lv^\nu = 0 \quad \text{in} \ \ \tilde{Q}\ , \tag{4.1}$$

$$v^\nu(0,t) = v^\nu(1,t) = v^\nu_{yy}(0,t) = v^\nu_{yy}(1,t) = 0\,, \qquad t \in [0,T]\ , \tag{4.2}$$

$$v^\nu(y,0) = v^\nu_0(y)\,, \qquad y \in (0,1)\ . \tag{4.3}$$

Multiplying (4.1) by $v^\nu(t)$, and acting as in Section 3, we obtain the estimate

$$|v^\nu(t)|^2 + \int_0^T |v^\nu_{yy}(\tau)|^2\,d\tau \ \le\ C(|v^\nu_0|^2)\ .$$

Therefore,

$$v^\nu \ \ \text{is bounded in} \ \ L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)) \tag{4.4}$$

uniformly in ν. Now we can estimate the derivative v^ν_t directly from (4.1) and get that

$$v^\nu_t \ \ \text{is bounded in} \ \ L^2(0,T,H^{-2}(0,1))\ . \tag{4.5}$$

Taking into account compactness arguments and embedding results, we can see that v^ν converges strongly in $L^2(\tilde{Q})$, therefore, there exists a subsequence which converges a.e. in $\tilde{Q}$. Then $v^\nu v^\nu_x$ converges to $v v_x$ in the sense of distribuitions in $\tilde{Q}$. From (4.4) and (4.5), we conclude that v is a weak solution to the problem,

$$v_t + \frac{1}{\gamma(t)}\,v\,v_y - \frac{(y\,\gamma'(t) + \alpha_1'(t))}{\gamma(t)}\,v_y + \frac{\beta}{\gamma^2(t)}\,v_{yy} + \frac{\delta}{\gamma^4(t)}\,v_{yyyy} \ = \ 0, \qquad \text{in} \ \ L^2(0,T;H^{-2}(0,1))\,, \tag{4.6}$$

$$v(y,0) = v_0(y)\,, \qquad y \in (0,1)\ . \tag{4.7}$$

Proof of uniqueness: Let v_1, v_2 be two solutions of system (4.6)–(4.7), corresponding to the same initial data v_0, and $z = v_1 - v_2$.

Obviously,

$$z \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))\,, \qquad z_t \in L^2(0,T;H^{-2}(0,1))$$

and

$$\int_0^t (z_\tau, w)(\tau)\,d\tau + \int_0^t \frac{1}{\gamma(\tau)}\,([v_1 v_{1y} - v_2 v_{2y}], w)\,(\tau)\,d\tau\; -$$
$$-\int_0^t \left(\left[\frac{(y\,\gamma'(\tau) + \alpha_1'(\tau))}{\gamma(\tau)}\,z_y - \frac{\beta}{\gamma^2(\tau)}\,z_{yy}\right],\,w\right)(\tau)\,d\tau + \int_0^t \frac{\delta}{\gamma^4(\tau)}(z_{yy}, w_{yy})(\tau)\,d\tau \;=\; 0\,,$$

where w is an arbitrary function from $L^2(0,T;(W_2(0,1))$. Replacing w by z, we come to the equality,

$$|z(t)|^2 + \int_0^t ([v_1^2 - v_2^2]_y, z)(\tau)\,d\tau + \int_0^t \frac{\gamma'(\tau)}{\gamma(\tau)}\,|z(\tau)|^2\,d\tau\; - \tag{4.8}$$
$$-\;2\int_0^t \frac{\beta}{\gamma^2(\tau)}\,|z_y(\tau)|^2\,d\tau \;+\; 2\int_0^t \frac{\delta}{\gamma^4(\tau)}\,|z_{yy}(\tau)|^2\,d\tau \;=\; 0\;.$$

Since

$$\begin{aligned} |([v_1^2 - v_2^2]_y, z)(t)| \;&=\; |([v_1^2 - v_2^2], z_y)(t)| \\ &=\; |(z[v_1 + v_2], z_y)(t)| \\ &\le\; \max_{y\in[0,1]} |v_1(t) + v_2(t)|\,|z(t)|\,|z_y(t)| \\ &\le\; C\,(|v_{1y}(t)| + |v_{2y}(t)|)\,|z(t)|\,|z_y(t)|\;, \end{aligned}$$

we obtain from (4.8)

$$\begin{aligned} |z(t)|^2 + 2\,\delta \int_0^t \frac{1}{\gamma^4(\tau)}\,|z_{yy}(\tau)|^2\,d\tau \;&\le \\ \le\; C\int_0^t &\Big(|v_{1y}(\tau)|^2 + |v_{2y}(\tau)|^2\Big)\,|z(\tau)|\,|z_y(\tau)|\,d\tau \\ +\; \frac{1}{\delta_0}&\int_0^t |\gamma'(\tau)|\,|z(\tau)|^2\,d\tau \;+\; \frac{\beta}{\delta_0^2}\int_0^t |z_y(\tau)|^2\,d\tau\;. \end{aligned} \tag{4.9}$$

Using Ehrling and Young inequalities, we obtain

$$|z(t)|^2 + 2\,\delta \int_0^t \frac{1}{\gamma^4(\tau)}\,|z_{yy}(\tau)|^2\,d\tau \;\le$$
$$\le\; \varepsilon\int_0^t |z_{yy}(\tau)|^2\,d\tau \;+\; C_\varepsilon\int_0^t \Big(1 + |v_{1y}(\tau)|^2 + |v_{2y}(\tau)|^2\Big)\,|z(\tau)|^2\,d\tau\;,$$

where ε is an arbitrary positive number. Choosing $\varepsilon \le \frac{2\delta}{\gamma^4(t)}$, $t \in [0,T]$, we come to the inequality,

$$|z(t)|^2 \le C \int_0^t \Big(1 + |v_{1y}(\tau)|^2 + |v_{2y}(\tau)|^2\Big) \, |z(\tau)|^2 \, d\tau \ .$$

Since v_1 and v_2 are solutions to (4.6)–(4.7), by Gronwall's lemma we conclude that $|w(t)| = 0$. ∎

5 – Smooth solutions

In this section we prove that if v_0 is more regular, then solutions of system (2.4) – (2.6) are also more regular. We introduce the notation,

$$\partial_y^k = \frac{\partial^k}{\partial y^k} \,, \qquad \partial_t^l = \frac{\partial^l}{\partial t^l} \ .$$

Theorem 5.1. *Let $k \ge 4$ be a natural number, $v_0 \in W_k(0,1)$ and $\alpha_1, \alpha_2 \in \mathcal{C}^{1+[\frac{k}{4}]}[0,\infty)$. Then there exists a unique solution to (2.4)–(2.6) such that*

$$v \in L^\infty(0,T; W_k(0,1)) \cap L^2(0,T; W_{k+2}(0,1)) \ , \tag{5.1}$$

$$\partial_t^l v \in L^\infty(0,T; W_{k-4l}(0,1)) \cap L^2(0,T; W_{k-4l+2}(0,1)) \ , \tag{5.2}$$

for $l = 0, ..., \left[\frac{k}{4}\right]$.

Proof: Considering approximate solutions to (2.4)–(2.6), we can suppose by induction that

$$v^N \text{ is bounded in } L^\infty(0,T; W_{k-1}(0,1)) \cap L^2(0,T; W_{k+1}(0,1)) \,, \qquad k \ge 4 \ .$$

By Theorem 4.1, the hypothesis of induction is true for $k = 3$, and we must prove it for $k = k+1$. Exploiting the basis $\{w_j\}$, we multiply (3.2) by $(-1)^k \lambda_j^k \, g_j^N(t)$. Summing over j, we come to the inequality,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\partial_y^k v^N(t)|^2 + \frac{\delta}{\gamma^4(t)} \, |\partial_y^{k+2} v^N(t)|^2 \ \le \\ &\qquad \le \ \frac{1}{\gamma(t)} \, |(\partial_y^{k-2}(v^N v_y^N), \partial_y^{k+2} v^N)(t)| \ + \ \frac{2\,\delta_1}{\delta_0} \, |\partial_y^k v^N(t)| \, |\partial_y^{k+1} v^N(t)| \\ &\qquad\quad + \ \frac{\beta}{\delta_0^2} \, |\partial_y^{k+1} v^N(t)|^2 \ . \end{aligned} \tag{5.3}$$

The first term in the right-hand side of (5.3) is estimated as follows,

$$\begin{aligned}
\frac{1}{\gamma(t)}\,|(\partial_y^{k-2}(v^N v_y^N), \partial_y^{k+2} v^N)(t)| \;\leq& \\
&\leq\; C\sum_{s=0}^{k-2} |(\partial_y^{k-2-s} v^N \partial_y^{s+1} v^N, \partial_y^{k+2} v^N)(t)| \\
&\leq\; C\sum_{s=0}^{k-2} |\partial_y^{s+1} v^N(t)|_{L^\infty(0,1)}\,|\partial_y^{k-2-s} v^N(t)|\,|\partial_y^{k+2} v^N(t)| \\
&\leq\; C\sum_{s=0}^{k-2} |\partial_y^{s+1} v^N(t)|^{\frac{1}{2}}\,|\partial_y^{s+2} v^N(t)|^{\frac{1}{2}}\,|\partial_y^{k-2-s} v^N(t)|\,|\partial_y^{k+2} v^N(t)|\;.
\end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}
&|\partial_y^{s+1} v^N(t)| \leq C, && s = 0, ..., k-2\;, \\
&|\partial_y^{s+2} v^N(t)| \leq C, && s = 0, ..., k-3\;, \\
&|\partial_y^{k-2-s} v^N(t)| \leq C\,, && s = 0, ..., k-2\;,
\end{aligned}$$

where C does not depend on N. Then

$$\frac{1}{\gamma(t)}\,|(\partial_y^{k-2}(v^N v_x^N), \partial_y^{k+2} v^N)(t)| \;\leq\; C_\varepsilon(1 + |\partial_y^k v^N(t)|^2) + \varepsilon|\partial_y^{k+2} v^N(t)|^2\;,$$

where ε is an arbitrary positive number. On the other hand,

$$\left(\frac{\beta}{\delta_0^2} + \frac{\delta_1}{\delta_0}\right)|\partial_y^{k+1} v^N(t)|^2 \;\leq\; C_\varepsilon|v^N(t)|^2 + \varepsilon\,|\partial_y^{k+2} v^N(t)|^2\;.$$

Using the two last inequalities, we reduce (5.3) to the form,

$$\frac{1}{2}\frac{d}{dt}|\partial_y^k v^N(t)|^2 + \beta|\partial_y^{k+2} v^N(t)|^2 \;\leq\; C_\varepsilon(1 + |\partial_y^k v^N(t)|^2) + 3\,\varepsilon\,|\partial_y^{k+2} v^N(t)|^2\;.$$

Choosing $\varepsilon > 0$ sufficiently small, we have

$$\frac{d}{dt}|\partial_y^k v^N(t)|^2 + |\partial_y^{k+2} v^N(t)|^2 \;\leq\; C(1 + |\partial_y^k v^N(t)|^2)\;.$$

Integrating from 0 to t and exploiting the Gronwall lemma, we obtain

$$|\partial_y^k v^N(t)|^2 + \int_0^T |\partial_y^{k+2} v^N(\tau)|^2 d\tau \;\leq\; C(|v_0|^2_{W_k(0,1)})\;.$$

This implies that

$$v^N \text{ is bounded in } L^\infty(0,T;W_k(0,1)) \cap L^2(0,T;W_{k+2}(0,1)), \quad \forall\, k \geq 2 .$$

Passing to the limit as $N \to \infty$ in (3.2), we obtain that

$$v \in L^\infty(0,T;W_k(0,1)) \cap L^2(0,T;W_{k+2}(0,1)) , \quad k \geq 2 , \tag{5.4}$$

and satisfies the equation

$$v_t = -\frac{1}{\gamma(t)}\, v\, v_y + \frac{(\gamma'(t) + \alpha_1'(t))}{\gamma(t)}\, v_y - \frac{\beta}{\gamma^2(t)}\, v_{yy} - \frac{\delta}{\gamma^4(t)}\, v_{yyyy} \quad \text{in } \tilde{Q} \tag{5.5}$$

and the initial condition

$$v(y,0) = v_0(y) , \quad y \in (0,1) .$$

If $k \geq 4$, we obtain directly from (5.4) and (5.5) that

$$v_t \in L^\infty(0,T;W_{k-4}(0,1)) \cap L^2(0,T;W_{k-2}(0,1)) .$$

From this and (5.4) we can rewrite (5.5) as the following ordinary differential equation

$$v_t = F(x,t) ,$$

where

$$F \in L^\infty(0,T;W_{k-4}(0,1)) \cap L^2(0,T;W_{k-2}(0,1)) .$$

It follows that

$$F_t \in L^\infty(0,T;W_{k-8}(0,1)) \cap L^2(0,T;W_{k-6}(0,1)) ,$$

hence

$$v_{tt} \in L^\infty(0,T;W_{k-8}(0,1)) \cap L^2(0,T;W_{k-6}(0,1)) .$$

By induction, we obtain

$$\partial_t^l v \in L^\infty(0,T;W_{k-4l}(0,1)) \cap L^2(0,T;W_{k-4l+2}(0,1)) , \quad l = 1, ..., \left[\tfrac{k}{4}\right] .$$

This proves Theorem 5.1. ∎

Being solutions to a parabolic problem, solutions of (2.4)–(2.6) are smooth for $t > 0$. Exploiting Galerkin approximations and the mean value theorem for integrals, we can prove the following result:

Theorem 5.2. *Let $v_0 \in L^2(0,1)$. Then there exists a unique weak solution to problem (2.4)–(2.6)*

$$v \in L^\infty(0,T;L^2(0,1) \cap L^2(0,T;H^{-2}(0,1)) ,$$

$$v_t \in L^2(0,T;H^{-2}(0,1)) ,$$

such that for any $\theta > 0$ and any natural k,

$$v \in L^\infty(\theta,T;W_{k-4l}(0,1)) \cap L^2(\theta,T;W_{k-4l+2}(0,1)) , \qquad l = 0,...,\left[\tfrac{k}{4}\right] .$$

Proof: If $v_0 \in L^2(0,1)$, then acting as in Section 3, we obtain the estimate,

$$|v(t)|^2 + \int_0^t |v_{yy}(\tau)|^2 \, d\tau \le C(|v_0|^2) , \qquad t \in (0,T) . \tag{5.6}$$

Hence, for any $\nu \in (0,T)$ and $t \in (0,\nu)$,

$$\int_0^\nu |v_{yy}(\tau)|^2 \, d\tau \le C .$$

By the mean value theorem for integrals, there exists $t_1 \in (0,\nu)$ such that

$$\nu |v_{yy}(t_1)|^2 \le C . \tag{5.7}$$

Multiplying (5.5) by v_{yyyy}, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_{yy}(t)|^2 + \frac{1}{\gamma^4(t)}(v v_y, v_{yyyy})(t) - \frac{y\,\gamma'(t) + \alpha_1'(t)}{\gamma(t)}\,(v_y, v_{yyyy})(t) \,-\\ - \frac{\beta}{\gamma^2(t)}\,|v_{yy}(t)|^2 + \frac{\delta}{\gamma^4(t)}\,|v_{yyyy}(t)|^2 = 0 .\end{aligned}$$

Taking into account (5.6), we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}|v_{yy}(t)|^2 + C_0 |v_{yyyy}(t)|^2 \le C .$$

Hence,

$$\int_{t_1}^t \left[\frac{1}{2}\frac{d}{dt}|v_{yy}(\tau)|^2 + C_0|v_{yyyy}(\tau)|^2\right] d\tau \le C(t-t_1) , \qquad t > t_1 ,$$

that is,

$$\frac{1}{2}\,|v_{yy}(t)|^2 - \frac{1}{2}\,|v_{yy}(t_1)|^2 + C_0 \int_{t_1}^t |v_{yyyy}(\tau)|^2 d\tau \le C(t-t_1) .$$

Then (5.7) implies that

$$\frac{1}{2}\,|v_{yy}(t)|^2 + C_0 \int_{t_1}^{t} |v_{yyyy}(\tau)|^2 d\tau \;\leq\; \frac{C}{\nu} + C(t-t_1)\,, \qquad t \in (t_1, T)\ . \tag{5.8}$$

Let $\nu_1 > \nu$. From (5.8), we get

$$\int_{t_1}^{\nu_1} |v_{yyyy}(\tau)|^2 d\tau \;\leq\; C\ .$$

By the mean value theorem for integrals, there exists $t_2 \in [t_1, \nu_1]$ such that

$$(\nu_1 - t_1)\, |v_{yyyy}(t_2)|^2 \;\leq\; C\ .$$

Repeating this procedure, we prove Theorem 5.2. ∎

6 – Stability

It is well-known that solutions of a parabolic equation

$$u_t + Au \;=\; 0$$

are stable as $t \to +\infty$ provided that A is a positive operator. In our case, A is nonlinear and depends on parameters $\gamma(t), \beta, \delta$. But it is possible to find sufficient conditions which guarantee asymptotic decay of $v(y,t)$:

Theorem 6.1. *Let $v(y,t)$ be a strong solution to (2.4)–(2.6) and for large t the following conditions hold:*

1) $\sup_{t\in\mathbb{R}^+}(\gamma(t)) < \infty$,

2) $\delta - \beta\gamma^2(t) \geq \sigma > 0$,

3) $2\,\lambda_1(\delta - \beta\gamma^2(t)) - \gamma^3(t)\,\gamma'(t) \geq \sigma_1 > 0$,

where λ_1 is the first eigenvalue in (3.1). *Then there exists a constant $\theta > 0$ such that*

$$|v(t)|^2 \leq |v_0|^2\, e^{-\theta t} \qquad \textit{as}\ \ t \to \infty\ .$$

Proof: Multiplying (2.4) by v, we obtain

$$\frac{d}{dt}|v(t)|^2 + \frac{\gamma'(t)}{\gamma(t)}|v(t)|^2 - \frac{2\beta}{\gamma^2(t)}|v_y(t)|^2 + \frac{\delta}{\gamma^4(t)}|v_{yy}|^2 \;=\; 0\ .$$

Using (3.7), we get

$$\frac{d}{dt}|v(t)|^2 + \frac{\gamma'(t)}{\gamma(t)}|v(t)|^2 + \frac{2}{\gamma^4(t)}\Big(\delta - \gamma^2(t)\beta - \gamma(t)\eta\Big)\,|v_{yy}(t)|^2 \;\le\; 0 \;.$$

If $\delta - \beta\gamma^2(t) \ge \sigma > 0,\ \forall t \in \mathbb{R}^+$, then

$$\frac{d}{dt}|v(t)|^2 + \frac{\gamma'(t)}{\gamma(t)}|v(t)|^2 + \frac{2\sigma}{\gamma^4(t)}|v_{yy}(t)|^2 \;\le\; 0 \;. \tag{6.1}$$

Because λ_1 is the first eingenvalue in (3.1), we have

$$|v_{yy}(t)|^2 \ge \lambda_1 |v(t)|^2 \;,$$

and we obtain from (6.1) that

$$\frac{d}{dt}|v(t)|^2 + \left(\frac{2\sigma\lambda_1}{\gamma^4(t)} + \frac{\gamma'(t)}{\gamma(t)}\right)|v(t)|^2 \;\le\; 0 \;.$$

From conditions 2), 3) of Theorem 6.1, it follows

$$\frac{d}{dt}|v(t)|^2 + \theta|v(t)|^2 \le 0\,, \qquad \theta > 0 \;,$$

therefore,

$$|v(t)|^2 \le |v_0|^2\, e^{-\theta t}\,, \qquad t > 0 \;. \;\blacksquare$$

We proved our results on the existence, uniqueness and stability of solutions for the transformed problem (2.4)–(2.6). Since the transformation $(x,t) \leftrightarrow (y,t)$ is a diffeomorphism, the same results hold for the original problem (2.1)–(2.3).

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