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WEIGHTED BOUNDEDNESS FOR COMMUTATORS OF PARAMETERIZED LITTLEWOOD–PALEY OPERATORS AND AREA INTEGRALS

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ABSTRACT. We establish the boundedness for commutators of parameterized Littlewood–Paley operators and area integrals on weighted Lebesgue spaces $L^p(\omega)$ when $1 < p < \infty$, where the kernel satisfies certain logarithmic type Lipschitz condition. Moreover, the weighted endpoint estimates when $p = 1$ are also obtained.

1. Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n (n \geqslant 2)$ equipped with normalized Lebesgue measure. Let Ω be a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \tag{1.1}$$

The parameterized area integral $\mu^{\rho}_{\Omega,S}$ and parameterized Littlewood–Paley operator $\mu^{*,\rho}_{\lambda}$ are defined by

$$\mu^{\rho}_{\Omega,S}(f)(x) = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\, dz \right|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

$$\mu^{*,\rho}_{\lambda}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\, dz \right|^2 \frac{dy\, dt}{t^{n+1}} \right)^{\frac{1}{2}},$$

respectively, where $\rho > 0$, $\lambda > 1$ and $\Gamma(x) = \{(t, y) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$.

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Define the Hilbert spaces as follows.

$$\mathcal{H}_1=\left\{h:\|h\|_{\mathcal{H}_1}=\left(\int_0^\infty\int_{|y|<1}|h(t,y)|^2\frac{dy\,dt}{t}\right)^{\frac{1}{2}}<\infty\right\},$$

$$\mathcal{H}_2=\left\{h:\|h\|_{\mathcal{H}_2}=\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{1}{1+|y|}\right)^{\lambda n}|h(t,y)|^2\frac{dy\,dt}{t}\right)^{\frac{1}{2}}<\infty\right\},$$

where $\lambda>1$. Then

$$\mu^\rho_{\Omega,S}(f)(x)=\|\phi_{t,y}(f)(x)\|_{\mathcal{H}_1},\ \mu^{*,\rho}_\lambda(f)(x)=\|\phi_{t,y}(f)(x)\|_{\mathcal{H}_2},$$

where $\phi(x)=\frac{\Omega(x)}{|x|^{n-\rho}}\chi_{\{x:|x|<1\}}$, and $\phi_{t,y}(f)(x)=\int t^{-n}\phi(\frac{x-z}{t}-y)f(z)\,dz$.

The commutators of $\mu^\rho_{\Omega,S}$ and $\mu^{*,\rho}_\lambda$ are defined by

$$\mu^{\rho,b}_{\Omega,S}(f)(x)=\left(\iint_{\Gamma(x)}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$\begin{aligned}\mu^{*,\rho}_{\lambda,b}(f)(x)=&\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\right.\\&\left.\times\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(x)-b(z))f(z)\,dz\right|^2\frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},\end{aligned}$$

respectively.

Denote $f_Q=\frac{1}{|Q|}\int_Q f(y)\,dy$. For $\delta>0$, we define

$$M_\delta(f)=\left[M(|f|^\delta)\right]^{\frac{1}{\delta}},\ M^\sharp_\delta(f)=\left[M^\sharp(|f|^\delta)\right]^{\frac{1}{\delta}},$$

where M is the Hardy–Littlewood maximal operator and $M^\sharp$ is the Fefferman–Stein sharp function. The corresponding dyadic maximal operators are denoted by M^Δ_δ and $M^{\sharp,\Delta}_\delta$, respectively.

A function $A:[0,\infty)\to[0,\infty)$ is said to be a Young function if it is continuous, convex and increasing satisfying $A(0)=0$ and $A(t)\to\infty$ as $t\to\infty$. The complementary Young function $\bar{A}(t)$ of the Young function $A(t)$ is defined by

$$\bar{A}(s)=\sup_{0\leqslant t<\infty}[st-A(t)],\quad 0\leqslant s<\infty.$$

As an example, $\Phi_m(t)=t(1+\log^+t)^m, 1\leqslant m<\infty$, is a Young function with its complementary $\bar{\Phi}_m(t)\approx e^{t^{1/m}}$. If A is a Young function, then the Luxembury norm of f on a cube $Q\subset\mathbb{R}^n$ is defined by

$$\|f\|_{A,Q}=\inf\left\{\lambda>0:\frac{1}{|Q|}\int_Q A\left(\frac{|f(y)|}{\lambda}\right)dy\leqslant 1\right\}.$$

If $A(t)=\Phi_1(t)$, we denote

$$\|f\|_{L\log L,Q}=\|f\|_{\Phi_1,Q},\quad \|f\|_{expL,Q}=\|f\|_{\bar{\Phi}_1,Q},\quad M_{L\log L}f(x)=\sup_{Q\ni x}\|f\|_{L\log L,Q}.$$

For the Luxembury norm, there is the following generalized Hölder inequality.

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leqslant \|f\|_{A,Q}\|g\|_{\bar{A},Q}.$$

Let us recall the definition of A_p weight class. A locally integrable nonnegative function ω is said to belong to $A_p(1 < p < \infty)$, if there is a constant $C > 0$ such that

$$\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}dx\right)^{p-1} \leqslant C,$$

where Q denotes a cube in $\mathbb{R}^n$. The smallest constant C such that the above inequality holds is called the A_p constant of ω and denoted by $[\omega]_{A_p}$. A weight ω is said to be in the class A_1 if there is a positive constant C such that $M\omega(x) \leqslant C\omega(x)$, *a.e.* $x \in \mathbb{R}^n$. We denote by $[\omega]_{A_1}$ the infimum of all these C. A weight ω is in the class A_∞ if there are positive constants C, ϵ such that

$$\frac{\omega(E)}{\omega(Q)} \leqslant C\left(\frac{|E|}{|Q|}\right)^{\epsilon},$$

for all cubes Q and all measurable sets $E \subset Q$. We denote by $[\omega]_{A_\infty}$ the infimum of all these C.

Inspired by Hörmander's work [**7**] on the parameterized Marcinkiewicz integral, the parameterized Littlewood–Paley g^*_λ function $\mu^{*,\rho}_\lambda$ and parameterized area integral $\mu^\rho_{\Omega,S}$ were discussed by Sakamoto and Yabuta [**12**] in 1999. In [**12**], the authors studied the $L^p(1 < p < \infty)$ boundedness with kernel satisfying the Lip_α condition. In 2002, Ding, Lu and Yabuta [**2**] proved the $L^p(2 \leqslant p < \infty)$ boundedness of $\mu^{*,\rho}_\lambda$ and $\mu^\rho_{\Omega,S}$ with kernel satisfying a weaker $Llog^+L(S^{n-1})$ condition.

Torchinsky and Wang [**14**] considered the weighted L^p boundedness of $\mu^{*,\rho}_\lambda$ and $\mu^\rho_{\Omega,S}$ with kernel satisfying the Lip_α condition. In 1999, Ding, Fan and Pan [**1**] improved Torchinsky and Wang's result in [**14**] and gave the weighted L^p boundedness of $\mu^{*,\rho}_\lambda$ and $\mu^\rho_{\Omega,S}$ when $\Omega \in L^q(S^{n-1})(q > 1)$. In 2002, Duoandikoetxea and Seijo [**5**] studied the weighed L^p boundedness of $\mu^{*,\rho}_\lambda$ and $\mu^\rho_{\Omega,S}$ with rough kernel. In 2004, Xue [**15**] proved the weighed L^p boundedness of $\mu^{*,\rho}_\lambda$ and $\mu^\rho_{\Omega,S}$ with Ω satisfying the L^2-Dini condition.

Lee and Rim [**8**], in 2004, established the logarithmic type Lipschitz condition

$$|\Omega(y_1) - \Omega(y_2)| \leqslant \frac{C}{\left(\log\frac{2}{|y_2-y_1|}\right)^{\alpha}}, \tag{1.2}$$

for any $y_1, y_2 \in S^{n-1}$, where $\alpha > 1$, and proved the type (L^∞, BMO) and (L^p, L^p) boundedness of the Marcinkiewicz integral with kernel satisfying the condition (1.2). In 2012, Lin, Liu and Gao [**9**] gave the endpoint estimate of $\mu^{*,\rho}_\lambda$.

Theorem 1.1. [**9**] *Let $n \geqslant 2$, $\Omega \in L^2(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha > \frac{3}{2}$, then for $\rho > n/2$, $\lambda > 2$, there exists a constant $C > 0$, such that for all $\beta > 0$ and $f \in L^1(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : |\mu^{*,\rho}_\lambda(f)(x)| > \beta\}| \leqslant C\|f\|_{L^1}/\beta.$$

In 2013, the authors in [**10**] discussed the operators $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with kernel satisfying (1.2) on weak Hardy spaces. Recently the authors in [**11**] gave the weighted L^p boundedness of $\mu_{\Omega,S}^\rho$ and $\mu_\lambda^{*,\rho}$ with kernel satisfying (1.2).

THEOREM 1.2. [**11**] *Let $\Omega \in L^2(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha > \frac{3}{2}$. Suppose $\omega \in A_p$, then for $\rho > n/2$, $\lambda > 2$ and $1 < p < \infty$,*

$$\|\mu_{\Omega,S}^\rho(f)\|_{p,\omega} \leqslant C\|f\|_{p,\omega}, \quad \|\mu_\lambda^{*,\rho}(f)\|_{p,\omega} \leqslant C\|f\|_{p,\omega}.$$

On the other hand, the boundedness of the commutator has also received increasing attentions. Torchinsky and Wang [**14**] in 1990 proved that if $b \in BMO$, then the commutator of the Marcinkiewicz integral $[b, \mu_\Omega]$ is bounded on weighted spaces $L^p(\omega)$ for $1 < p < \infty$ and $\omega \in A_p$. In 2002, Ding, Lu and Yabuta [**2**] gave the weighted L^p boundedness of the higher order commutator $\mu_{\Omega,b}^m$ for rough Marcinkiewicz integral. In 2004, Ding, Lu and Zhang [**3**] gave the endpoint weighted esimates for the higher order commutator $\mu_{\Omega,b}^m$. In 2007, Ding and Xue [**4**] gave the weighted boundedness and the weak $L \log L$ estimates for the commutators of parameterized Littlewood–Paley operators and area integrals with kernel satisfying the L^2-Dini condition.

Inspired by the above results, in this paper, we will focus on the weighted L^p $(1 < p < \infty)$ boundedness and the weighted endoint estimates $(p = 1)$ for the commutators $\mu_{\Omega,S}^{\rho,b}$ and $\mu_{\lambda,b}^{*,\rho}$, where the kernel satisfies the logarithmic type Lipschitz condition (1.2).

2. Main results

Now, we state our main results as follows.

THEOREM 2.1. *Suppose that $\rho > n/2$, $\lambda > 2$, $\Omega \in L^2(S^{n-1})$ is a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha > \frac{3}{2}$. If $b \in BMO$, then for $1 < p < \infty$ and $\omega \in A_p$, there exists a constant $C > 0$ such that for any $f \in L^p(\omega)$,*

$$\|\mu_{\Omega,S}^{\rho,b}(f)\|_{p,\omega} \leqslant C\|f\|_{p,\omega}, \quad \|\mu_{\lambda,b}^{*,\rho}(f)\|_{p,\omega} \leqslant C\|f\|_{p,\omega}.$$

THEOREM 2.2. *Suppose that $\rho > n/2$, $\lambda > 2$, $\Omega \in L^2(S^{n-1})$ is a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha > \frac{5}{2}$. If $b \in BMO$ and $\omega \in A_1$, then there exists a constant $C > 0$ such that for any $\beta > 0$ and each smooth function f with compact support, the following inequalities hold*

$$\omega(\{x \in \mathbb{R}^n : \mu_{\Omega,S}^{\rho,b}(f)(x) > \beta\}) \leqslant C \int_{\mathbb{R}^n} \frac{|f(x)|}{\beta}\left(1 + \log^+ \frac{|f(x)|}{\beta}\right)\omega(x)\, dx,$$

$$\omega(\{x \in \mathbb{R}^n : \mu_{\lambda,b}^{*,\rho}(f)(x) > \beta\}) \leqslant C \int_{\mathbb{R}^n} \frac{|f(x)|}{\beta}\left(1 + \log^+ \frac{|f(x)|}{\beta}\right)\omega(x)\, dx.$$

3. Some lemmas

In order to prove the main results, we need the following necessary lemmas.

Lemma 3.1. [**16**] *Suppose $f \in BMO$. There exist constants C_1, $C_2 > 0$, depending only on the dimension n, such that for $0 < C < \frac{C_2}{\|f\|_*}$, every cube Q in $\mathbb{R}^n$, we have*

$$\int_Q e^{C|f(x)-f_Q|}dx \leqslant C_1 C\left(\frac{C_2}{\|f\|_*} - C\right)^{-1}|Q|.$$

Lemma 3.2. *Let $1 < p < \infty$ and $\lambda' > 0$. Then, when $b(x) \in BMO$ with $\|b\|_* < \min\{\frac{C_2}{\lambda'}, \frac{C_2(p-1)}{\lambda'}\}$, where C_2 is the constant in Lemma 3.1, we have $e^{\lambda' b(x)} \in A_p$.*

Proof. We have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q e^{\lambda' b(x)}dx\right)\left(\frac{1}{|Q|}\int_Q \left(e^{\lambda' b(x)}\right)^{-\frac{1}{p-1}}dx\right)^{p-1}\\
&=\left(\frac{1}{|Q|}\int_Q e^{\lambda'(b(x)-b_Q)}dx\right)\left(\frac{1}{|Q|}\int_Q \left(e^{-\lambda'(b(x)-b_Q)}\right)^{\frac{1}{p-1}}dx\right)^{p-1}\\
&\leqslant\left(\frac{1}{|Q|}\int_Q e^{\lambda'|b(x)-b_Q|}dx\right)\left(\frac{1}{|Q|}\int_Q e^{\frac{\lambda'}{p-1}|b(x)-b_Q|}dx\right)^{p-1} := I_Q.
\end{aligned}$$

Let $\lambda_0 = \frac{\lambda'}{p-1}$. If $1 < p < 2$, then $\lambda_0 > \lambda'$. By taking $C = \lambda_0$ in Lemma 3.1, we have

$$\begin{aligned}
I_Q &\leqslant \left(\frac{1}{|Q|}\int_Q e^{\lambda_0|b(x)-b_Q|}dx\right)\left(\frac{1}{|Q|}\int_Q e^{\lambda_0|b(x)-b_Q|}dx\right)^{p-1}\\
&= \left(\frac{1}{|Q|}\int_Q e^{\lambda_0|b(x)-b_Q|}dx\right)^{p} \leqslant \left(\frac{C_1\lambda_0}{\frac{C_2}{\|b\|_*} - \lambda_0}\right)^p.
\end{aligned}$$

If $p \geqslant 2$, then $\lambda_0 \leqslant \lambda'$. By taking $C = \lambda'$ in Lemma 3.1, we have

$$\begin{aligned}
I_Q &\leqslant \left(\frac{1}{|Q|}\int_Q e^{\lambda'|b(x)-b_Q|}dx\right)\left(\frac{1}{|Q|}\int_Q e^{\lambda'|b(x)-b_Q|}dx\right)^{p-1}\\
&= \left(\frac{1}{|Q|}\int_Q e^{\lambda'|b(x)-b_Q|}dx\right)^{p} \leqslant \left(\frac{C_1\lambda'}{\frac{C_2}{\|b\|_*} - \lambda'}\right)^p.
\end{aligned}$$

The two cases imply the desired result. □

Remark 3.1. It follows from Lemma 3.2 that if $1 < p < \infty$, $\lambda' > 0$ and $a(x), b(x) \in BMO$ with $\|a\|_* \leqslant \|b\|_* < \min\left\{\frac{C_2}{\lambda'}, \frac{C_2(p-1)}{\lambda'}\right\}$, then $e^{\lambda' a(x)} \in A_p$ and the A_p constant of $e^{\lambda' a(x)}$ satisfies

$$\left[e^{\lambda' a(x)}\right]_{A_p} \leqslant \begin{cases} \left(\frac{C_1\lambda_0}{\frac{C_2}{\|b\|_*}-\lambda_0}\right)^p, & 1 < p < 2,\\ \left(\frac{C_1\lambda'}{\frac{C_2}{\|b\|_*}-\lambda'}\right)^p, & 2 \leqslant p < \infty. \end{cases}$$

Lemma 3.3. *Suppose $b \in BMO$. There is a positive constant C such that for all ball $B \subset \mathbb{R}^n$,*

$$\|b - b_B\|_{expL,B} \leqslant C\|b\|_*.$$

Proof. When $C > \frac{1}{C_2}$, by Lemma 3.1, we have

$$\frac{1}{|B|}\int_B e^{\frac{|b(x)-b_B|}{C\|b\|_*}}dx \leqslant C_1\frac{1}{C\|b\|_*}\left(\frac{C_2}{\|b\|_*} - \frac{1}{C\|b\|_*}\right)^{-1} = \frac{C_1}{C_2C-1}.$$

Taking $C \geqslant \frac{C_1+1}{C_2}$, we have $\frac{C_1}{C_2C-1} \leqslant 1$, then $\frac{1}{|B|}\int_B e^{\frac{|b(x)-b_B|}{C\|b\|_*}}dx \leqslant 1$. By the definition of $\|b - b_B\|_{expL,B}$, there is $\|b - b_B\|_{expL,B} \leqslant C\|b\|_*$. □

Lemma 3.4. [**6**] *Let $0 < r < l < \infty$. For each function f, define*

$$\|f\|_{WL^l} = \sup_{t>0} t|\{x : |f(x)| > t\}|^{\frac{1}{l}}, N_{l,r}(f) = \sup_E \frac{\|f\chi_E\|_r}{\|\chi_E\|_s}, \frac{1}{s} = \frac{1}{r} - \frac{1}{l},$$

where the supremum is taken over all the measurable sets E with $0 < |E| < \infty$. Then

$$\|f\|_{WL^l} \leqslant N_{l,r}(f) \leqslant \left(\frac{l}{l-r}\right)^{\frac{1}{r}}\|f\|_{WL^l}.$$

Lemma 3.5. [**3**] (1) *Let M^Δ, $M^{\sharp,\Delta}$ be the dyadic Hardy–Littlewood maximal operator and the dyadic sharp function, respectively. If $\omega \in A_\infty$, then there exists a positive dimensional constant C for which the following good-λ inequality holds. For all λ, $\varepsilon > 0$,*

$$\begin{aligned}&\omega(\{y \in \mathbb{R}^n : M^\Delta f(y) > \lambda, M^{\sharp,\Delta} f(y) \leqslant \varepsilon\lambda\})\\ &\quad\leqslant C[\omega]_{A_\infty}\varepsilon\omega(\{y \in \mathbb{R}^n : M^\Delta f(y) > \lambda/2\}).\end{aligned}$$

(2) *If $\varphi : (0,\infty) \to (0,\infty)$ is a doubling function and $\delta > 0$, then there exists a positive constant C such that*

$$\sup_{\lambda>0}\varphi(\lambda)\omega(\{y \in \mathbb{R}^n : M_\delta^\Delta f(y) > \lambda\}) \leqslant C[\omega]_{A_\infty}\sup_{\lambda>0}\varphi(\lambda)\omega(\{y \in \mathbb{R}^n : M_\delta^{\sharp,\Delta} f(y) > \lambda\})$$

for all functions f such that the left side is finite.

Lemma 3.6. [**3**] *There exists a constant $C > 0$ such that for any weight ω and all $\beta > 0$,*

$$\omega(\{y : M^{m+1}f(y) > \beta\}) \leqslant C\int_{\mathbb{R}^n}\frac{|f(y)|}{\beta}\left(1 + \log^+\frac{|f(y)|}{\beta}\right)^m M\omega(y)\,dy$$

for every locally integrable function f.

Lemma 3.7. *As for $|y - z| \geqslant 4r$, there is*

$$\int_{|y-z|}^\infty \frac{(\log\frac{t}{r})^{4+2\epsilon}}{t^{2\rho-n+1}}dt \leqslant C\frac{\left[\log(\frac{|y-z|}{r})\right]^{4+2\epsilon}}{(|y-z|)^{2\rho-n}},$$

where $r > 0$, $0 < \epsilon < \rho - \frac{n}{2}$ and $\rho > \frac{n}{2}$.

The proof of this lemma is similar to that of Lemma 2.1.2 in [**15**], so we omit the details.

LEMMA 3.8. [**11**] *There exists a constant $C > 0$ such that for any $z \in (8B^*)^c$, $|y-z| \geqslant 6r$,*

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega - x_0 + y - z)}{|\omega - x_0 + y - z|^{n-\rho}}\right| \leqslant \frac{C(1+|\Omega(y-z)|)}{|y-z|^{n-\rho}\big(\log\frac{|y-z|}{r}\big)^{\alpha}},$$

where $\Omega \in L^2(S^{n-1})$ is a homogeneous function of degree zero satisfying (1.1) *and* (1.2) *with $\alpha > \frac{3}{2}$; B is a ball with center at $\bar{x}$ and radius r_0; B^* is a ball with center at $\bar{x}$ and radius $r = 2r_0$ and x_0, $\omega \in B$.*

LEMMA 3.9. *Let $b \in BMO$, $0 < \delta < l < 1$. Suppose that $\rho > n/2$, $\lambda > 2$, $\Omega \in L^2(S^{n-1})$ is a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha > \frac{5}{2}$. Then for any smooth function with compact support f, there exists a positive constant $0 < C = C_\delta$ such that*

$$M_\delta^{\sharp,\Delta}\big(\mu_{\lambda,b}^{*,\rho}(f)\big)(x) \leqslant C\|b\|_*\big(M_l^{\Delta}(\mu_\lambda^{*,\rho}(f))(x) + M^2(f)(x)\big).$$

PROOF. Given $x \in \mathbb{R}^n$, let $Q = Q(\bar{x}, r_1)$ be a dyadic cube centered at $\bar{x}$ with half side length r_1 and $x \in Q$. Let B be a ball centered at $\bar{x}$ and with radius $r_0 = \sqrt{n}r_1$, and $B^* = B(\bar{x}, r)$ with $r = 2r_0$. Decompose $f = f\chi_{8B^*} + f(1-\chi_{8B^*}) := f_1 + f_2$. Take $C_Q = \frac{1}{|Q|}\int_Q \mu_\lambda^{*,\rho}[(b-b_{B^*})f_2](u)\,du$; then

$$\begin{aligned}&|\mu_{\lambda,b}^{*,\rho}(f)(u) - C_Q|\\ &\leqslant |b(u)-b_{B^*}|\mu_\lambda^{*,\rho}(f)(u) + \mu_\lambda^{*,\rho}((b-b_{B^*})f_1)(u) + |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u) - C_Q|,\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \Big||\mu_{\lambda,b}^{*,\rho}(f)(u)|^\delta - |C_Q|^\delta\Big|du\right)^{\frac{1}{\delta}}\\
&\quad\leqslant C_\delta\left(\frac{1}{|Q|}\int_Q |(b(u)-b_{B^*})\mu_\lambda^{*,\rho}(f)(u)|^\delta du\right)^{\frac{1}{\delta}}\\
&\qquad + C_\delta\left(\frac{1}{|Q|}\int_Q |\mu_\lambda^{*,\rho}((b-b_{B^*})f_1)(u)|^\delta du\right)^{\frac{1}{\delta}}\\
&\qquad + C_\delta\left(\frac{1}{|Q|}\int_Q |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u) - C_Q|^\delta du\right)^{\frac{1}{\delta}}\\
&\quad := C_\delta(I_1 + I_2 + I_3).\end{aligned}$$

As for I_1, we can choose $1 < r < \min\{\frac{l}{\delta}, \frac{1}{1-\delta}\}$, then by Hölder's inequality,

$$\begin{aligned}I_1 &\leqslant \left(\frac{1}{|Q|}\int_Q |b(u)-b_{B^*}|^{\delta r'} du\right)^{\frac{1}{\delta r'}}\left(\frac{1}{|Q|}\int_Q |\mu_\lambda^{*,\rho}(f)(u)|^{\delta r} du\right)^{\frac{1}{\delta r}}\\
&\leqslant C\|b\|_*\left(\frac{1}{|Q|}\int_Q |\mu_\lambda^{*,\rho}(f)(u)|^l du\right)^{\frac{1}{l}} \leqslant C\|b\|_* M_l^{\Delta}(\mu_\lambda^{*,\rho}(f))(x).\end{aligned}$$

As for I_2, applying Lemma 3.4 with $\frac{1}{s}=\frac{1}{\delta}-1$, Theorem 1.1, Hölder's generalized inequality and Lemma 3.3, we have

$$\begin{aligned}
I_2 &\leqslant \left(\frac{1}{|Q|}\right)^{\frac{1}{\delta}}\left(\frac{1}{1-\delta}\right)^{\frac{1}{\delta}}\|\mu_\lambda^{*,\rho}((b-b_{B^*})f_1)\|_{WL^1}\|\chi_Q\|_s \\
&\leqslant \frac{C}{|8B^*|}\int_{8B^*}|b(u)-b_{B^*}||f(u)|du \\
&\leqslant \|b-b_{B^*}\|_{expL,8B^*}\|f\|_{L\log L,8B^*} \\
&\leqslant C\|b\|_* M_{L\log L}(f)(x).
\end{aligned} \tag{3.1}$$

Note that $M^2(f)(x)\approx M_{L\log L}(f)(x)$, we have $I_2\leqslant C\|b\|_*M^2(f)(x)$.

Finally, let us estimate I_3. Since $f\in L^p$, and $\mu_\lambda^{*,\rho}$ is L^p bounded for $1<p<\infty$ by Theorem 1.2, then

$$\int_Q|\mu_\lambda^{*,\rho}(f_2)(u)|du \leqslant |Q|^{\frac{1}{p'}}\Big(\int_Q|\mu_\lambda^{*,\rho}(f_2)(u)|^p du\Big)^{\frac{1}{p}} \leqslant C|Q|^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^n}|f(u)|^p du\Big)^{\frac{1}{p}}.$$

This fact shows that $\mu_\lambda^{*,\rho}(f_2)(u)<\infty$ a.e. on Q, so except a subset E with measure zero, for all $u\in Q\smallsetminus E$, $\mu_\lambda^{*,\rho}(f_2)(u)<\infty$. Thus,

$$\begin{aligned}
I_3 &\leqslant \frac{1}{|Q|}\int_Q|\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u)-(\mu_\lambda^{*,\rho}((b-b_{B^*})f_2))_Q|du \\
&\leqslant \frac{1}{|Q|^2}\int_{Q\smallsetminus E}\int_{Q\smallsetminus E}|\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(u)-\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(v)|dvdu.
\end{aligned}$$

Next we will prove the following fact. For any $x_0,w\in Q\smallsetminus E$,

$$\begin{aligned}
J &= |\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(x_0)-\mu_\lambda^{*,\rho}((b-b_{B^*})f_2)(w)| \\
&\leqslant Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz + Cr^{\varepsilon/2}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}}dz \\
&\quad + Cr^{\rho-\frac{n}{2}}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}dz \\
&\quad + C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{2+\varepsilon}}dz \\
&:= T_1+T_2+T_3+T_4.
\end{aligned} \tag{3.2}$$

write

$$\begin{aligned}
J &= \big|\|\phi_{t,y}((b-b_{B^*})f_2)(x_0)\|_{\mathcal{H}_2}-\|\phi_{t,y}((b-b_{B^*})f_2)(w)\|_{\mathcal{H}_2}\big| \\
&\leqslant \|\phi_{t,y}((b-b_{B^*})f_2)(x_0)-\phi_{t,y}((b(z)-b_{B^*})f_2)(w)\|_{\mathcal{H}_2} \\
&\leqslant \Big(\int_0^\infty\int_{|y|<1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\Big) \\
&\qquad \times(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t}\Big)^{\frac{1}{2}}
\end{aligned}$$

$$+\bigg(\int_0^\infty\int_{|y|\geqslant 1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\Big)$$
$$\times(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t}\bigg)^{\frac{1}{2}}:=J_1+J_2.$$

Since $\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\leqslant 1$, there is

$$J_1\leqslant\bigg(\int_0^\infty\int_{|y|<1}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\|\frac{\omega-z}{t}-y|\geqslant 1}}t^{-n}\phi\Big(\frac{x_0-z}{t}-y\Big)(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t}\bigg)^{\frac{1}{2}}$$
$$+\bigg(\int_0^\infty\int_{|y|<1}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|\geqslant 1\\|\frac{\omega-z}{t}-y|<1}}t^{-n}\phi\Big(\frac{\omega-z}{t}-y\Big)$$
$$\times(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t}\bigg)^{\frac{1}{2}}$$
$$+\bigg(\int_0^\infty\int_{|y|<1}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\|\frac{\omega-z}{t}-y|<1}}t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{\omega-z}{t}-y\Big)\Big)$$
$$\times(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t}\bigg)^{\frac{1}{2}}.$$

Use the transform $y\to\frac{x_0-y'}{t}$ (we still use y instead y'), then

$$J_1\leqslant\bigg(\int_0^\infty\int_{|x_0-y|<t}\Big|\int_{\substack{|y-z|<t\\|\omega-x_0+y-z|\geqslant t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac{1}{2}}$$
$$+\bigg(\int_0^\infty\int_{|x_0-y|<t}\Big|\int_{\substack{|y-z|\geqslant t\\|\omega-x_0+y-z|<t}}\frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}}$$
$$\times(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac{1}{2}}$$
$$+\bigg(\int_0^\infty\int_{|x_0-y|<t}\Big|\int_{\substack{|y-z|<t\\|\omega-x_0+y-z|<t}}\Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}}\Big)$$
$$\times(b(z)-b_{B^*})f_2(z)\,dz\Big|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac{1}{2}}$$
$$:=J_{1.1}+J_{1.2}+J_{1.3}.$$

Take $0<\varepsilon<\min\big\{\frac{1}{2},\rho-\frac{n}{2},\alpha-\frac{5}{2},\frac{(\lambda-2)}{2}n\big\}$ (we always restrict that ε satisfies this in the whole proof of this lemma). As for $J_{1.1}$, it follows from the Minkowski inequality that

$$\begin{aligned}J_{1.1} &\leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg[\Bigg(\iint_{\substack{y\in 2B^*\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\geqslant t}} \\ &\quad + \iint_{\substack{y\in (2B^*)^c\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\geqslant t}} \Bigg) \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Bigg]^{\frac12} dz \\ &\leqslant J_{1.1.1}+J_{1.1.2},\end{aligned}$$

where

$$J_{1.1.1} = \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg(\iint_{\substack{y\in 2B^*\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\geqslant t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Bigg)^{\frac12} dz,$$

$$J_{1.1.2} = \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg(\iint_{\substack{y\in (2B^*)^c\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\geqslant t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Bigg)^{\frac12} dz.$$

For $J_{1.1.1}$, since $y\in 2B^*$, $z\in(8B^*)^c$, then $|y-z|\sim|x_0-z|\sim|\omega-x_0+y-z|$. We have

$$\begin{aligned}J_{1.1.1} &\leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg(\int_{y\in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \\ &\qquad \times \int_{|y-z|<t\leqslant|\omega-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy\Bigg)^{\frac12} dz \\ &\leqslant C\int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg(\int_{y\in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{n+2\rho+1}} dy\Bigg)^{\frac12} dz \\ &\leqslant C\int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}} \Bigg(\int_{|y-z|>6r} \frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy\Bigg)^{\frac12} dz \qquad (3.3) \\ &\leqslant Cr^{\varepsilon}\int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}} dz.\end{aligned}$$

For $J_{1.1.2}$, we have

$$\begin{aligned}J_{1.1.2} &\leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ &\qquad \times \Bigg(\iint_{\substack{y\in (2B^*)^c,\ |y-z|<t\\ |x_0-y|<t,\ 2|y-z|\geqslant|z-x_0|\\ |\omega-x_0+y-z|\geqslant t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Bigg)^{\frac12} dz \\ &\quad + \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\end{aligned}$$

$$\times \left(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |x_0-y|<t,\ 2|y-z|<|z-x_0|\\ |\omega-x_0+y-z|\geqslant t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz$$

$$:= J_{1.1.2'} + J_{1.1.2''}.$$

First we give the estimate of $J_{1.1.2'}$.

$$\begin{aligned} J_{1.1.2'} &\leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ &\quad \times \left(\int_{\substack{y\in(2B^*)^c\\ 2|y-z|\geqslant|z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-z|}^{|\omega-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\ &\leqslant C \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ &\quad \times \left(\int_{\substack{y\in(2B^*)^c\\ 2|y-z|\geqslant|z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \\ &\leqslant C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}} \left(\int_{|y-z|>3r} \frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \right)^{\frac{1}{2}} dz. \end{aligned}$$

Similarly to the estimate of (3.3), we have

$$J_{1.1.2'} \leqslant Cr^{\varepsilon} \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}} dz.$$

Then we give the estimate of $J_{1.1.2''}$.

$$\begin{aligned} J_{1.1.2''} &\leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ &\quad \times \left(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |x_0-y|<t,\ 2|y-z|<|z-x_0|\\ |\omega-x_0+y-z|\geqslant t,\ |y-z|<2r}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &\quad + \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ &\quad \times \left(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |x_0-y|<t,\ 2|y-z|<|z-x_0|\\ |\omega-x_0+y-z|\geqslant t,\ |y-z|\geqslant 2r}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}} \right)^{\frac{1}{2}} dz \\ &:= J_{1.1.2.1''} + J_{1.1.2.2''}. \end{aligned}$$

For $J_{1.1.2.1''}$, since $|y-z| < \frac{|z-x_0|}{2}$, then $|y-x_0| > \frac{|z-x_0|}{2}$ and we get

$$\begin{aligned} J_{1.1.2.1''} &\leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \\ &\quad \times \left(\int_{\substack{|y-z|<2r\\ |y-x_0|>\frac{|z-x_0|}{2}}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-x_0|}^{\infty} \frac{dt}{t^{n+2\rho+1}} dy \right)^{\frac{1}{2}} dz \end{aligned}$$

$$\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}\bigg(\int_{|y-z|<2r}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\bigg)^{\frac12}dz$$
$$\leqslant Cr^{\rho-\frac{n}{2}}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}dz.$$

For $J_{1.1.2.2''}$, since $t>|y-x_0|>\frac{|z-x_0|}{2}$, and $|y-z|\sim|\omega-x_0+y-z|$, we have

$$J_{1.1.2.2''}\leqslant\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{(\frac{|z-x_0|}{2})^{n+\varepsilon}}\bigg(\int_{\substack{y\in(2B^*)^c\\ 2|y-z|<|z-x_0|\\ |y-z|\geqslant 2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}$$
$$\times\int_{|y-z|<t\leqslant|\omega-x_0+y-z|}\frac{dt}{t^{2\rho-n+1-2\varepsilon}}dy\bigg)^{\frac12}dz$$
$$\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}\bigg(\int_{|y-z|\geqslant 2r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\bigg)^{\frac12}dz.$$

Similarly to the estimate of (3.3), we have

$$J_{1.1.2.2''}\leqslant Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz.$$

Combining the estimates of $J_{1.1.1}$, $J_{1.1.2'}$, $J_{1.1.2.1''}$ and $J_{1.1.2.2''}$, we obtain

$$J_{1.1}\leqslant Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz+Cr^{\rho-\frac{n}{2}}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}dz.$$

Similarly as we deal with $J_{1.1}$, we can get

$$J_{1.2}\leqslant Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz+Cr^{\rho-\frac{n}{2}}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}dz.$$

Next we give the estimate of $J_{1.3}$. Apply the Minkowski inequality to $J_{1.3}$ and divide the region by $|y-z|\geqslant 6r$ and $|y-z|<6r$. When $|y-z|<6r$, we have $y\in(2B^*)^c$, so

$$J_{1.3}\leqslant\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c\\ |y-z|<t\\ |x_0-y|<t\\ |y-z|<6r\\ |\omega-x_0+y-z|<t}}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$-\frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}}\bigg|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}dz$$
$$+\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{|y-z|\geqslant 6r\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|<t}}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$-\frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}}\bigg|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}dz:=J_{1.3.1}+J_{1.3.2}.$$

When $z \in (8B^*)^c$ and $|y-z| < 6r$, there are $|y-x_0| \sim |z-x_0|$ and $|\omega - x_0 + y - z| \leqslant |\omega - x_0| + |y-z| < 8r$. Then

$$
\begin{aligned}
J_{1.3.1} \leqslant C \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg(\int_{\substack{y\in(2B^*)^c,\ |y-z|<6r \\ |x_0-y|<t,\ |\omega-x_0+y-z|<8r}} \Bigg(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \\
+ \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} \Bigg) \int_{|y-x_0|}^{\infty} \frac{dt}{t^{n+2\rho+1}} dy \Bigg)^{\frac{1}{2}} dz \\
\leqslant C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \Bigg(\int_{|y-z|<6r} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \Bigg)^{\frac{1}{2}} dz \\
+ C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \\
\times \Bigg(\int_{|\omega-x_0+y-z|<8r} \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} dy \Bigg)^{\frac{1}{2}} dz \\
\leqslant C r^{\rho-\frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\rho+\frac{n}{2}}} dz.
\end{aligned}
$$

Next we estimate $J_{1.3.2}$. Note that $|z-x_0| \leqslant |x_0-y|+|y-z| < 2t$, so $t > \frac{|z-x_0|}{2}$.

$$
\begin{aligned}
J_{1.3.2} \leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)| \Bigg(\iint_{\substack{t>|z-x_0|/2,\ |y-z|<t \\ |y-x_0|<t,\ |y-z|\geqslant 6r \\ |\omega-x_0+y-z|<t}} \Bigg| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\
- \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \Bigg|^2 \frac{(\log \frac{t}{r})^{4+2\varepsilon} dy\, dt}{t^{2\rho-n+1} t^{2n} (\log \frac{t}{r})^{4+2\varepsilon}} \Bigg)^{\frac{1}{2}} dz \\
\leqslant C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \Bigg(\int_{|y-z|\geqslant 6r} \Bigg| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\
- \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \Bigg|^2 \int_{|y-z|}^{\infty} \frac{(\log \frac{t}{r})^{4+2\varepsilon}}{t^{2\rho-n+1}} dt dy \Bigg)^{\frac{1}{2}} dz. \qquad (3.4)
\end{aligned}
$$

By Lemma 3.7 and Lemma 3.8, there is

$$
\begin{aligned}
J_{1.3.2} \leqslant C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{2r})^{2+\varepsilon}} \\
\times \Bigg(\int_{|y-z|\geqslant 6r} \frac{(1+|\Omega(y-z)|)^2}{|y-z|^n (\log \frac{|y-z|}{r})^{2\alpha-4-2\varepsilon}} dy \Bigg)^{\frac{1}{2}} dz \qquad (3.5) \\
\leqslant C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz.
\end{aligned}
$$

Combining the estimates of $J_{1.3.1}$ and $J_{1.3.2}$, we obtain

$$
J_{1.3} \leqslant C r^{\rho-\frac{n}{2}} \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} dz + C \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz.
$$

As for J_2,

$$J_2 \leqslant \bigg(\int_0^\infty\int_{|y|\geqslant 1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\bigg|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\|\frac{w-z}{t}-y|\geqslant 1}} t^{-n}$$

$$\times\,\phi\Big(\frac{x_0-z}{t}-y\Big)(b(z)-b_{B^*})f_2(z)\,dz\bigg|^2\frac{dy\,dt}{t}\bigg)^{\frac12}$$

$$+\bigg(\int_0^\infty\int_{|y|\geqslant 1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\bigg|\int_{\substack{|\frac{x_0-z}{t}-y|\geqslant 1\\|\frac{w-z}{t}-y|<1}} t^{-n}$$

$$\times\,\phi\Big(\frac{w-z}{t}-y\Big)(b(z)-b_{B^*})f_2(z)\,dz\bigg|^2\frac{dy\,dt}{t}\bigg)^{\frac12}$$

$$+\bigg(\int_0^\infty\int_{|y|\geqslant 1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\bigg|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\|\frac{w-z}{t}-y|<1}} t^{-n}$$

$$\times\bigg(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\bigg)(b(z)-b_{B^*})f_2(z)\,dz\bigg|^2\frac{dy\,dt}{t}\bigg)^{\frac12}.$$

Using the trasform $y\to\frac{x_0-y'}{t}$ again (we still use y instead y'), we have

$$J_2 \leqslant \bigg(\int_0^\infty\int_{|x_0-y|\geqslant t}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\bigg|\int_{\substack{|y-z|<t\\|w-x_0+y-z|\geqslant t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$\times\,(b(z)-b_{B^*})f_2(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}$$

$$+\bigg(\int_0^\infty\int_{|x_0-y|\geqslant t}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\bigg|\int_{\substack{|y-z|\geqslant t\\|w-x_0+y-z|<t}}\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}$$

$$\times\,(b(z)-b_{B^*})f_2(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}$$

$$+\bigg(\int_0^\infty\int_{|x_0-y|\geqslant t}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\bigg|\int_{\substack{|y-z|<t\\|w-x_0+y-z|<t}}\bigg(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg)(b(z)-b_{B^*})f_2(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}$$

$$:= J_{2.1}+J_{2.2}+J_{2.3}.$$

For $J_{2.1}$, we claim that $y\in(2B^*)^c$. Otherwise if $y\in 2B^*$, then $t\leqslant|x_0-y|<4r$. But $z\in(8B^*)^c$ and $t>|y-z|>6r$. Thus by the Minkowski inequality, we get

$$J_{2.1}\leqslant\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\int_0^\infty\int_{\substack{|x_0-y|\geqslant t\\ y\in(2B^*)^c\\|y-z|<t\\|w-x_0+y-z|\geqslant t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}$$

$$\times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz \leqslant J_{2.1.1} + J_{2.1.2},$$

where

$$J_{2.1.1} = \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\Big(\int_0^\infty \int_{\substack{|y-z|<8r,\ |x_0-y|\geqslant t\\ y\in(2B^*)^c,\ |y-z|<t\\ |w-x_0+y-z|\geqslant t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}$$
$$\times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz,$$

$$J_{2.1.2} = \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\Big(\int_0^\infty \int_{\substack{|y-z|\geqslant 8r,\ |x_0-y|\geqslant t\\ y\in(2B^*)^c,\ |y-z|<t\\ |w-x_0+y-z|\geqslant t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}$$
$$\times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz.$$

For $J_{2.1.1}$, since $|y-z|<8r$, $z\in(8B^*)^c$ and $y\in(2B^*)^c$, then $|y-x_0|\sim|z-x_0|$. So

$$J_{2.1.1} \leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\Big(\int_0^\infty \int_{\substack{|y-z|<8r,\ |x_0-y|\geqslant t\\ y\in(2B^*)^c,\ |y-z|<t}} \Big(\frac{1}{t+|x_0-y|}\Big)^{2n+2\varepsilon}$$
$$\times \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n-2n-2\varepsilon} t^{2n+2\varepsilon} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz$$
$$\leqslant C\int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\Big(\int_0^\infty \int_{\substack{|y-z|<8r\\ |x_0-y|\geqslant t\\ y\in(2B^*)^c\\ |y-z|<t}} \frac{1}{|z-x_0|^{2n+2\varepsilon}}$$
$$\times \frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}} \frac{dy\,dt}{t^{1-\varepsilon}}\Big)^{\frac{1}{2}} dz$$
$$\leqslant C\int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}}\Big(\int_{|y-z|<8r} \frac{|\Omega(y-z)|^2}{|z-x_0|^{\varepsilon}|y-z|^{n-\varepsilon}}$$
$$\times \Big(\int_0^{|x_0-y|} \frac{1}{t^{1-\varepsilon}} dt\Big) dy\Big)^{\frac{1}{2}} dz$$
$$\leqslant Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz.$$

For $J_{2.1.2}$, there is

$$J_{2.1.2} \leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\Big(\int_0^\infty \int_{\substack{2|y-z|\geqslant|z-x_0|\\ |x_0-y|\geqslant t,\ y\in(2B^*)^c\\ |y-z|<t,\ |y-z|\geqslant 8r\\ |w-x_0+y-z|\geqslant t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}$$
$$\times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz$$

$$+\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\int_0^\infty\int_{\substack{2|y-z|<|z-x_0|\\|x_0-y|\geqslant t,\ y\in(2B^*)^c\\|y-z|<t,\ |y-z|\geqslant 8r\\|w-x_0+y-z|\geqslant t}}\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}$$

$$\times\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}dz:=J_{2.1.2'}+J_{2.1.2''}.$$

As for $J_{2.1.2'}$, we have

$$J_{2.1.2'}\leqslant\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\int_{\substack{y\in(2B^*)^c\\|y-z|\geqslant 8r\\2|y-z|\geqslant|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}$$

$$\times\int_{|y-z|}^{|w-x_0+y-z|}\frac{dt}{t^{n+2\rho+1}}dy\bigg)^{\frac12}dz$$

$$\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}\bigg(\int_{|y-z|\geqslant 8r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\bigg)^{\frac12}dz$$

$$\leqslant Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz.$$

For $J_{2.1.2''}$, by $|z-x_0|>2|y-z|$, we have $|y-x_0|>|z-x_0|/2$ and $|y-z|\sim|w-x_0+y-z|$. Then

$$J_{2.1.2''}\leqslant\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c,\ |y-z|\geqslant 8r\\|y-x_0|\geqslant|z-x_0|/2\\|y-z|<t,\ |w-x_0+y-z|\geqslant t}}\frac{t^{2n+2\varepsilon}}{|x_0-y|^{2n+2\varepsilon}}$$

$$\times\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n-2n-2\varepsilon}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12}dz$$

$$\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}\bigg(\int_{\substack{y\in(2B^*)^c,\ |y-z|\geqslant 8r\\|y-x_0|\geqslant|z-x_0|/2}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}$$

$$\times\bigg(\int_{|y-z|}^{|w-x_0+y-z|}\frac{1}{t^{2\rho-n-2\varepsilon+1}}dt\bigg)dy\bigg)^{\frac12}dz$$

$$\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}\bigg(\int_{|y-z|\geqslant 8r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\bigg)^{\frac12}dz.$$

Similarly to the estimate of (3.3), we have $J_{2.1.2''}\leqslant Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz$.

Combining the estimates of $J_{2.1.1}$, $J_{2.1.2'}$ and $J_{2.1.2''}$, we obtain

$$J_{2.1}\leqslant Cr^{\varepsilon/2}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}}dz+Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz.$$

Similarly as $J_{2.1}$, we can obtain

$$J_{2.2}\leqslant Cr^{\varepsilon/2}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}}dz+Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon}}dz.$$

Finally, we deal with the last part $J_{2.3}$. By the Minskowski inequality,

$$J_{2.3} \leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{|y-z|<6r,\ |x_0-y|\geqslant t\\ |y-z|<t,\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \times \bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2 \frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12} dz$$
$$+ \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{|y-z|\geqslant 6r,\ |x_0-y|\geqslant t\\ |y-z|<t,\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \times \bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2 \frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12} dz := J_{2.3.1} + J_{2.3.2}.$$

For $J_{2.3.1}$, it follows from $|y-z|<6r$ and $z\in(8B^*)^c$ that $|y-\bar{x}|\geqslant|z-\bar{x}|-|y-z|>2r$. We can get $y\in(2B^*)^c$ and $|w-x_0+y-z|\leqslant|w-x_0|+|y-z|<8r$.

$$J_{2.3.1} \leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c,\ |x_0-y|\geqslant t\\ |y-z|<6r,\ |y-z|<t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12} dz$$
$$+ \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{y\in(2B^*)^c,\ |x_0-y|\geqslant t\\ |w-x_0+y-z|<8r\\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \times \frac{|\Omega(w-x_0+y-z)|^2}{|w-x_0+y-z|^{2n-2\rho}} \frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12} dz := J_{2.3.1'} + J_{2.3.1''}.$$

Estimate $J_{2.3.1'}$ and $J_{2.3.1''}$ by the similar method as we deal with $J_{2.1.1}$. There is

$$J_{2.3.1} \leqslant Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}} dz.$$

As for $J_{2.3.2}$, we have

$$J_{2.3.2} \leqslant \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{2|y-z|\geqslant|z-x_0|\\ |y-z|\geqslant 6r,\ |x_0-y|\geqslant t\\ |y-z|<t,\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \times \bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2 \frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12} dz$$
$$+ \int_{(8B^*)^c} |b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{2|y-z|<|z-x_0|\\ |y-z|\geqslant 6r,\ |x_0-y|\geqslant t\\ |y-z|<t,\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \times \bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2 \frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac12} dz$$
$$:= J_{2.3.2'} + J_{2.3.2''}.$$

For $J_{2.3.2'}$, there is

$$\begin{aligned}J_{2.3.2'} &\leqslant C\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\\ &\quad\times\bigg(\int_{|y-z|\geqslant 6r}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2\\ &\quad\times\bigg(\int_{max\{|y-z|,|z-x_0|/2\}}^{\infty}\frac{(\log\frac{t}{r})^{4+2\varepsilon}dt}{t^{2\rho-n+1}|z-x_0|^{2n}(\log\frac{|z-x_0|}{2r})^{4+2\varepsilon}}\bigg)dy\bigg)^{\frac{1}{2}}dz\\ &\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{2r})^{2+\varepsilon}}\\ &\quad\times\bigg(\int_{|y-z|\geqslant 6r}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2\\ &\quad\times\bigg(\int_{|y-z|}^{\infty}\frac{(\log\frac{t}{r})^{4+2\varepsilon}dt}{t^{2\rho-n+1}}\bigg)dy\bigg)^{\frac{1}{2}}dz.\end{aligned}$$

By the estimate of (3.4), we get $J_{2.3.2'}\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{2+\varepsilon}}dz$.

For $J_{2.3.2''}$, denote $C(\varepsilon)=e^{(4+2\varepsilon)/\varepsilon}$. Since $2|y-z|<|z-x_0|$, then $|x_0-y|\geqslant|z-x_0|-|y-z|>\frac{|z-x_0|}{2}$, thus

$$\begin{aligned}J_{2.3.2''} &\leqslant\int_{(8B^*)^c}|b(z)-b_{B^*}||f(z)|\bigg(\iint_{\substack{|y-z|\geqslant 6r,\ |x_0-y|\geqslant t\\ |x_0-y|>|z-x_0|/2\\ |y-z|<t}}\\ &\qquad\times\frac{t^{\lambda n}}{(t+|x_0-y|)^{\lambda n-2n+2n}}\frac{(\log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}{(\log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}\\ &\qquad\times\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2\frac{dy\,dt}{t^{n+2\rho+1}}\bigg)^{\frac{1}{2}}dz\\ &\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{2r})^{2+\varepsilon}}\\ &\qquad\times\bigg(\int_{|y-z|\geqslant 6r}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2\\ &\qquad\times\bigg(\int_{|y-z|}^{|x_0-y|}\frac{t^{\lambda n}(\log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}\bigg)dy\bigg)^{\frac{1}{2}}dz.\end{aligned}$$

Notice that the function $G(s)=\frac{(\log s)^{4+2\varepsilon}}{s^{\varepsilon}}$ is decreasing when $s\geqslant e^{(4+2\varepsilon)/\varepsilon}$ and

$$\frac{t+|y-x_0|+C(\varepsilon)r}{r}\geqslant\frac{|y-z|+C(\varepsilon)r}{r}\geqslant C(\varepsilon)=e^{(4+2\varepsilon)/\varepsilon},$$

then

$$\frac{[\log(\frac{t+|y-x_0|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{(\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{\varepsilon}}\leqslant\frac{[\log(\frac{|y-z|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{(\frac{|y-z|+C(\varepsilon)r}{r})^{\varepsilon}}.$$

Since $t+|y-x_0|\sim t+|y-x_0|+C(\varepsilon)r$ and $0<\varepsilon<\min\{\frac{1}{2},\frac{(\lambda-2)n}{2},\rho-\frac{n}{2},\alpha-\frac{5}{2}\}$, then

$$\begin{aligned}&\int_{|y-z|}^{|x_0-y|}\frac{t^{\lambda n}(\log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{4+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}\\ &\qquad\leqslant C\int_{|y-z|}^{\infty}\frac{[\log(\frac{|y-z|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{(|y-z|+C(\varepsilon)r)^{\varepsilon}}\frac{dt}{t^{2\rho-n+1-\varepsilon}}\\ &\qquad\leqslant C\frac{[\log(\frac{|y-z|+C(\varepsilon)r}{r})]^{4+2\varepsilon}}{|y-z|^{2\rho-n}}.\end{aligned}$$

Since $|y-z|\geqslant 6r$, there exists a constant $l\geqslant 1$ such that $|y-z|+C(\varepsilon)r\leqslant 2^l|y-z|$. Hence

$$\begin{aligned}J_{2.3.2''}\leqslant{}& C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{2r})^{2+\varepsilon}}\bigg(\int_{|y-z|\geqslant 6r}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\ &-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\bigg|^2\frac{(\log\frac{2^l|y-z|}{r})^{4+2\varepsilon}}{|y-z|^{2\rho-n}}dy\bigg)^{\frac{1}{2}}dz\\ \leqslant{}& C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{2r})^{2+\varepsilon}}\\ &\times\bigg(\int_{|y-z|\geqslant 6r}\frac{(1+|\Omega(y-z)|)^2}{|y-z|^n(\log\frac{|y-z|}{r})^{2\alpha-4-2\varepsilon}}dy\bigg)^{\frac{1}{2}}dz.\end{aligned}$$

By the estimate of (3.5), we get $J_{2.3.2''}\leqslant C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{2+\varepsilon}}dz$.

Combining the estimates of $J_{2.3.1}$, $J_{2.3.2'}$ and $J_{2.3.2''}$, we obtain

$$J_{2.3}\leqslant Cr^{\varepsilon/2}\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^{n+\varepsilon/2}}dz+C\int_{(8B^*)^c}\frac{|b(z)-b_{B^*}||f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{2+\varepsilon}}dz.$$

Then we complete the proof of (3.2). Next we will show that $T_i\leqslant C\|b\|_*M^2(f)(x)$, for $i=1,2,3,4$.

Denote $B_j=\{z:|z-\bar{x}|<2^jr\}$, then $|b_{B_{j+1}}-b_{B^*}|\leqslant j\|b\|_*$. Since $M^2(f)\approx M_{L\log L}(f)$, by Lemma 3.3 we get

$$\begin{aligned}T_1&\leqslant Cr^{\varepsilon}\sum_{j=3}^{\infty}\int_{2^jr\leqslant|z-\bar{x}|<2^{j+1}r}\frac{|b(z)-b_{B^*}||f(z)|}{|z-\bar{x}|^{n+\varepsilon}}dz\\ &\leqslant C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\bigg(\int_{B_{j+1}}\frac{|b(z)-b_{B_{j+1}}||f(z)|}{(2^jr)^n}dz+|b_{B_{j+1}}-b_{B^*}|M(f)(x)\bigg)\\ &\leqslant C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\big(\|b-b_{B_{j+1}}\|_{expL,B_{j+1}}\|f\|_{L\log L,B_{j+1}}+j\|b\|_*M(f)(x)\big)\\ &\leqslant C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\big(\|b\|_*M_{L\log L}(f)(x)+j\|b\|_*M(f)(x)\big)\leqslant C\|b\|_*M^2(f)(x).\end{aligned}$$

Taking $\varepsilon/2$ and $\rho - \frac{n}{2}$ instead of ε in the above inequality respectively, we get $T_2 \leqslant C\|b\|_* M^2(f)(x)$ and $T_3 \leqslant C\|b\|_* M^2(f)(x)$.

Similarly to the way in estimating T_1, we obtain

$$\begin{aligned} T_4 &\leqslant C \sum_{j=3}^{\infty} \int_{2^j r \leqslant |z-\bar{x}| < 2^{j+1} r} \frac{|b(z) - b_{B^*}||f(z)|}{|z - x_0|^n (\log \frac{|z-\bar{x}|}{r})^{2+\varepsilon}} dz \\ &\leqslant C \sum_{j=3}^{\infty} \frac{1}{j^{2+\varepsilon}} \big(\|b\|_* M_{L \log L}(f)(x) + j\|b\|_* M(f)(x)\big) \\ &\leqslant C\|b\|_* M^2(f)(x). \end{aligned}$$

So we have $J \leqslant C\|b\|_* M^2(f)(x)$ and $I_3 \leqslant C\|b\|_* M^2(f)(x)$. Combining the estimates of I_1, I_2 and I_3, we have

$$\begin{aligned} M_\delta^{\sharp,\Delta}\big(\mu_{\lambda,b}^{*,\rho}(f)\big)(x) &= \big[M^{\sharp,\Delta}\big(|\mu_{\lambda,b}^{*,\rho}(f)|^\delta\big)(x)\big]^{\frac{1}{\delta}} \\ &\leqslant \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q \Big||\mu_{\lambda,b}^{*,\rho}(f)|^\delta - |C_Q|^\delta\Big| du\right)^{\frac{1}{\delta}} \\ &\leqslant C\|b\|_* \big(M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) + M^2(f)(x)\big), \end{aligned}$$

where the supremum is taken over all dyadic cubes Q with $x \in Q$. □

Lemma 3.10. *Suppose that* $0 < \delta < 1$, $\rho > n/2$, $\lambda > 2$, *and* $\Omega \in L^2(S^{n-1})$ *is a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with* $\alpha > \frac{5}{2}$. *Then for any smooth function with compact support* f, *there exists a positive constant* $0 < C = C_\delta$ *such that*

$$M_\delta^{\sharp,\Delta}\big(\mu_\lambda^{*,\rho}(f)\big)(x) \leqslant CM(f)(x).$$

Proof. Let f_1, f_2, Q and B^* be the same as in the proof of Lemma 3.9. Then applying Lemma 3.4 with $\frac{1}{s} = \frac{1}{\delta} - 1$ and Theorem 1.1, similarly to get (3.1), we have

$$\left(\frac{1}{|Q|} \int_Q |\mu_\lambda^{*,\rho}(f_1)(y)|^\delta dy\right)^{\frac{1}{\delta}} \leqslant \frac{C}{|8B^*|} \int_{8B^*} |f(y)| dy \leqslant CM(f)(x).$$

Since $f \in L^p$ for $1 < p < \infty$, and $\mu_\lambda^{*,\rho}$ is L^p bounded, then $\mu_\lambda^{*,\rho}(f_2)(u) < \infty$ a.e. on Q, so except a subset E with measure zero, for all $u \in Q \setminus E$, $\mu_\lambda^{*,\rho}(f_2)(u) < \infty$. Next we will prove the following fact. For any $x_0, w \in Q \setminus E$,

$$J = \big|\mu_\lambda^{*,\rho}(f_2)(x_0) - \mu_\lambda^{*,\rho}(f_2)(w)\big| \leqslant CM(f)(x).$$

In fact, similarly to (3.2), we know that

$$\begin{aligned} J &\leqslant Cr^\varepsilon \int_{(8B^*)^c} \frac{|f(z)|}{|z - x_0|^{n+\varepsilon}} dz + Cr^{\varepsilon/2} \int_{(8B^*)^c} \frac{|f(z)|}{|z - x_0|^{n+\varepsilon/2}} dz \\ &\quad + Cr^{\rho - \frac{n}{2}} \int_{(8B^*)^c} \frac{|f(z)|}{|z - x_0|^{\frac{n}{2}+\rho}} dz + C \int_{(8B^*)^c} \frac{|f(z)|}{|z - x_0|^n (\log \frac{|z-x_0|}{r})^{2+\varepsilon}} dz \end{aligned}$$

$$
\begin{aligned}
&\leqslant Cr^{\varepsilon}\sum_{j=3}^{\infty}\int_{2^jr\leqslant|z-\bar{x}|<2^{j+1}r}\frac{|f(z)|}{|z-\bar{x}|^{n+\varepsilon}}dz\\
&\quad+Cr^{\varepsilon/2}\sum_{j=3}^{\infty}\int_{2^jr\leqslant|z-\bar{x}|<2^{j+1}r}\frac{|f(z)|}{|z-\bar{x}|^{n+\varepsilon/2}}dz\\
&\quad+Cr^{\rho-\frac{n}{2}}\sum_{j=3}^{\infty}\int_{2^jr\leqslant|z-\bar{x}|<2^{j+1}r}\frac{|f(z)|}{|z-\bar{x}|^{\frac{n}{2}+\rho}}dz\\
&\quad+C\sum_{j=3}^{\infty}\int_{2^jr\leqslant|z-\bar{x}|<2^{j+1}r}\frac{|f(z)|}{|z-\bar{x}|^{n}(\log\frac{|z-\bar{x}|}{r})^{2+\varepsilon}}dz\\
&\leqslant C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon}}\frac{1}{(2^{j+1}r)^n}\int_{|z-\bar{x}|<2^{j+1}r}|f(z)|dz\\
&\quad+C\sum_{j=3}^{\infty}\frac{1}{2^{j\varepsilon/2}}\frac{1}{(2^{j+1}r)^n}\int_{|z-\bar{x}|<2^{j+1}r}|f(z)|dz\\
&\quad+C\sum_{j=3}^{\infty}\frac{1}{2^{j(\rho-\frac{n}{2})}}\frac{1}{(2^{j+1}r)^n}\int_{|z-\bar{x}|<2^{j+1}r}|f(z)|dz\\
&\quad+C\sum_{j=3}^{\infty}\frac{1}{j^{2+\varepsilon}}\frac{1}{(2^{j+1}r)^n}\int_{|z-\bar{x}|<2^{j+1}r}|f(z)|dz\leqslant CM(f)(x).
\end{aligned}
$$

Let $C_Q=\big(\mu_{\lambda}^{*,\rho}(f_2)\big)_Q$. Since $\big|\mu_{\lambda}^{*,\rho}(f)(u)-C_Q\big|\leqslant|\mu_{\lambda}^{*,\rho}(f_1)(u)|+|\mu_{\lambda}^{*,\rho}(f_2)(u)-C_Q|$, then

$$
\begin{aligned}
&M_{\delta}^{\sharp,\Delta}\Big(\mu_{\lambda}^{*,\rho}(f)\Big)(x)\\
&\quad\leqslant C_{\delta}\sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q|\mu_{\lambda}^{*,\rho}(f_1)(u)|^{\delta}du\right)^{\frac{1}{\delta}}\\
&\qquad+C_{\delta}\sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q\big|\mu_{\lambda}^{*,\rho}(f_2)-C_Q\big|^{\delta}du\right)^{\frac{1}{\delta}}\\
&\quad\leqslant CM(f)(x)+C\sup_{Q\ni x}\frac{1}{|Q|^2}\int_{Q\setminus E}\int_{Q\setminus E}|\mu_{\lambda}^{*,\rho}(f_2)(u)-\mu_{\lambda}^{*,\rho}(f_2)(v)|dvdu\\
&\quad\leqslant CM(f)(x).
\end{aligned}
$$

□

For $b\in BMO$, let $b_k(x)=b(x)$ if $|b(x)|\leqslant k$, $b_k(x)=k$ if $b(x)>k$ and $b_k(x)=-k$ if $b(x)<-k$ for $k=1,2,3,\cdots$. Then $b_k\in L^{\infty}$ and $\|b_k\|_*\leqslant\|b\|_*$. The following lemma shows that $\mu_{\lambda,b_k}^{*,\rho}(f)$ can be controlled by the maximal operator.

LEMMA 3.11. [**4**] *Suppose that $\Omega\in L^2(S^{n-1})$ is a homogeneous function of degree zero. If $\rho>n/2$, $\lambda>2$, $\mathrm{supp}f\subset B(0,R)$ and $|x|\geqslant 2R$, then for any k, there exists a constant C independent of k, f, R and x such that $\mu_{\lambda,b_k}^{*,\rho}(f)(x)\leqslant CkMf(x)$.*

REMARK 3.2. By checking the proof of Lemma 3.7 in [**4**], we find out that the condition "$\Omega \in L^2(S^{n-1})$ and Ω is a homogeneous function of degree zero" is sufficient to get the desired result.

4. Proof of theorems

First, we give the proof of Theorem 2.1 as follows.

PROOF. It is easy to check that $\mu^{\rho,b}_{\Omega,S} \leqslant 2^{\lambda n/2}\mu^{*,\rho}_{\lambda,b}$, so we only give the proof of Theorem 2.1 for $\mu^{*,\rho}_{\lambda,b}$.

Since $\omega \in A_p$, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in A_p$, then by Theorem 1.2, we have

$$\|\mu^{*,\rho}_{\lambda}(\phi)\|_{p,\omega^{1+\varepsilon}} \leqslant \bar{C}_1\|\phi\|_{p,\omega^{1+\varepsilon}}, \text{ for } \phi \in L^p(\omega^{1+\varepsilon}). \tag{4.1}$$

Take $\lambda' = p(1+\varepsilon)/\varepsilon$ and $\eta = \min\{C_2/\lambda', C_2(p-1)/\lambda'\}$, where C_2 is the constant in Lemma 3.1. Without loss of generality we may assume that $\|b\|_* < \eta$. Otherwise we take $0 < \delta < \eta$ and set $b_0(x) = \delta b(x)/\|b\|_*$, then $\|b_0\|_* = \delta < \eta$ and $\mu^{*,\rho}_{\lambda,b}(f)(x) = (\|b\|_*/\delta)\mu^{*,\rho}_{\lambda,b_0}(f)(x)$. Therefore, it suffices to consider $\mu^{*,\rho}_{\lambda,b_0}(f)(x)$. By Lemma 3.2, we have $e^{p(1+\varepsilon)b(x)/\varepsilon} \in A_p$ for $b(x) \in BMO$ with $\|b\|_* < \eta$. Since $b(x) \in BMO$ implies that $tb(x) \in BMO$ with $\|tb\|_* \leqslant \|b\|_*$ for $|t| \leqslant 1$, by Remark 3.1 we have

$$e^{p(1+\varepsilon)tb(x)/\varepsilon} \in A_p, \text{ for } b(x) \in BMO \text{ with } |b|_* < \eta \text{ and } |t| \leqslant 1, \tag{4.2}$$

where $[e^{p(1+\varepsilon)tb(x)/\varepsilon}]_{A_p}$ can be dominated by a constant independent of t.

By (4.2) and Theorem 1.2, we know that for any $\phi \in L^p(e^{p(1+\varepsilon)b(x)cos\theta/\varepsilon})$ and $\theta \in [0, 2\pi]$,

$$\|\mu^{*,\rho}_{\lambda}(\phi)\|_{p,e^{p(1+\varepsilon)b(x)cos\theta/\varepsilon}} \leqslant \bar{C}_2\|\phi\|_{p,e^{p(1+\varepsilon)b(x)cos\theta/\varepsilon}}, \tag{4.3}$$

where $\bar{C}_2$ depends on n, p, b, Ω, but not on θ and ϕ.

Applying the Stein-Weiss interpolation theorem with change of measure in [**13**] between (4.1) and (4.3), we have for any $\theta \in [0, 2\pi]$ and $\phi \in L^p(\omega e^{pb(x)cos\theta})$,

$$\|\mu^{*,\rho}_{\lambda}(\phi)\|_{p,\omega e^{pb(x)cos\theta}} \leqslant \bar{C}\|\phi\|_{p,\omega e^{pb(x)cos\theta}}, \tag{4.4}$$

where $\bar{C} = \max\{\bar{C}_1, \bar{C}_2\}$ depending only on n, p, b, ω, Ω, but not on θ and ϕ.

Denote $F(y) = e^{y[b(x)-b(z)]}$, $y \in \mathbb{C}$, then by the analyticity of $F(y)$ on $\mathbb{C}$ and the Cauchy integration formula, we have

$$b(x) - b(z) = F'(0) = \frac{1}{2\pi i}\int_{|y|=1}\frac{F(y)}{y^2}dy = \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}(b(x)-b(z))}e^{-i\theta}d\theta. \tag{4.5}$$

By (4.5) and the Minkowski inequality we get

$$\mu^{*,\rho}_{\lambda,b}(f)(x) = \bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\bigg|\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{t^{\rho}}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$\times\, e^{e^{i\theta}(b(x)-b(z))}e^{-i\theta}f(z)\,dz d\theta\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi}\bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\bigg|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$\times e^{-e^{i\theta}b(z)}f(z)\,dz\bigg|^2\frac{dy\,dt}{t^{n+1}}\bigg)^{\frac{1}{2}}\big|e^{e^{i\theta}b(x)}\big|\big|e^{-i\theta}\big|d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\mu_\lambda^{*,\rho}(f^\theta)(x)e^{b(x)cos\theta}d\theta,$$

where $f^\theta(z) = f(z)e^{-e^{i\theta}b(z)}$ for $\theta \in [0, 2\pi]$. Then by the Minkowski inequality and (4.4) we get

$$\begin{aligned}\|\mu_{\lambda,b}^{*,\rho}(f)\|_{p,\omega} &\leqslant \bigg(\int_{\mathbb{R}^n}\bigg|\frac{1}{2\pi}\int_0^{2\pi}\mu_\lambda^{*,\rho}(f^\theta)(x)e^{b(x)cos\theta}d\theta\bigg|^p\omega(x)\,dx\bigg)^{\frac{1}{p}}\\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_{\mathbb{R}^n}\big|\mu_\lambda^{*,\rho}(f^\theta)(x)\big|^p\omega(x)e^{pb(x)cos\theta}dx\Big)^{\frac{1}{p}}d\theta\\ &\leqslant C\int_0^{2\pi}\Big(\int_{\mathbb{R}^n}\big|f^\theta(x)\big|^p\omega(x)e^{pb(x)cos\theta}dx\Big)^{\frac{1}{p}}d\theta = C\|f\|_{p,\omega}.\end{aligned}$$

Thus we complete the proof of Theorem 2.1. □

Now, we turn to the proof of Theorem 2.2.

PROOF. Since $\mu_{\Omega,S}^{\rho,b} \leqslant 2^{\lambda n/2}\mu_{\lambda,b}^{*,\rho}$, we only give the proof of Theorem 2.2 for $\mu_{\lambda,b}^{*,\rho}$. Let $\Phi(t) = t(1+\log^+ t)$. We first prove the following inequality

$$\begin{aligned}&\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : \mu_{\lambda,b}^{*,\rho}(f)(x) > t\})\\ &\leqslant C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x)) > t\}).\end{aligned}\tag{4.6}$$

If $\|b\|_* = 0$, (4.6) holds obviously. As below we may assume $\|b\|_* > 0$. Denote

$$L_{\delta,b}(f) = \sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t\}),$$

then it is easy to see that

$$\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : \mu_{\lambda,b}^{*,\rho}(f)(x) > t\}) \leqslant L_{\delta,b}(f).\tag{4.7}$$

First we will prove that for any $0<\delta<1$ and $r>0$, there is

$$L_{\delta,b}(f) \leqslant C_\delta r L_{\delta,b}(f) + C\Phi\bigg(\frac{\|b\|_*}{r^{1/\delta}}\bigg)\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x)) > t\}).\tag{4.8}$$

By Lemma 3.5(1), we get for any $t>0$,

$$\begin{aligned}&\omega(\{x\in\mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t\})\\ &\leqslant \omega(\{x\in\mathbb{R}^n : M^\Delta([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(x) > t^\delta, M^{\sharp,\Delta}([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(x) \leqslant rt^\delta\})\\ &\quad + \omega(\{x\in\mathbb{R}^n : M^{\sharp,\Delta}([\mu_{\lambda,b}^{*,\rho}(f)]^\delta)(x) > rt^\delta\})\end{aligned}$$

$$\leqslant Cr\omega(\{x\in\mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t/2^{\frac{1}{\delta}}\}) + \omega(\{x\in\mathbb{R}^n : M_\delta^{\sharp,\Delta}(\mu_{\lambda,b}^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t\}).$$

Since $0<\delta<1$, we can choose an l satisfying $0<\delta<l<1$. By Lemma 3.9,

$$\begin{aligned}&\omega(\{x\in\mathbb{R}^n : M_\delta^{\sharp,\Delta}(\mu_{\lambda,b}^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t\})\\ &\leqslant \omega(\{x\in\mathbb{R}^n : M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\})\\ &\quad+\omega(\{x\in\mathbb{R}^n : M^2(f)(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\}).\end{aligned}$$

Note that $\Phi(ab)\leqslant\Phi(a)\Phi(b)$ for $a,b\geqslant 0$, and Φ is increasing and doubling, so we have

$$\begin{aligned}&\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t\})\\ &\leqslant \frac{Cr}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b}^{*,\rho}(f))(x) > t/2^{\frac{1}{\delta}}\})\\ &\quad+\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\})\\ &\quad+\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x) > r^{\frac{1}{\delta}}t/(2C\|b\|_*)\})\\ &\leqslant C_\delta rL_{\delta,b}(f)+\Phi\left(\frac{2C\|b\|_*}{r^{1/\delta}}\right)\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_l^\Delta(\mu_\lambda^{*,\rho}(f))(x) > t\})\\ &\quad+\Phi\left(\frac{2C\|b\|_*}{r^{1/\delta}}\right)\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x) > t\}).\end{aligned}$$

Applying Lemma 3.5(2) and Lemma 3.10, we have

$$\begin{aligned}L_{\delta,b}(f)&\leqslant C_\delta rL_{\delta,b}(f)+C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right)\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x) > t\})\\ &\quad+C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right)\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_l^{\sharp,\Delta}(\mu_\lambda^{*,\rho}(f))(x) > t\})\\ &\leqslant C_\delta rL_{\delta,b}(f)+C\Phi\left(\frac{\|b\|_*}{r^{1/\delta}}\right)\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x) > t\}).\end{aligned}$$

Thus we obtain the result of (4.8). Next we will show that

$$L_{\delta,b}(f)\leqslant C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x) > t\}). \tag{4.9}$$

For $b\in BMO$, let b_k be the same as in Lemma 3.11. Then $b_k\in L^\infty$ and $\|b_k\|_*\leqslant\|b\|_*$. Since f is smooth with compact support, we may assume $\operatorname{supp} f\subset B(0,R)$. Then by Lemma 3.11, Theorem 2.1, Lemma 3.6 and $t\Phi(1/t)\geqslant 1$, we get

$$\begin{aligned}&\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M_\delta^\Delta(\mu_{\lambda,b_k}^{*,\rho}(f))(x) > t\})\\ &\leqslant \frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M(\chi_{B(0,2R)}\mu_{\lambda,b_k}^{*,\rho}(f))(x) > t/2\})\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M(\chi_{B^c(0,2R)}\mu^{*,\rho}_{\lambda,b_k}(f))(x)>t/2\})\\
&\leqslant \frac{C}{t\Phi(1/t)}\int_{B(0,2R)}|\mu^{*,\rho}_{\lambda,b_k}(f)(x)|\omega(x)\,dx\\
&\quad+\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x)>t/(Ck)\})\\
&\leqslant C\omega(B(0,2R))^{\frac12}\bigg(\int_{B(0,2R)}|\mu^{*,\rho}_{\lambda,b_k}(f)(x)|^2\omega(x)\,dx\bigg)^{\frac12}\\
&\quad+\frac{C}{\Phi(1/t)}\int_{\mathbb{R}^n}\Phi\bigg(\frac{Ck|f(x)|}{t}\bigg)\omega(x)\,dx\\
&\leqslant C\omega(B(0,2R))^{\frac12}\bigg(\int_{B(0,R)}|f(x)|^2\omega(x)\,dx\bigg)^{\frac12}+C_k\int_{B(0,R)}\Phi(|f(x)|)\omega(x)\,dx.
\end{aligned}
$$

So $L_{\delta,b_k}(f)<\infty$. Then choose an $r>0$ with $r<1/C_\delta$, applying (4.8) for b_k, we have

$$(1-C_\delta r)L_{\delta,b_k}(f)\leqslant C_{\delta,r,\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x)>t\}).$$

That is

$$L_{\delta,b_k}(f)\leqslant C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x)>t\}), \tag{4.10}$$

where C is independent of k. Thus we get (4.9) by letting $k\to\infty$ in (4.10). By (4.7) and (4.9), we prove that (4.6) holds.

For $\beta=1$, applying (4.6) and Lemma 3.6, we obtain

$$
\begin{aligned}
&\omega(\{x\in\mathbb{R}^n : \mu^{*,\rho}_{\lambda,b}(f)(x)>1\})\\
&\leqslant \sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : \mu^{*,\rho}_{\lambda,b}(f)(x)>t\})\\
&\leqslant C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\omega(\{x\in\mathbb{R}^n : M^2(f)(x)>t\})\\
&\leqslant C_{\|b\|_*}\sup_{t>0}\frac{1}{\Phi(1/t)}\int_{\mathbb{R}^n}\frac{|f(x)|}{t}\bigg(1+\log^+\frac{|f(x)|}{t}\bigg)\omega(x)\,dx\\
&\leqslant C_{\|b\|_*}\int_{\mathbb{R}^n}\Phi(|f(x)|)\omega(x)\,dx\\
&= C_{\|b\|_*}\int_{\mathbb{R}^n}|f(x)|\big(1+\log^+|f(x)|\big)\omega(x)\,dx.
\end{aligned}
$$

Then by homogeneity, we complete the proof of Theorem 2.2. □

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