PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 100(114) (2016), 17–48 DOI: 10.2298/PIM1614017M

A SEQUENTIAL APPROACH TO ULTRADISTRIBUTION SPACES

Snježana Maksimović, Svetlana Mincheva-Kamińska, Stevan Pilipović, and Petar Sokoloski

Abstract. We introduce and investigate two types of the space $\mathcal{U}^*$ of s-ultradistributions meant as equivalence classes of suitably defined fundamental sequences of smooth functions; we prove the existence of an isomorphism between $\mathcal{U}^*$ and the respective space $\mathcal{D}'^*$ of ultradistributions: of Beurling type if $* = (p!^t)$ and of Roumieu type if $* = \{p!^t\}$. We also study the spaces $\mathcal{T}^*$ and $\tilde{\mathcal{T}}^*$ of t-ultradistributions and $\tilde{t}$-ultradistributions, respectively, and show that these spaces are isomorphic with the space $\mathcal{S}'^*$ of tempered ultradistributions both in the Beurling and the Roumieu case.

1. Introduction

That distributions based by Sobolev [**29**] and Schwartz [**28**] on functional analysis can be founded on a more elementary sequential approach was remarked by Mikusiński already in [**18**] and [**19**]. This idea was accomplished by him in cooperation with Sikorski in [**20**, **21**] and then, in the extended form, together with the third author Antosik in [**1**].

Roumieu and Beurling in [**27**] and [**2**] introduced two types of ultradistribution spaces, substantially larger than the space of distributions. However only the famous papers [**12**–**14**] of Komatsu which substantially extended the knowledge on the structure of these spaces gave impulse to an intensive development of the theory of ultradistributions of both types in various directions. In particular, the theory became an important tool of microlocal analysis.

Similarly as in the case of distributions one can expect that an ultradistribution can also be viewed as, in a sense, a limit of a sequence of functions or, more precisely, as an equivalent class of sequences of smooth functions, suitably approximating it. Our aim in this paper is to provide a sequential approach to the theory of non-quasi-analytic ultradistributions of both Beurling and Roumieu types [**12**]. Analogously to the sequential theory of distributions [**1**], we introduce sequential

2010 *Mathematics Subject Classification:* Primary 46F05; Secondary 46F10.
Key words and phrases: fundamental sequences, Hermite expansions.
Communicated by Gradimir Milovanović.

ultadistributions, called shortly s-ultadistributions, as equivalence classes of fundamental sequences of smooth functions which, however, are defined by means of ultradifferential operators instead of differential operators. The difference is intrinsic and requires distinct techniques: instead of polynomials we have to use functions of sub-exponential growth and apply their specific properties. In showing that the sequential approach is equivalent to the classical approach to ultradistribution theory (see [**12**–**14**]) one needs to know intrinsic structures of ultradistributions as well as of tempered ultradistributions (see [**5**–**10**, **17**–**25**]); the equivalence of the two approaches will be proved through Hermite expansions and certain structural properties of tempered ultradistributions.

In order to simplify our exposition we will consider only the Gevrey sequence of functions of the form $M_p = p!^t$ ($p \in \mathbb{Z}_+$) for $t > 1$; they satisfy all conditions usually assumed for a general sequence $(M_p)_p$. Therefore we use the simplified symbols $\mathcal{D}^{(t)}(\Omega)$, $\mathcal{D}^{\{t\}}(\Omega)$ for the spaces $\mathcal{D}^{(M_p)}(\Omega)$, $\mathcal{D}^{\{M_p\}}(\Omega)$ of test functions on an open set $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{S}^{(t)}(\mathbb{R}^d)$, $\mathcal{S}^{\{t\}}(\mathbb{R}^d)$ for the spaces $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ of test functions on $\mathbb{R}^d$, respectively. This concerns also their duals, i.e., $\mathcal{D}'^{(t)}(\Omega)$, $\mathcal{D}'^{\{t\}}(\Omega)$ are the spaces of ultradistributions of Beurling and Roumieu type on the set Ω and $\mathcal{S}'(\mathbb{R}^{(t)})d$, $\mathcal{S}'^{\{t\}}(\mathbb{R}^d)$ are the spaces of tempered ultradistributions of Beurling and Roumieu type on $\mathbb{R}^d$, respectively. We traditionally use the upper index * for a common notation of the considered spaces both in the Beurling and Roumieu cases, i.e., $\mathcal{D}^*(\Omega)$, $\mathcal{S}^*(\mathbb{R}^d)$, $\mathcal{D}'^*(\Omega)$, $\mathcal{S}'^*(\mathbb{R}^d)$ are common symbols for the pairs of spaces listed above. The mentioned spaces were investigated in [**5**, **10**, **12**, **16**, **25**] and in many other papers. It should be noted that another approach to the theory of ultradistributions was developed by D. Vogt, R. Meise and their collaborators. There exists an extensive literature in this direction with many applications; we refer here just to a few of them (and references therein): [**3**, **4**, **23**, **30**].

Our approach to ultradistributions is similar to that presented in [**1**] for distributions. We begin with the definition of a special kind of fundamental sequences of smooth functions and the corresponding equivalence classes called s-ultradistributions which are elements of the space that we denote by $\mathcal{U}^*(\Omega)$. This is done in sections 2 and 3 together with an analysis of operations on s-ultradistributions, the convergence structure in $\mathcal{U}^*(\Omega)$ and actions of s-ultradistributions on test functions belonging to $\mathcal{D}^*(\Omega)$. In section 4 we introduce the spaces $\mathcal{T}^*$ and $\tilde{\mathcal{T}}^*$ of t-and $\tilde{t}$-ultradistributions, respectively. Again we discuss their structure, the convergence in them and actions of considered tempered ultradistributions on elements of the respective spaces of test functions. It is well-known (see e.g. [**8**]) that there exists a topological isomorphism between the space $\mathcal{S}^*(\mathbb{R}^d)$ and the Köthe echelon space $\mathbf{s}^*$ of sequences of sub-exponential growth. Using this fact we prove in section 5 that the spaces $\mathcal{T}^*$ and $\tilde{\mathcal{T}}^*$ are topologically isomorphic with the space $\mathcal{S}'^*(\mathbb{R}^d)$. Applying the results of section 5, we prove in section 6 the existence of a sequential topological isomorphism between the spaces $\mathcal{U}^*(\Omega)$ and $\mathcal{D}'^*(\Omega)$.

1.1. Preliminaries. The sets of all positive integers, nonnegative integers, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$, respectively.

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}_0^d$ and $\lambda \in \mathbb{R}$, we use the following notation

$$x + y := (x_1 + y_1, \dots, x_d + y_d) \in \mathbb{R}^d; \quad x + \lambda := (x_1 + \lambda, \dots, x_d + \lambda) \in \mathbb{R}^d;$$
$$\lambda x := (\lambda x_1, \dots, \lambda x_d) \in \mathbb{R}^d; \quad x \leqslant y \quad \text{if } x_j \leqslant y_j \quad \text{for } j = 1, \dots, d;$$
$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}; \quad \alpha! := \alpha_1! \dots \alpha_d!; \quad \binom{\alpha}{\beta} := \prod_{j=1}^d \binom{\alpha_j}{\beta_j} \quad \text{for } \alpha \leqslant \beta;$$
$$|x| := (x_1^2 + \dots + x_d^2)^{1/2}; \quad |\alpha| := \alpha_1 + \dots + \alpha_d;$$
$$D^\alpha = D_x^\alpha := D_1^{\alpha_1} \dots D_d^{\alpha_d}, \quad \text{where} \quad D_j^{\alpha_j} := (-i\partial/\partial x_j)^{\alpha_j} \quad (j = 1, \dots, d)$$

and the following summation notation

$$\sum_{|\alpha|=0}^{\infty} := \sum_{\alpha \in \mathbb{N}_0^d}; \qquad \sum_{0 \leqslant \alpha \leqslant \beta} := \sum_{\alpha_1=0}^{\beta_1} \cdots \sum_{\alpha_d=0}^{\beta_d}.$$

The symbol X° for $X \subset \mathbb{R}^d$ means the interior of X and the symbol $K \Subset V$ for an open $V \subset \mathbb{R}^d$ means that K is a compact subset of V. By Ω we denote a fixed open subset of $\mathbb{R}^d$. By $\mathcal{C}(\mathbb{R}^d)$ and $\mathcal{C}(K)$ for $K \Subset \Omega$ we denote the sets of all continuous functions on $\mathbb{R}^d$ and K, respectively, and by $\xrightarrow{\mathcal{C}(\mathbb{R}^d)}$ and $\xrightarrow{\mathcal{C}(K)}$ the uniform convergences on $\mathbb{R}^d$ and K of sequences of functions in $\mathcal{C}(\mathbb{R}^d)$ and $\mathcal{C}(K)$, respectively; the latter is the convergence in the Banach space $\mathcal{C}(K)$ with the supremum norm $\|\cdot\|_\infty$. We denote a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of numbers (functions, distributions, ultradistributions) shorter by (α_n) or $(\alpha_n)_n$ and the mapping $\Omega \ni x \mapsto F(x)$ by F or $F(x)$. The norm in $L^2(\mathbb{R}^d)$ is denoted by $\|\cdot\|_2$ and the convergence in $L^2(\mathbb{R}^d)$ by $\xrightarrow{2}$. The support of a function (distribution, ultradistribution) f by $\operatorname{supp} f$. A function f is called compactly supported if there is a $K \Subset \mathbb{R}^d$ such that $\operatorname{supp} f \subset K$. For the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we use the two symbols: $\mathcal{F}(\varphi) = \widehat{\varphi} := \int_{\mathbb{R}^d} \varphi(x) e^{-i\langle x, \cdot\rangle} dx$; clearly, $\mathcal{F}(D^\alpha \varphi) = \xi^\alpha \widehat{\varphi}$ $(\xi \in \mathbb{R}^d)$. For the properties of the spaces of test functions $\mathcal{D}(\Omega)$, $\mathcal{S}(\mathbb{R}^d)$ and their duals $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\mathbb{R}^d)$ we refer to [**28**].

We recall some notions from [**12**]. By the *associated function*, corresponding to the Gevrey sequence $(p!^t)_p$ for a fixed $t > 1$, we mean the following function: $M(\rho) := \sup_{p\in\mathbb{N}_0} \log_+ \rho^p/p!^t = e^{k\rho^{1/t}}$ for $\rho > 0$, where $k > 0$ is an appropriate constant. Denote by $\mathcal{R}$ the set of all sequences (r_p) of positive numbers strictly increasing to infinity. By the (r_p)-*associated function*, corresponding to $(r_p) \in \mathcal{R}$, we mean the function: $N_{(r_p)}(\rho) := \sup_{p\in\mathbb{N}_0} \log_+ \rho^p/N_p$ for $\rho > 0$, where $(N_p)_{p\in\mathbb{N}_0}$ is defined by means of (r_p) as follows

$$N_0 := 1; \quad N_p := p!^t R_p, \quad \text{where} \quad R_p := \prod_{j=1}^p r_j, \quad \text{for } p \in \mathbb{N}. \tag{1.1}$$

Note that if $(r_p) \in \mathcal{R}$, then for every $k > 0$ there is a $\rho_0 > 0$ such that $N_{(r_p)}(\rho) \leqslant e^{k\rho^{1/t}}$ for $\rho > \rho_0$ (see [**12**]).

Let $K \Subset \Omega$ and $h > 0$. We recall the definitions of some spaces of test functions [**12**]

$$\begin{aligned}
\mathcal{E}^{t,h}(K) &:= \Big\{\varphi \in \mathcal{C}^\infty(\Omega)\colon P_{h,K}(\varphi) := \sup_{x\in K, \alpha\in\mathbb{N}_0^d} \frac{|D^\alpha\varphi(x)|}{h^{|\alpha|}\alpha!^t} < \infty\Big\};\\
\mathcal{D}_K^{t,h} &:= \mathcal{E}^{t,h}(K) \cap \{\varphi \in \mathcal{C}^\infty(\Omega) : \operatorname{supp}\varphi \subset K\};\\
\mathcal{D}_K^{(t)} &:= \varprojlim_{h\to 0} \mathcal{D}_K^{t,h}; \quad \mathcal{D}^{(t)}(\Omega) := \varinjlim_{K\Subset\Omega} \mathcal{D}_K^{(t)};\\
\mathcal{D}_K^{\{t\}} &:= \varinjlim_{h\to\infty} \mathcal{D}_K^{t,h}; \quad \mathcal{D}^{\{t\}}(\Omega) := \varinjlim_{K\Subset\Omega} \mathcal{D}_K^{\{t\}}.
\end{aligned}$$

As already said, we use the common symbol $\mathcal{D}^*(\Omega)$ for the spaces $\mathcal{D}^{(t)}(\Omega)$ and $\mathcal{D}^{\{t\}}(\Omega)$ and $\mathcal{D}'^*(\Omega)$ for their duals. Recall now (see [**5**]) the definitions of the spaces $\mathcal{S}^{(t)}(\mathbb{R}^d)$ and $\mathcal{S}^{\{t\}}(\mathbb{R}^d)$, invariant under the Fourier transform

$$\begin{aligned}
&\mathcal{S}_h^t(\mathbb{R}^d) := \Big\{f \in \mathcal{S}(\mathbb{R}^d)\colon \exists C > 0\ \forall \alpha,\beta \in \mathbb{N}_0^d\ \frac{\|x^\alpha\partial^\beta f\|_2}{h^{|\alpha+\beta|}\alpha!^t\beta!^t} \leqslant C\Big\};\\
&\mathcal{S}^{(t)}(\mathbb{R}^d) := \varprojlim_{h\to 0} \mathcal{S}_h^t(\mathbb{R}^d); \qquad \mathcal{S}^{\{t\}}(\mathbb{R}^d) := \varinjlim_{h\to\infty} \mathcal{S}_h^t(\mathbb{R}^d).
\end{aligned}$$

Notice that the space $\mathcal{S}^{(t)}(\mathbb{R}^d)$ is nontrivial if $t > 1/2$, while $\mathcal{S}^{\{t\}}(\mathbb{R}^d)$ is nontrivial if $t \geqslant 1/2$. The spaces $\mathcal{S}^{(t)}(\mathbb{R}^d)$ and $\mathcal{S}^{\{t\}}(\mathbb{R}^d)$ are denoted commonly by $\mathcal{S}^*(\mathbb{R}^d)$ and their duals by $\mathcal{S}'^*(\mathbb{R}^d)$.

The Hermite polynomials H_n and the corresponding Hermite functions h_n are defined on $\mathbb{R}$ by

$$H_n(x) := (-1)^n e^{x^2}\Big(\frac{d}{dx}\Big)^n(e^{-x^2}), \quad h_n(x) := (2^n n!\sqrt{\pi})^{-\frac{1}{2}} e^{-x^2/2} H_n(x)$$

for $x \in \mathbb{R}$, $n \in \mathbb{N}_0$. The d-dimensional Hermite functions h_n are defined by

$$h_n(x) := h_{n_1}(x_1)\dots h_{n_d}(x_d), \quad x = (x_1,\dots,x_d) \in \mathbb{R}^d, \qquad n \in \mathbb{N}_0^d.$$

They form an orthonormal basis for $L^2(\mathbb{R}^d)$ and are the eigenfunctions of the product $H = \prod_{i=1}^d(-\partial^2/\partial x_i^2 + x_i^2)$ of the one-dimensional Hermite harmonic oscillators, so that $H^\alpha = \prod_{i=1}^d(-\partial^2/\partial x_i^2 + x_i^2)^{\alpha_i}$ and

$$H^\alpha h_k(x) = (2k+1)^\alpha h_k(x) = \prod_{i=1}^d (2k_i+1)^{\alpha_i} h_k(x), \qquad x \in \mathbb{R}^d, \quad \alpha, k \in \mathbb{N}_0^d.$$

Note that H is a self-adjoint operator. For $f \in \mathcal{S}(\mathbb{R}^d)$, the Hermite coefficients are $c_k = \int_{\mathbb{R}^d} f h_k = (f, h_k)_{L^2}$ for $k \in \mathbb{N}_0^d$.

We have

$$\mathcal{D}^*(\mathbb{R}^d) \hookrightarrow \mathcal{S}^*(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'^*(\mathbb{R}^d) \hookrightarrow \mathcal{D}'^*(\mathbb{R}^d),$$

where the symbol $\hookrightarrow$ means that the identity mapping is a continuous and dense embedding.

A sequence (δ_n) of the form $\delta_n := n^d\varphi(n\cdot), n \in \mathbb{N}$, where $\varphi \in \mathcal{D}^*(\mathbb{R}^d)$, $\varphi = 1$ in $B(0, 1/2)$ and $\varphi = 0$ out of $B(0, 1)$ ($B(x_0, r)$ denotes the closed ball with the center at x_0 and radius r) is called a *delta sequence in* $\mathcal{D}'^*(\mathbb{R}^d)$.

1.2. Ultradifferential operators. We recall the definitions and some results related to ultradifferential operators from [**12**–**16**]. A formal expression $P(D) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ ($a_\alpha \in \mathbb{C}$), corresponding to the function $P(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ ($z \in \mathbb{C}^d$), is called an *ultradifferential operator of the Beurling class* $(p!^t)$ *resp. of the Roumieu class* $\{p!^t\}$) if it satisfies the condition

$$\exists h > 0\ \exists C > 0 \text{ (resp. } \forall h > 0\ \exists C > 0)\ \forall \alpha \in \mathbb{N}_0^d \quad |a_\alpha| \leqslant \frac{Ch^{|\alpha|}}{(\alpha!)^t};$$

in the Roumieu case, the condition can be expressed in the equivalent form

$$\exists (r_p) \in \mathcal{R}\ \exists C > 0\ \forall \alpha \in \mathbb{N}_0^d \quad |a_\alpha| \leqslant \frac{C}{(\alpha!)^t R_{|\alpha|}},$$

where $R_{|\alpha|}$ is defined in (1.1) and $t > 1$ will be fixed throughout the paper. We use the common term *ultradifferential operator of the class* $*$ in both cases of Beurling and Roumieu.

If $P(D)$ is an ultradifferential operator of the Beurling class (resp. of the Roumieu class), then the function $P(z)$ satisfies, by [**12**, Proposition 4.5], the estimate

$$\exists h > 0\ \exists C > 0 \text{ (resp. } \forall h > 0,\ \exists C > 0)\ \forall z \in \mathbb{C}^d \quad |P(z)| \leqslant Ce^{h|z|^{1/t}};$$

in the Roumieu case, the estimate can be written in the equivalent form

$$\exists (r_p) \in \mathcal{R}\ \exists C > 0\ \forall \xi \in \mathbb{R}^d \quad |P(\xi)| \leqslant Ce^{c(|\xi|)^{1/t}},$$

where c is the *subordinate function* of (r_p), i.e., an increasing function on $[0, \infty)$ such that $c(0) = 0$ and $c(\rho)/\rho \to 0$ as $\rho \to \infty$ corresponding to (r_p) by means of the identity: $M(c(\rho)) = N_{(r_p)}(\rho)$ ($\rho > 0$), where $N_{(r_p)}$ is the (r_p)-associated function (see [**12**]).

We denote by $\mathcal{P}^{(t)}$ (resp. $\mathcal{P}^{\{t\}}$) the class of ultradifferential operators $P_r(D)$ of Beurling type (resp. $P_{(r_p)}(D)$ of Roumieu type) of the form

$$P_r(D) = \Big(1 + \sum_{j=1}^{d} D_j^2\Big)^l \prod_{p=1}^{\infty} \Big[1 + \Big(\sum_{j=1}^{d} D_j^2\Big)/r^2p^{2t}\Big] \quad \Big(= \sum_{p=0}^{\infty} a_p D^p\Big), \tag{1.2}$$

$$P_{(r_p)}(D) = \Big(1 + \sum_{j=1}^{d} D_j^2\Big)^l \prod_{p=1}^{\infty} \Big[1 + \Big(\sum_{j=1}^{d} D_j^2\Big)/r_p^2p^{2t}\Big] \quad \Big(= \sum_{p=0}^{\infty} b_p D^p\Big), \tag{1.3}$$

where $r > 0$, $(r_p) \in \mathcal{R}$ and $l \geqslant 0$. Replacing D_j by ξ_j in (1.2) and (1.3) we get the *ultra-polynomials* $P_r(\xi)$ and $P_{(r_p)}(\xi)$ of the Beurling and Roumieu type corresponding to the ultradifferential operators $P_r(D)$ and $P_{(r_p)}(D)$, respectively. They can be described in the following way (see [**12**]):

Ultra-polynomials of Beurling type are of sub-exponential growth, i.e., there are constants $C_1, C_2, C > 0$ and $h_1, h_2, h > 0$ such that

$$C_1 e^{h_1|\xi|^{1/t}} \leqslant |P_r(\xi)| \leqslant C_2 e^{h_2|\xi|^{1/t}}, \quad \xi \in \mathbb{R}^d, \tag{1.4}$$

and

$$|a_p| \leqslant Ch^p/p!^t, \quad p \in \mathbb{N}_0.$$

The description of ultra-polynomials in the Roumieu case is more difficult. One can prove, similarly to the Beurling case, that for a given $(r_p) \in \mathcal{R}$ and its subordinate function c there exists a constant $C > 0$ such that

$$Ce^{c(|\xi|)^{1/t}} \leqslant |P_{(r_p)}(\xi)|, \quad \xi \in \mathbb{R}^d. \tag{1.5}$$

To get a suitable upper estimate we have to find a sequence $(r_{0,p}) \in \mathcal{R}$ and its subordinate function c_0 such that the inequality

$$(1 + |\xi|^2)^l |P_{(r_p)}(\xi)| \leqslant C_0 e^{c_0(|\xi|)^{1/t}}, \quad \xi \in \mathbb{R}^d. \tag{1.6}$$

holds for some $C_0 > 0$ and all $l \geqslant 0$. For this aim we use the following property of a subordinate function which is a consequence of Lemma 3.12 (see also Lemma 3.10) in [**12**].

Let c be an arbitrary subordinate function and put $\tilde{c} := 2c$. There exists a sequence $(r_p^0) \in \mathcal{R}$ such that the (r_p^0)-associated function $N_{(r_p^0)}$ and the subordinate function c_0 corresponding to (r_p^0) satisfy the inequality

$$M(\tilde{c}(\rho)) \leqslant N_{(r_p^0)}(\rho) = M(c_0(\rho)), \quad \rho > 0,$$

Consequently,

$$c_0(\rho) \geqslant \tilde{c}(\rho) = 2c(\rho), \quad \rho > 0.$$

The above remarks can be formulated as follows:

Lemma 1.1. *For an arbitrary subordinate function c, corresponding to some sequence $(r_p) \in \mathcal{R}$, and $h > 0$ there exist a sequence $(r_p^0) \in \mathcal{R}$ and its subordinate function c_0 such that*

$$c_0(\rho) \geqslant hc(\rho), \quad \rho > 0.$$

In particular, for a given subordinate function c there exists another subordinate function c_0 (both corresponding to appropriate sequences from $\mathcal{R}$) such that $c_0(\rho) \geqslant c(2\rho)$ for all $\rho > 0$.

In Subsection 4.2 we will need also the following lemma.

Lemma 1.2. (a) *If $P_r \in \mathcal{P}^{(t)}$* (resp. $P_{(r_p)} \in \mathcal{P}^{\{t\}}$), *then there exist $r_0 > 0$* (resp. $(r_p^0) \in \mathcal{R}$), *$C > 0$ and $\varepsilon > 0$ such that*

$$|D^\alpha P_r(x)| \leqslant \frac{C\alpha!}{\varepsilon^{|\alpha|}} e^{r_0|x|^{1/t}}, \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d$$

$$\Big(\text{resp. } |D^\alpha P_{(r_p)}(x)| \leqslant \frac{C\alpha!}{\varepsilon^{|\alpha|}} e^{c_{(r_p^0)}(|x|)^{1/t}}, \quad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d\Big),$$

where $c_{(r_p^0)}$ is the subordinate function corresponding to the sequence (r_p^0).

(b) *If* $P_{\tilde r} \in \mathcal{P}^{(t)}$ (resp. $P_{(\tilde r_p)} \in \mathcal{P}^{\{t\}}$), *then there exist* $\tilde r_0 > 0$ (resp. $(\tilde r^0_p) \in \mathcal{R}$), $C > 0$ *and* $\varepsilon > 0$ *such that*

$$|D^\alpha(1/P_r(x))| \leqslant \frac{C\alpha!}{\varepsilon^{|\alpha|}} e^{-\tilde r_0 |x|^{1/t}}, \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d$$

$$\Big(\textit{resp. } |D^\alpha(1/P_{(\tilde r_p)}(x))| \leqslant \frac{C\alpha!}{\varepsilon^{|\alpha|}} e^{-c_{(\tilde r^0_p)}(|x|)^{1/t}}, \quad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d\Big),$$

where $c_{(\tilde r^0_p)}$ *is the subordinate function corresponding to the sequence* $(\tilde r^0_p)$.

Proof. In the proof of both parts we use Lemma 1.1 and the Cauchy integral formula for ultra-polynomials on the circles $K(x,\varepsilon)$ around $x \in \mathbb{R}^d$. The proof of (b) follows from the estimate

$$|D^\alpha(1/P_{(\tilde r_p)}(x))| \leqslant \frac{C\alpha!}{\varepsilon^{|\alpha|}} e^{-N_{(\tilde r_p)}(|x|)}, \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d,$$

shown in [**26**, Lemma 2.1], because we can construct $c_{(\tilde r^0_p)}$ such that

$$N_{(\tilde r_p)}(|x|) = M(c_{(\tilde r_p)}(|x|)) = Cc_{(\tilde r_p)}(|x|)^{1/t} = c_{(\tilde r^0_p)}(|x|)^{1/t}$$

for some $C > 0$ and $|x| > 0$, in view of Lemma 3.10 in [**12**] and Lemma 1.1. The proof of (a) follows from Proposition 4.5 in [**12**]; see also the last part of the proof of Theorem 10.2 in [**12**]. □

The symbol $\mathcal{P}^*$ will be common for the classes $\mathcal{P}^{(t)}$ and $\mathcal{P}^{\{t\}}$ of ultradifferential operators of Beurling and Roumieu types and $\mathcal{P}^{2*}$ will mean the space of t-ultradistributions of both types. The corresponding spaces of ultradifferentiabile functions will be denoted by $\mathcal{P}^*_u$ and $\mathcal{P}^{2*}_u$, respectively. This notation looks complicated but it helps to distinct the different use of P: $P(D)$, $P(x)$, $P(\xi)$. To simplify the exposition we will usually consider ultradifferential operators of the form (1.2) and (1.3), but in some proofs we need their general form.

Denote by μ_β the operator acting on measurable functions G as follows: $(\mu_\beta G)(\xi) := (i\xi)^\beta G(\xi)$ for $\xi \in \mathbb{R}^d$ and $\beta \in \mathbb{N}_0^d$; in particular $\mu_0 G = G$. We will use in the sequel the following assertion: for arbitrary $\beta \in \mathbb{N}_0^d$, $q \in [1,\infty]$ and $P_r(D) \in \mathcal{P}^{(t)}$ with $r > 0$ (resp. $P_{(r_p)}(D) \in \mathcal{P}^{\{t\}}$ with $(r_p) \in \mathcal{R}$) there exists $P_{\tilde r}(D) \in \mathcal{P}^{(t)}$ with $\tilde r > r$ (resp. $P_{(\tilde r_p)} \in \mathcal{P}^{\{t\}}$ with $(\tilde r_p) \in \mathcal{R}$, $r_p/\tilde r_p \downarrow 0$ as $p \to \infty$) such that

$$\tag{1.7} \begin{gathered} \mu_\beta \frac{P_r}{P_{\tilde r}} \in L^q(\mathbb{R}^d) \quad \text{and} \quad \mathcal{F}^{-1}\Big(\mu_\beta \frac{P_r}{P_{\tilde r}}\Big) \in L^q(\mathbb{R}^d) \\ \Big(\text{resp. } \mu_\beta \frac{P_{(r_p)}}{P_{(\tilde r_p)}} \in L^q(\mathbb{R}^d) \quad \text{and} \quad \mathcal{F}^{-1}\Big(\mu_\beta \frac{P_{(r_p)}}{P_{(\tilde r_p)}}\Big) \in L^q(\mathbb{R}^d)\Big) \end{gathered}$$

and, analogously, with $\mathcal{P}^{2*}_u$ instead of $\mathcal{P}^*_u$, the following one

$$\tag{1.8} \Big(\frac{P_r(2\alpha+1)}{P_{\tilde r}(2\alpha+1)}\Big)_{\alpha \in \mathbb{Z}^d} \in l^q \quad \Big(\text{resp. } \Big(\frac{P_{(r_p)}(2\alpha+1)}{P_{(\tilde r_p)}(2\alpha+1)}\Big)_{\alpha \in \mathbb{Z}^d} \in l^q\Big),$$

where $P(2\alpha+1) := \sum_{|k|=0}^\infty a_k \prod_{i=1}^d (2\alpha_i+1)^{k_i}$ for $\alpha \in \mathbb{N}_0^d$.

The following well-known assertions will also be used in the sequel; their proofs can be found e.g. in [**5**, **25**].

Lemma 1.3. (a) *A smooth function φ on $\mathbb{R}^d$ belongs to $\mathcal{S}^*(\mathbb{R}^d)$ iff for arbitrary $P \in \mathcal{P}^*$ and $P_1 \in \mathcal{P}^*_u$ we have $\|P_1 P(-D)\varphi\|_2 < \infty$.*

(b) *If $\varphi_n \in \mathcal{S}^*(\mathbb{R}^d)$ $(n \in \mathbb{N}_0)$ and $\varphi_n \xrightarrow{\mathcal{S}^*} \varphi_0$ as $n \to \infty$, then for arbitrary $P \in \mathcal{P}^*$ and $P_1 \in \mathcal{P}^*_u$ we have $P_1 P(-D)\varphi_n \xrightarrow{\mathcal{S}^*} P_1 P(-D)\varphi_0$ as $n \to \infty$.*

(c) *If $\varphi_n \in \mathcal{S}^*(\mathbb{R}^d)$ $(n \in \mathbb{N}_0)$ and $\varphi_n \xrightarrow{\mathcal{S}^*} \varphi_0$ as $n \to \infty$, then for every $P \in \mathcal{P}^*$ we have $P(H)\varphi_n \xrightarrow{\mathcal{S}^*} P(H)\varphi_0$ as $n \to \infty$.*

2. Fundamental sequences

Let us recall that Schwartz distributions in the sequential approach presented in [**1**] are equivalence classes of fundamental sequences of smooth functions defined with the use of derivatives of finite order. We introduce s-ultradistributions of Beurling and Roumieu type in a similar way, but our fundamental sequences are defined by means of the ultradifferential operators $P_r(D)$ and $P_{(r_p)}(D)$, respectively, instead of finite order differential operators.

If $P \in \mathcal{P}^*$ and F is an integrable function compactly supported, then $P(z)\mathcal{F}(F)(z)$ $(z \in \mathbb{C}^d)$ is an entire function of sub-exponential growth on $\mathbb{R}^d$. If the inverse Fourier transform $\mathcal{F}^{-1}(P\widehat{F})$ is a locally integrable function, then we define

$$P(D)F(x) := \mathcal{F}^{-1}(P\widehat{F})(x), \quad x \in \mathbb{R}^d \tag{2.1}$$

to give the meaning for the formal acting of the ultradifferential operator $P(D)$ on a compactly supported smooth function. If F is a compactly supported smooth function such that $\operatorname{supp} F \subset K_1 \Subset \Omega$ and $P \in \mathcal{P}^*$ is of the form $P(D) := \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$, then the left hand side of (2.1) is meant as follows

$$P(D)F(x) := \lim_{k\to\infty} P_k(D)F(x), \quad x \in K, \tag{2.2}$$

where $P_k(D) := \sum_{|\alpha|=0}^{k} a_\alpha D^\alpha$ and the limit in (2.2) is assumed to exist for every $x \in K$ and to be a smooth function on K. In this case the limit defines $f(x) = P(D)F(x)$ for $x \in K$ and gives the meaning of (2.3) below.

Definition 2.1. A sequence (f_n) of smooth functions defined on an open set $\Omega \subset \mathbb{R}^d$ is called *s-fundamental* (of type $*$, i.e., of Beurling or Roumieu type, respectively) in Ω if for arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ there exist an ultradifferential operator $P(D) \in \mathcal{P}^*$, a sequence (F_n) of smooth functions on Ω and a continuous function F_0 on Ω such that

$$\begin{aligned} &f_n = P(D)F_n \text{ on } K\ (n \in \mathbb{N}), \quad \operatorname{supp} F_n \subset K_1\ (n \in \mathbb{N}_0) \\ &\text{and } F_n \xrightarrow{\mathcal{C}(K)} F_0 \text{ as } n \to \infty. \end{aligned} \tag{2.3}$$

The equality in (2.3) is meant in the sense of (2.2). In the sequel, for a given $K \Subset \Omega$ we will always take a set $K_1 \Subset \Omega$ with $K \Subset K_1^\circ$ which is sufficiently close to K, not referring explicitly about it (one can show, taking an appropriate cut-of function, that the definition does not depend on the choice of K_1).

REMARK 2.1. 1° One can consider in (2.3) all ultradifferential operators of class $*$, not only belonging to $\mathcal{P}^*$, which gives a more general form of the definition. Since both formulations are equivalent, we will use the above one for simplicity.

2° Let Ω_1, Ω be open sets in $\mathbb{R}^d$ such that $\Omega_1 \subset \Omega$. If a sequence (f_n) is s-fundamental in Ω, then it is s-fundamental in Ω_1.

3° Let (f_n) be a sequence of smooth functions in Ω. If for every open set $\Omega_0 \subset \Omega$ the sequence $(f_n|\Omega_0)$ is s-fundamental in Ω_0, then (f_n) is s-fundamental in Ω.

DEFINITION 2.2. Let (f_n) and (g_n) be s-fundamental sequences in an open set Ω. We write $(f_n) \sim (g_n)$ if for arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$, there exist an ultradifferential operator $P \in \mathcal{P}^*$ and sequences (F_n), (G_n) of smooth functions on Ω such that

$$f_n = P(D)F_n,\ g_n = P(D)G_n \text{ on } K\ (n \in \mathbb{N}),$$
$$\operatorname{supp} F_n,\ \operatorname{supp} G_n \subset K_1\ (n \in \mathbb{N}),\ \text{and } F_n \overset{\mathcal{C}(K)}{\rightarrow\leftarrow} G_n \text{ as } n \to \infty,$$

where the symbol $F_n \overset{\mathcal{C}(K)}{\rightarrow\leftarrow} G_n$ means that (F_n) and (G_n) converge in $\mathcal{C}(K)$ to a common continuous function H on Ω.

REMARK 2.2. It is clear that if (f_n) and (g_n) are s-fundamental sequences in Ω, then $(f_n) \sim (g_n)$ iff the sequence $f_1, g_1, f_2, g_2, f_3, g_3, \dots$ is s-fundamental in Ω.

Obviously, the relation $\sim$ is reflexive and symmetric. To prove its transitivity we need some auxiliary statements.

PROPOSITION 2.1. *Fix $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$. Assume that (f_n) satisfies Definition 2.1 in the Roumieu case, i.e., $f_n = P_{(r_p)}(D)F_n$ on K, $\operatorname{supp} F_n \subset K_1$ for $n \in \mathbb{N}_0$ and $F_n \xrightarrow{\mathcal{C}(K)} F_0$ as $n \to \infty$ for a sequence (F_n) of smooth functions and a continuous function F_0 on Ω. Then there are a $P_{(\tilde{r}_p)} \in \mathcal{P}^{\{t\}}$, where $(\tilde{r}_p) \in \mathcal{R}$ with $r_p/\tilde{r}_p \downarrow 0$ as $p \to \infty$ and $\mathcal{F}^{-1}(P_{(r_p)}/P_{(\tilde{r}_p)}) \in L^1(\mathbb{R}^d)$, smooth functions $\tilde{F}_n$ $(n \in \mathbb{N})$ and a continuous function $\tilde{F}_0$ on Ω such that*

$$f_n = P_{(\tilde{r}_p)}(D)\tilde{F}_n \text{ on } K\ (n \in \mathbb{N}), \quad \operatorname{supp} \tilde{F}_n \subset K_1\ (n \in \mathbb{N}_0)$$
$$\text{and } \tilde{F}_n \xrightarrow{\mathcal{C}(K)} \tilde{F}_0 \text{ as } n \to \infty.$$

The same assertion holds in the Beurling case (with the corresponding notation).

PROOF. By (1.7), for arbitrary $\beta \in \mathbb{N}_0^d$ (in particular, for $\beta = 0$) we can find $(\tilde{r}_p) \in \mathcal{R}$ with $r_p/\tilde{r}_p \downarrow 0$ as $p \to \infty$ such that

$$\mu_\beta \frac{P_{(r_p)}}{P_{(\tilde{r}_p)}} \in L^1(\mathbb{R}^d) \text{ and } \mathcal{F}^{-1}\Big(\mu_\beta \frac{P_{(r_p)}}{P_{(\tilde{r}_p)}}\Big) \in L^1(\mathbb{R}^d).$$

Define

$$\tilde{F}_n := \kappa_K \Big[\mathcal{F}^{-1}\Big(\frac{P_{(r_p)}}{P_{(\tilde{r}_p)}}\Big) * (\kappa_K F_n)\Big], \quad n \in \mathbb{N}_0, \tag{2.4}$$

where κ_K is a smooth function in $\mathcal{D}^{\{t\}}(\Omega)$ such that

$$\kappa_K(x) = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \in K_1^c. \end{cases} \tag{2.5}$$

Then $\operatorname{supp} \tilde{F}_n \subset K_1$ for $n \in \mathbb{N}_0$ and

$$\begin{aligned} P_{(\tilde{r}_p)}(D)\tilde{F}_n(x) &= P_{(\tilde{r}_p)}(D)\Big[\mathcal{F}^{-1}\Big(\frac{P_{(r_p)}}{P_{(\tilde{r}_p)}}\Big) * (\kappa_K F_n)\Big](x) \\ &= \big[\mathcal{F}^{-1}(P_{(r_p)}) * (\kappa_K F_n)\big](x) = P_{(r_p)}(D)(\kappa_K F_n)(x) = f_n(x) \end{aligned}$$

for $x \in K$ and $n \in \mathbb{N}$. Since

$$\mathcal{F}^{-1}\big(P_{(r_p)}/P_{(\tilde{r}_p)}\big) \in L^1(\mathbb{R}^d) \quad \text{and} \quad \kappa_K F_n \xrightarrow{\mathcal{C}(K)} \kappa_K F_0 \quad \text{as } n \to \infty,$$

it follows that $\tilde{F}_n \xrightarrow{\mathcal{C}(K)} \tilde{F}_0$ as $n \to \infty$. □

PROPOSITION 2.2. *If for arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ there are a $P \in \mathcal{P}^*$, smooth functions F_n $(n \in \mathbb{N})$ and a continuous function F_0 on Ω with* $\operatorname{supp} F_n \subset K_1$ *$(n \in \mathbb{N}_0)$ such that $F_n \xrightarrow{\mathcal{C}(K)} F_0$ and $P(D)F_n(x) \to 0$ for $x \in K$ as $n \to \infty$, then $F_0 = 0$ on Ω. In particular, if F is a smooth function on Ω and $P(D)F(x) = 0$ for $x \in \Omega$, then $F = 0$ on Ω.*

PROOF. Fix $K_1 \Subset \Omega$, $K \Subset K_1^\circ$ and κ_K as in (2.5). By the assumption,

$$P(D)(\kappa_K F_n)(x) \to 0 \ (x \in K), \quad \kappa_K F_n \xrightarrow{\mathcal{C}(K)} \kappa_K F_0 \quad \text{as } n \to \infty. \tag{2.6}$$

Since $\lim_{m\to\infty}[P(D) - P_m(D)](\kappa_K F_n) = 0$ in $\mathbb{R}^d$, it follows from (2.6) that

$$\lim_{n\to\infty}\lim_{m\to\infty} P_m(\xi)\widehat{\kappa_K F_n}(\xi) = \lim_{n\to\infty} P(\xi)\widehat{\kappa_K F_n}(\xi) = 0, \qquad \xi \in \mathbb{R}^d.$$

and

$$\lim_{n\to\infty} P(\xi)\big[\widehat{\kappa_K F_n}(\xi) - \widehat{\kappa_K F_0}(\xi)\big] = 0, \qquad \xi \in \mathbb{R}^d.$$

This implies $P(\xi)\widehat{\kappa_K F_0}(\xi) = 0$ for $\xi \in \mathbb{R}^d$, so $\kappa_K(x)F_0(x) = 0$ for $x \in K$. Hence $F_0 = 0$ in K and thus $F_0 = 0$ in Ω, since $K \Subset \Omega$ was arbitrarily chosen. The particular case is clear if we take $F_n = F$ for $n \in \mathbb{N}$. □

PROPOSITION 2.3. *Fix $K, K_1 \Subset \Omega$ such that $K \Subset K_1^\circ$. Assume that, for sequences $(r_p), (\tilde{r}_p) \in \mathcal{R}$, smooth functions F_n, $\tilde{F}_n$ on Ω $(n \in \mathbb{N})$ and continuous functions F_0, $\tilde{F}_0$ on Ω, we have*

$$\begin{aligned} &f_n = P_{(r_p)}(D)F_n \quad \text{on } K \ (n \in \mathbb{N}), \quad \operatorname{supp} F_n \subset K_1 \ (n \in \mathbb{N}_0), \\ &F_n \xrightarrow{\mathcal{C}(K)} F_0 \quad \text{as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} &f_n(x) = P_{(\tilde{r}_p)}(D)\,\tilde{F}_n \quad \text{on } K \ (n \in \mathbb{N}), \quad \operatorname{supp} \tilde{F}_n \subset K_1 \ (n \in \mathbb{N}_0), \\ &\tilde{F}_n \xrightarrow{\mathcal{C}(K)} \tilde{F}_0 \quad \text{as } n \to \infty. \end{aligned}$$

Then there are a sequence $(\bar r_p) \in \mathcal{R}$, $r_p/\bar r_p \downarrow 0$, $\tilde r_p/\bar r_p \downarrow 0$ *as* $p \to \infty$ *such that*

$$\mathcal{F}^{-1}\Big(\frac{P_{(r_p)}}{P_{(\bar r_p)}}\Big),\ \mathcal{F}^{-1}\Big(\frac{P_{(\tilde r_p)}}{P_{(\bar r_p)}}\Big) \in L^1(\mathbb{R}^d), \tag{2.7}$$

and sequences $(F_{n,1})$, $(F_{n,2})$ *of smooth functions and continuous functions* $F_{0,1}$, $F_{0,2}$ *on* Ω, *with* $\operatorname{supp} F_{n,1}$, $\operatorname{supp} F_{n,2} \subset K_1$ $(n \in \mathbb{N}_0)$, *such that*

$$f_n = P_{(\bar r_p)}(D)F_{n,j} \quad on\ K\ (n \in \mathbb{N}), \quad F_{n,j} \xrightarrow{\mathcal{C}(K)} F_{0,j} \quad as\ n \to \infty \tag{2.8}$$

for $j = 1, 2$. *Moreover* $F_{0,1} = F_{0,2}$ *in* Ω.

The same assertion also holds in the Beurling case (with the corresponding notation).

Proof. The existence of $(\bar r_p) \in \mathcal{R}$ satisfying (2.7) follows from (1.7). Define

$$F_{n,1} := \kappa_K\Big[\mathcal{F}^{-1}\Big(\frac{P_{(r_p)}}{P_{(\bar r_p)}}\Big) * (\kappa_K F_n)\Big], \quad F_{n,2} := \kappa_K\Big[\mathcal{F}^{-1}\Big(\frac{P_{(\tilde r_p)}}{P_{(\bar r_p)}}\Big) * (\kappa_K \tilde F_n)\Big],$$

where κ_K is a smooth function in $\mathcal{D}^{\{t\}}(\Omega)$ which satisfies (2.5). Using Proposition 2.1, one can deduce (2.8). Finally, by Proposition 2.2, we conclude that $F_{0,1} = F_{0,2}$ in Ω. □

Proposition 2.4. *Relation* $\sim$ *introduced in Definition* 2.2 *is transitive.*

Proof. We will prove the assertion only in the Roumieu case; the proof in the Beurling case is similar. Suppose that $(f_n) \sim (g_n)$ and $(g_n) \sim (h_n)$ and fix $K, K_1 \Subset \Omega$ so that $K \Subset K_1^\circ$. Now select $\tilde K \Subset \Omega$ such that $K \Subset \tilde K^\circ$ and $\tilde K \Subset K_1^\circ$.

By the assumption and Definition 2.2, there exist $(r_p), (\tilde r_p) \in \mathcal{R}$ and sequences (F_n), (G_n), $(\tilde G_n)$, (H_n) of smooth functions on Ω such that

$$f_n = P_{(r_p)}(D)F_n,\ g_n = P_{(r_p)}(D)G_n \text{ on } K\ (n \in \mathbb{N}),$$
$$\operatorname{supp} F_n,\ \operatorname{supp} G_n \subset \tilde K\ (n \in \mathbb{N}), \quad F_n \overset{\mathcal{C}(K)}{\rightarrow\leftarrow} G_n \text{ as } n \to \infty$$

and

$$g_n = P_{(\tilde r_p)}(D)\tilde G_n,\ h_n = P_{(\tilde r_p)}(D)H_n \text{ on } \tilde K\ (n \in \mathbb{N}),$$
$$\operatorname{supp} \tilde G_n,\ \operatorname{supp} H_n \subset K_1\ (n \in \mathbb{N}), \quad \tilde G_n \overset{\mathcal{C}(\tilde K)}{\longrightarrow\!\!\longleftarrow} H_n \text{ as } n \to \infty.$$

In view of Proposition 2.3, there exist an appropriate $(\bar r_p) \in \mathcal{R}$ and convergent sequences $(F_{n,1})$, $(G_{n,1})$, $(\tilde G_{n,1})$, $(H_{n,1})$ of smooth functions, all having supports contained in K_1, such that

$$f_n = P_{(\bar r_p)}(D)F_{n,1}, \quad g_n = P_{(\bar r_p)}(D)G_{n,1} = P_{(\bar r_p)}(D)\tilde G_{n,1}, \quad h_n = P_{(\bar r_p)}(D)H_{n,1}$$

on K. Moreover $F_{n,1} \overset{\mathcal{C}(K)}{\rightarrow\leftarrow} G_{n,1}$ and $\tilde G_{n,1} \overset{\mathcal{C}(K)}{\rightarrow\leftarrow} H_{n,1}$ as $n \to \infty$.

If we put now

$$\tilde H_{n,1}(x) := G_{n,1}(x) - \tilde G_{n,1}(x) + H_{n,1}(x), \qquad x \in K,$$

then $h_n = P_{(\bar r_p)}(D)\tilde H_{n,1}$ in K and $F_{n,1} \overset{\mathcal{C}(K)}{\rightarrow\leftarrow} \tilde H_{n,1}$ as $n \to \infty$, which means that $(f_n) \sim (h_n)$. This completes the proof. □

2.1. Sequential ultradistributions.

Definition 2.3. Let (f_n) be a s-fundamental sequence (of type $*$) in an open set $\Omega \subset \mathbb{R}^d$. The class of all s-fundamental sequences equivalent to (f_n) with respect to the relation $\sim$ is called a *sequential ultradistributon* or, shortly, *s-ultradistribution* (of type $*$) and denoted by $f = [f_n]$. The set of all s-ultradistributions (of type $*$) on Ω is denoted by $\mathcal{U}^*(\Omega)$.

Remark 2.3. 1° By Proposition 2.2, $f = [f_n] = 0$ on Ω for $f \in \mathcal{U}^*(\Omega)$ if for arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ there exist a sequence (F_n) of smooth functions on Ω and an ultradifferential operator $P \in \mathcal{P}^*$ such that

$$f_n = P(D)F_n \text{ on } K \ (n \in \mathbb{N}), \quad \operatorname{supp} F_n \subset K_1 \ (n \in \mathbb{N})$$
$$\text{and } F_n \xrightarrow{\mathcal{C}(K)} 0 \text{ as } n \to \infty.$$

2° Let $f \in \mathcal{U}^*(\Omega)$. If $f = 0$ on Ω_1 for every $\Omega_1 \subset \Omega$, then $f = 0$ on Ω.

Definition 2.4. By the support of an s-ultradistribution $f \in \mathcal{U}^*(\Omega)$ we mean the complement of the union of all open sets where $f = 0$. We say that an s-ultradistribution $f = [f_n]$ or a s-fundamental sequence (f_n) is compactly supported if there exists $K \Subset \mathbb{R}^d$ such that $\operatorname{supp} f_n \subset K$ for $n \in \mathbb{N}$. Then we write $\operatorname{supp} f \subset K$ or $\operatorname{supp}(f_n) \subset K$. In this case there exist a sequence of smooth functions (F_n), a continuous function F_0, an ultradifferential operator $P \in \mathcal{P}^*$ and $K_1 \Subset \Omega$ so that

$$f_n(x) = P(D)F_n(x) \text{ on } \mathbb{R}^d \ \ (n \in \mathbb{N}), \quad \operatorname{supp} F_n \subset K_1 \ (n \in \mathbb{N}_0)$$
$$\text{and } F_n \xrightarrow{\mathcal{C}(K)} F_0 \text{ as } n \to \infty.$$

Example 2.1. Let F be a compactly supported continuous function in $\mathbb{R}^d$, (δ_n) be a delta sequence in $\mathcal{D}'^*(\mathbb{R}^d)$ and let $F_n := F * \delta_n$ for $n \in \mathbb{N}$. Then (F_n) is a s-fundamental sequence on $\mathbb{R}^d$.

Example 2.2. Let (f_n) be a s-fundamental sequence on $\Omega \subseteq \mathbb{R}^d$ and (K_n) be an increasing sequence of compact sets such that $K_n \subset K_{n+1}^\circ$ for $n \in \mathbb{N}$. Put $\Omega := \bigcup_{n\in\mathbb{N}} K_n$ and consider the open sets

$$\Omega_n := \{x \in \Omega \colon d(x, \partial\Omega) > 1/n\}, \quad \Omega_{n,n} := \{x \in \Omega_n \colon d(x, \partial\Omega_n) > 1/n\}$$

and functions $\kappa_n \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\kappa_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_{n,n}, \\ 0, & \text{if } x \in \Omega_n^c, \end{cases}$$

for $n \in \mathbb{N}$. Then the sequence $(\tilde{f}_n)$, where

$$\tilde{f}_n(x) = \begin{cases} ((\kappa_n f_n) * \delta_n)(x), & \text{if } x \in \Omega_n, \\ 0, & \text{if } x \in \Omega_n^c, \end{cases}$$

is s-fundamental on Ω and $(f_n) \sim (\tilde{f}_n)$.

EXAMPLE 2.3. Let $f \in \mathcal{D}'(\Omega)$ for an open $\Omega \subseteq \mathbb{R}^d$, let Ω_n, $\Omega_{n,n}$ and κ_n be as in Example 2.2 and let $P_{(r_p)} \in \mathcal{P}^{\{t\}}$. Put

$$F_n(x) := \begin{cases} \mathcal{F}^{-1}(\widehat{f_n}/P_{(r_p)})(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \Omega_n^c, \end{cases} \tag{2.9}$$

for $n \in \mathbb{N}$, where $f_n := (\kappa_n f) * \delta_n = P_{(r_p)}(D)F_n$ and $P_{(r_p)}$ means in (2.9) the function corresponding to an ultra-differential operator $P_{(r_p)}$. Since $\widehat{\kappa_n f}$, for every $n \in \mathbb{N}$, is bounded by a polynomial, it follows from (1.6) that (f_n) is a s-fundamental sequence of the Roumieu type on Ω, that is $[f_n] \in \mathcal{U}^{\{t\}}(\Omega)$. In a similar way, we can also represent f as an element of $\mathcal{U}^{(t)}(\Omega)$ by the use $P_r(D)$ instead of $P_{(r_p)}(D)$.

REMARK 2.4. If (f_n) is a s-fundamental sequence in the sense of Definition 2.1, then $F_n \xrightarrow{\mathcal{C}(K)} F_0$ as $n \to \infty$ for every $K \Subset \Omega$. Examples 2.1 and 2.3 show that every s-fundamental sequence (f_n) for which (2.3) holds can be identified with the formal representation $f = P(D)F_0$ on K, since from the general theory of ultradistributions we know that for arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ there exist $P \in \mathcal{P}^*$ and $F_0 \in \mathcal{C}(\Omega)$ such that

$$f = P(D)F_0 \text{ on } K \quad \text{and} \quad \operatorname{supp} F_0 \subset K_1.$$

This will be justified by (3.3) and the last section.

2.2. Operations on s-ultradistributions. We start from the operations of addition and multiplication by a constant. Let $f, g \in \mathcal{U}^*(\Omega)$ and $\lambda \in \mathbb{C}$, where $f = [f_n]$, $g = [g_n]$ for some s-fundamental sequences (f_n), (g_n). Using Proposition 2.3, one can prove that $(f_n + g_n)$ and (λf_n) are s-fundamental sequences on Ω, so we may define s-ultradistributions $f + g := [f_n + g_n]$ and $\lambda f := [\lambda f_n]$. By Remark 2.2, the definitions are consistent. Consequently, $\mathcal{U}^*(\Omega)$ is a vector space.

Next consider the operation of differentiation. If $f = [f_n] \in \mathcal{U}^{\{t\}}(\Omega)$, i.e., (f_n) is a s-fundamental sequence of the Roumieu type, and let $\beta \in \mathbb{N}_0^d$. We will show that the sequence $(f_n^{(\beta)})$ is s-fundamental of the Roumieu type. For arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ take (F_n) and $P_{(r_p)}(D)$ according to Definition 2.1. Since $f_n^{(\beta)} = P_{(r_p)}(D)F_n^{(\beta)}$ on K, it follows from (1.7) as in the proof of Proposition 2.1 for $\beta = 0$ that there exist $P_{(\tilde{r}_p)} \in \mathcal{P}^{\{t\}}$, a sequence $(\tilde{F}_n)$ of smooth functions and a continuous function $\tilde{F}_0$ on Ω defined by (cf. (2.4))

$$\tilde{F}_n := \kappa_K \Big[\mathcal{F}^{-1}\Big(\mu_\beta \frac{P_{(r_p)}}{P_{(\tilde{r}_p)}}\Big) * (\kappa_K F_n)\Big], \quad n \in \mathbb{N}_0,$$

with $\operatorname{supp} \tilde{F}_n \subset K_1$ $(n \in \mathbb{N}_0)$, such that $P_{(r_p)}(D)F_n^{(\beta)} = P_{(\tilde{r}_p)}(D)\tilde{F}_n$ on K and $\tilde{F}_n \xrightarrow{\mathcal{C}(K)} \tilde{F}_0$ as $n \to \infty$. Consequently, $(f_n^{(\beta)})$ is s-fundamental and we define $f^{(\beta)} = [f_n^{(\beta)}]$. By Remark 2.2, the definition is consistent and $f^{(\beta)} \in \mathcal{U}^{\{t\}}(\Omega)$. An analogous assertion holds in the Beurling case.

Let us discuss now the operations of multiplication and convolution by a function from $\mathcal{E}^*(\Omega)$. We consider only the Roumieu case. The Beurling case is similar.

Fix $\omega \in \mathcal{E}^{\{t\}}(\Omega)$ and let $f = [f_n] \in \mathcal{U}^{\{t\}}(\Omega)$. We will show that the sequence (ωf_n) is s-fundamental, so one can define $\omega f := [\omega f_n] \in \mathcal{E}^{\{t\}}(\Omega)$ and the definition is consistent, by Remark 2.2. For every $K \Subset \Omega$ there exist $P_{(r_p)} \in \mathcal{P}^{\{t\}}$, a sequence (F_n) of smooth functions and a continuous function F_0 in Ω with $\operatorname{supp} F_n \subset K_1$ $(n \in \mathbb{N}_0)$ such that $\omega f_n = \omega P_{(r_p)}(D)F_n$ in K and $F_n \xrightarrow{\mathcal{C}(K)} F_0$, $n \to \infty$. We can assume that ω is compactly supported multiplying it by a cut-of function equal to 1 on K. We have

$$\widehat{\omega}(\xi) \leqslant Ce^{-h|\xi|^{1/t}} \quad \text{and} \quad |P_{(r_p)}(\xi)\widehat{F_n}| \leqslant C_1 e^{c(|\xi|)^{1/t}}, \qquad \xi \in \mathbb{R}^d$$

for some constants $C > 0$, $C_1 > 0$, $h > 0$ and a subordinate function c, in view of (1.6). By (1.7), there exists a $P_{(\tilde{r}_p)} \in \mathcal{P}^{\{t\}}$, where $(\tilde{r}_p) \in \mathcal{R}$ with $r_p/\tilde{r}_p \downarrow 0$ as $p \to \infty$, such that $(\widehat{\omega} * (P_{(r_p)}\widehat{F_n}))/P_{(\tilde{r}_p)} \in L^1(\mathbb{R}^d)$. Now, defining

$$G_n := \kappa_K \mathcal{F}^{-1}\left(\frac{\widehat{\omega} * (P_{(r_p)}\widehat{F_n})}{P_{(\tilde{r}_p)}}\right), \quad n \in \mathbb{N}_0,$$

we have $\omega f_n = P_{(\tilde{r}_p)}(D)G_n$ on K $(n \in \mathbb{N})$, $\operatorname{supp} G_n \subset K_1$ $(n \in \mathbb{N}_0)$ and $G_n \xrightarrow{\mathcal{C}(K)} G_0$ as $n \to \infty$. Hence, (ωf_n) is a s-fundamental sequence on Ω.

Moreover, if $f = [f_n]$ and $\omega \in \mathcal{D}^{\{t\}}(\Omega)$, then $\omega * f = [\omega * f_n]$, since $\omega * P(D)F_n = P(D)(\omega * F_n)$ on every compact set $K \subset \mathbb{R}^d$ for $n \in \mathbb{N}$.

More generally, we have the following assertion: if (f_n) and (g_n) are s-fundamental sequences on $\mathbb{R}^d$ and $\operatorname{supp}(g_n) \subset K_0 \Subset \mathbb{R}^d$, then $(f_n * g_n)$ is a s-fundamental sequence on $\mathbb{R}^d$.

3. Sequences of s-ultradistributions

Definition 3.1. Let $f^m \in \mathcal{U}^*(\Omega)$ for $m \in \mathbb{N}_0$, i.e., $f^m = [(f_n^m)_n]$, where $(f_n^m)_n$ means a s-fundamental sequence representing f^m for $m \in \mathbb{N}_0$. We say that the sequence (f^m) converges to f^0 in $\mathcal{U}^*(\Omega)$ and write $f^m \xrightarrow{s} f^0$ as $m \to \infty$ or $s\text{-}\lim_{m\to\infty} f^m = f^0$ if for arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ there exist a $P \in \mathcal{P}^*$, smooth functions F_n^m on Ω $(m \in \mathbb{N}_0, n \in \mathbb{N})$ and continuous functions F^m on Ω $(m \in \mathbb{N}_0)$, all supported by K_1, such that

$$f_n^m = P(D)F_n^m \text{ on } K \ (n \in \mathbb{N},\ m \in \mathbb{N}_0);$$

$$F_n^m \xrightarrow{\mathcal{C}(K)} F_n^0 \text{ as } m \to \infty \text{ uniformly in } n \in \mathbb{N};$$

$$F_n^m \xrightarrow{\mathcal{C}(K)} F^m \text{ as } n \to \infty \ (m \in \mathbb{N}_0) \quad \text{and} \quad F^m \xrightarrow{\mathcal{C}(K)} F^0 \text{ as } m \to \infty.$$

We know that the above assumptions imply that

$$\lim_{m\to\infty} \lim_{n\to\infty} F_n^m = \lim_{n\to\infty} \lim_{m\to\infty} F_n^m \text{ in } \mathcal{C}(K).$$

Theorem 3.1. *If the limit $s\text{-}\lim_{m\to\infty} f^m$ exists, then it is unique.*

Proof. Assume that $f^m \xrightarrow{s} f$ and $f^m \xrightarrow{s} g$, where $f = [f_n], g = [g_n] \in \mathcal{U}^*(\Omega)$ and $f^m = [(f_n^m)_n] \in \mathcal{U}^*(\Omega)$ for $m \in \mathbb{N}$. We will prove that $f = g$.

Fix arbitrary $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$. According to Definition 3.1, there exist $P, \tilde{P} \in \mathcal{P}^*$, smooth functions F_n^m, F_n, G_n^m, G_n, on Ω and continuous functions F^m, F, G^m, G on Ω $(n, m \in \mathbb{N})$, all supported by K_1, such that

$$f_n^m = P(D)F_n^m,\ f_n = P(D)F_n \text{ on } K\ (n, m \in \mathbb{N});$$

$$F_n^m \xrightarrow{\mathcal{C}(K)} F_n \text{ as } m \to \infty \text{ uniformly in } n \in \mathbb{N};\ F_n \xrightarrow{\mathcal{C}(K)} F \text{ as } n \to \infty;$$

$$F_n^m \xrightarrow{\mathcal{C}(K)} F^m \text{ as } n \to \infty\ (m \in \mathbb{N});\ F^m \xrightarrow{\mathcal{C}(K)} F \text{ as } m \to \infty;$$

and, on the other hand,

$$f_n^m = \tilde{P}(D)G_n^m,\ g_n = \tilde{P}(D)G_n \text{ on } K\ (n, m \in \mathbb{N});$$

$$G_n^m \xrightarrow{\mathcal{C}(K)} G_n \text{ as } m \to \infty \text{ uniformly in } n \in \mathbb{N};\ G_n \xrightarrow{\mathcal{C}(K)} G \text{ as } n \to \infty;$$

$$G_n^m \xrightarrow{\mathcal{C}(K)} G^m \text{ as } n \to \infty\ (m \in \mathbb{N});\ G^m \xrightarrow{\mathcal{C}(K)} G \text{ as } m \to \infty.$$

By Proposition 2.3, there exist an ultradifferential operator $\bar{P} \in \mathcal{P}^*$, smooth functions $\bar{F}_n^m$, $\bar{G}_n^m$, $\bar{F}_n$, $\bar{G}_n$ and $H_n^m := \bar{F}_n^m - \bar{G}_n^m$, $H_n := \bar{F}_n - \bar{G}_n$ on Ω as well as continuous functions $\bar{F}^m, \bar{G}^m$, $\bar{F}$, $\bar{G}$ and $H^m := \bar{F}^m - \bar{G}^m$, $H := \bar{F} - \bar{G}$ on Ω $(n, m \in \mathbb{N})$, all supported by K_1, such that

$$0 = \bar{P}(D)H_n^m,\ f_n - g_n = \bar{P}(D)H_n \text{ on } K\ (n, m \in \mathbb{N});$$

$$H_n^m \xrightarrow{\mathcal{C}(K)} H_n \text{ as } m \to \infty \text{ uniformly in } n \in \mathbb{N};\ H_n \xrightarrow{\mathcal{C}(K)} G \text{ as } n \to \infty;$$

$$H_n^m \xrightarrow{\mathcal{C}(K)} H^m \text{ as } n \to \infty\ (m \in \mathbb{N}) \text{ and } H^m \xrightarrow{\mathcal{C}(K)} H \text{ as } m \to \infty.$$

Hence $H_n^m = 0$ on K° for $n, m \in \mathbb{N}$, by Proposition 2.2. This implies that $H = 0$, i.e., $\bar{F} = \bar{G}$ on K°. Since $K \Subset \Omega$ was fixed arbitrarily, we conclude that $f = g$ on Ω. ☐

3.1. Action on test functions from $\mathcal{D}^*(\Omega)$. Let $f = [f_n] \in \mathcal{U}^*(\Omega)$, where (f_n) is a s-fundamental sequence satisfying Definition 2.1, i.e., for arbitrary $K, K_1 \Subset \Omega$ with $K \Subset K_1^\circ$ there exist $P(D) \in \mathcal{P}^*$, smooth functions F_n $(n \in \mathbb{N})$ and a continuous function F_0 on Ω such that

$$f_n = P(D)F_n \text{ on } K\ (n \in \mathbb{N}), \quad \operatorname{supp} F_n \subset K_1\ (n \in \mathbb{N}_0) \tag{3.1}$$
$$\text{and } F_n \xrightarrow{\mathcal{C}(K)} F_0 \text{ as } n \to \infty.$$

By the action of f on the test functions from $\mathcal{D}^*(\Omega)$ we mean the mapping

$$\mathcal{D}^*(\Omega) \ni \varphi \mapsto (f, \varphi)_{\mathcal{U}^*(\Omega)} \in \mathbb{R}, \tag{3.2}$$

where

$$(f, \varphi)_{\mathcal{U}^*(\Omega)} := \lim_{n\to\infty} \int_K f_n(x)\varphi(x)dx = \int_K F_0(x)[P(-D)\varphi](x)dx. \tag{3.3}$$

If, beside (3.1), we have

$$f_n = \tilde{P}(D)\tilde{F}_n \text{ on } K\ (n \in \mathbb{N}),\ \operatorname{supp} \tilde{F}_n \subset K_1\ (n \in \mathbb{N}_0)$$
$$\text{and } \tilde{F}_n \xrightarrow{\mathcal{C}(K)} \tilde{F}_0 \text{ as } n \to \infty$$

for some $\tilde{P}(D) \in \mathcal{P}^*$, smooth functions $\tilde{F}_n$ $(n \in \mathbb{N})$ and a continuous function $\tilde{F}_0$ on Ω, then

$$\lim_{n\to\infty} \int_K f_n(x)\varphi(x)dx = \int_K \tilde{F}_0(x)[\tilde{P}(-D)\varphi](x)dx,$$

i.e., definition in (3.3) is consistent.

Clearly, 3.2 is a linear mapping. To prove that mapping (3.2) is sequentially continuous we need the following result from [**12**]: if $\varphi_n \to \varphi_0$ in $\mathcal{D}^*(\Omega)$, then $P(D)\varphi_n \to P(D)\varphi_0$ in $\mathcal{D}^*(\Omega)$ for every $P(D) \in \mathcal{P}^*$.

THEOREM 3.2. (a) *Let $f \in \mathcal{U}^*(\Omega)$ and $\varphi_m \to \varphi_0$ as $m \to \infty$ in $\mathcal{D}^*(\Omega)$. Then $(f, \varphi_m)_{\mathcal{U}^*(\Omega)} \to (f, \varphi_0)_{\mathcal{U}^*(\Omega)}$ as $m \to \infty$.*

(b) *Let $f^m \to f^0$ as $m \to \infty$ in $\mathcal{U}^*(\Omega)$. Then $(f^m, \varphi)_{\mathcal{U}^*(\Omega)} \to (f^0, \varphi)_{\mathcal{U}^*(\Omega)}$ as $m \to \infty$ for every $\varphi \in \mathcal{D}^*(\Omega)$.*

PROOF. (a) Supose that $f = [f_n]$, where (f_n) is a s-fundamental sequence, i.e., (3.1) holds for given $K_1 \Subset \Omega$ and $K \Subset K_1^\circ$ and suitable $P(D) \in \mathcal{P}^*$ and F_n $(n \in \mathbb{N}_0)$. By (3.3), we have

$$\lim_{m\to\infty}[(f, (\varphi_m - \varphi_0)_{\mathcal{U}^*(\Omega)}] = \lim_{m\to\infty} \int_K F_0(x)[P(-D)(\varphi_n - \varphi_0)](x)dx = 0.$$

(b) Let $\varphi \in \mathcal{D}^*_K$. Using the notation of Definition 3.1 we obtain

$$\begin{aligned}\lim_{m\to\infty}(f^m - f^0, \varphi)_{\mathcal{U}^*(\Omega)} &= \lim_{m\to\infty}\lim_{n\to\infty}(f_n^m - f_n, \varphi)_{\mathcal{U}^*(\Omega)} \\ &= \lim_{m\to\infty}\lim_{n\to\infty}\int_K (F_n^m - F_n)(x)[P(-D)\varphi](x)dx.\end{aligned}$$

Since $F_n^m \xrightarrow{\mathcal{C}(K)} F_n$ as $m \to \infty$ uniformly in $n \in \mathbb{N}$ and $F_n \xrightarrow{\mathcal{C}(K)} F^0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_K (F_n^m - F_n)(x)[P(-D)\varphi](x)dx = 0. \qquad \square$$

4. Tempered sequential ultradistributions

4.1. *t*-Tempered sequential ultradistributions.

Recall that

$$H^\alpha = \prod_{i=1}^d (-\partial^2/\partial {x_i}^2 + x_i^2)^{\alpha_i}, \quad \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$$

and that the symbol $\mathcal{P}^{2*}$ means either $\mathcal{P}^{(2t)}$ or $\mathcal{P}^{\{2t\}}$. Let $P \in \mathcal{P}^{2*}$. We will use ultradifferential operators of the form $P(H) = \sum_{|\alpha|=0}^\infty a_\alpha H^\alpha$ *of Beurling class* $(p!^{2t})$ (resp. *of Roumieu class* $\{p!^{2t}\}$) such that

$$\exists h > 0\ \exists C > 0\ (\text{resp.}\,\forall h > 0\ \exists C > 0)\ \forall \alpha \in \mathbb{N}_0^d \quad |a_\alpha| \leqslant \frac{Ch^{|\alpha|}}{(\alpha!)^{2t}};$$

in the Roumieu case the given condition is equivalent to

$$\exists (r_p) \in \mathcal{R}\ \exists C > 0\ \forall \alpha \in \mathbb{N}_0^d \quad |a_\alpha| \leqslant \frac{C}{(\alpha!)^{2t} R_{|\alpha|}},$$

where $R_{|\alpha|}$ is defined in (1.1).

DEFINITION 4.1. A sequence (f_n) of functions from $L^2(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ is called *t-fundamental* (of type $*$) in $\mathbb{R}^d$ if there exist an ultradifferential operator $P\in\mathcal{P}^{2*}$, functions $F_n\in L^2(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d)$ of the form $F_n=\sum_{|\alpha|=0}^\infty c_{\alpha,n}h_\alpha$ with $(c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d}\in l^2$ for $n\in\mathbb{N}$ and a function $F_0\in L^2(\mathbb{R}^d)$ such that

$$f_n=P(H)F_n \text{ on } \mathbb{R}^d \quad\text{and}\quad F_n\xrightarrow{2}F_0 \quad\text{as } n\to\infty, \tag{4.1}$$

where $P(H)F_n$ in (4.1) is meant in $L^2(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ as follows

$$\lim_{k\to\infty}P_k(H)F_n=\lim_{k\to\infty}\sum_{|\alpha|=0}^k a_\alpha H^\alpha F_n=\sum_{|\alpha|=0}^\infty c_{\alpha,n}P(2\alpha+1)h_\alpha,\quad (n\in\mathbb{N}).$$

We will need later the following assertion which enables us to change in representations of t-fundamental sequences an L^2-convergent sequence with a sequence which converges both in $L^2(\mathbb{R}^d)$ and uniformly on $\mathbb{R}^d$.

LEMMA 4.1. *Assume that $F_n\in L^2(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ for $n\in\mathbb{N}_0$ and $F_n\xrightarrow{2}F_0$ as $n\to\infty$. Denote $\tilde{F}_n:=\mathcal{F}^{-1}(G_n)$, where $G_n(\xi):=(1+|\xi|)^{-d/2}\widehat{F}_n(\xi)$ for $\xi\in\mathbb{R}^d$ and $n\in\mathbb{N}_0$. Then $(\tilde{F}_n)$ is a bounded sequence of smooth functions such that $\tilde{F}_n\xrightarrow{2}\tilde{F}_0$ as $n\to\infty$ and $\tilde{F}_n\xrightarrow{\mathcal{C}(\mathbb{R}^d)}\tilde{F}_0$ as $n\to\infty$.*

PROOF. It is clear that the sequence $(\tilde{F}_n)$ is bounded and $\tilde{F}_n\xrightarrow{2}\tilde{F}_0$, due to the Parseval identity. By the Schwarz inequality, we have

$$\|\tilde{F}_n-\tilde{F}_0\|_\infty=\|\mathcal{F}^{-1}(G_n-G_0)\|_\infty\leqslant\Big(\int_{\mathbb{R}^d}(1+|\xi|^2)^{-d}d\xi\Big)^{1/2}\|\widehat{F}_n-\widehat{F}_0\|_2,$$

which proves the uniform convergence. □

DEFINITION 4.2. Let (f_n) and (g_n) be t-fundamental sequences. We write $(f_n)\sim_1(g_n)$ if there exist sequences (F_n), (G_n) of functions $F_n,G_n\in L^2(\mathbb{R}^d)\cap\mathcal{C}^\infty(\mathbb{R}^d)$ $(n\in\mathbb{N})$, both convergent in $L^2(\mathbb{R}^d)$, and an operator $P\in\mathcal{P}^{2*}$ such that

$$f_n=P(H)F_n,\ g_n=P(H)G_n \text{ on } \mathbb{R}^d \quad\text{and}\quad F_n-G_n\xrightarrow{2}0 \text{ as } n\to\infty.$$

The following two assertions will be used in the sequel.

PROPOSITION 4.1. *If there exist $P\in\mathcal{P}^{2*}$, functions $F_n\in L^2(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ for $n\in\mathbb{N}$ and $F_0\in L^2(\mathbb{R}^d)$ such that $F_n=\sum_{|\alpha|=0}^\infty c_{\alpha,n}h_\alpha$ with $(c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d}\in l^2$ for $n\in\mathbb{N}_0$ and, moreover, $F_n\xrightarrow{2}F_0$ and $P(H)F_n\xrightarrow{2}0$ as $n\to\infty$, then $F_0=0$ on $\mathbb{R}^d$. In particular, if $F\in L^2(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d)$ and $P(H)F=0$ in $L^2(\mathbb{R}^d)$, then $F=0$ on $\mathbb{R}^d$.*

PROOF. By the assumption, we have $(c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d}\to(c_{\alpha,0})_{\alpha\in\mathbb{N}_0^d}$ in l^2 and

$$P(H)F_n=\sum_{|\alpha|=0}^\infty P(2\alpha+1)c_{\alpha,n}h_\alpha\xrightarrow{2}0$$

as $n\to\infty$. Consequently, $(c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d}\to0$ in l^2 as $n\to\infty$. Hence $F_0=0$. The particular case is clear. □

Proposition 4.2. *Assume that* $f_n = P_{(r_p)}(H)F_n = P_{(\tilde r_p)}(H)\tilde F_n$ *on* $\mathbb{R}^d$ $(n \in \mathbb{N})$ *and* $F_n \xrightarrow{2} F_0$, $\tilde F_n \xrightarrow{2} \tilde F_0$ *as* $n \to \infty$ *for some* $P_{(r_p)}, P_{(\tilde r_p)} \in \mathcal{P}^{\{2t\}}$ *and functions* $F_n, \tilde F_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ *and* $F_0, \tilde F_0 \in L^2(\mathbb{R}^d)$ *of the form*

$$F_n = \sum_{|\alpha|=0}^{\infty} c_{\alpha,n} h_\alpha, \ \tilde F_n = \sum_{|\alpha|=0}^{\infty} \tilde c_{\alpha,n} h_\alpha \quad (n \in \mathbb{N}_0),$$

where $(c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d}$, $(\tilde c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d} \in l^2$ *for* $n \in \mathbb{N}_0$. *Then there are a* $P_{(\bar r_p)} \in \mathcal{P}^{\{2t\}}$, *where* $(\bar r_p) \in \mathcal{R}$ *with* $r_p/\bar r_p \downarrow 0$ *and* $\tilde r_p/\bar r_p \downarrow 0$ *as* $p \to \infty$, *such that*

$$\left(\frac{P_{(r_p)}(2\alpha+1)}{P_{(\bar r_p)}(2\alpha+1)}\right)_{\alpha\in\mathbb{N}_0^d} \in l^\infty, \quad \left(\frac{P_{(\tilde r_p)}(2\alpha+1)}{P_{(\bar r_p)}(2\alpha+1)}\right)_{\alpha\in\mathbb{N}_0^d} \in l^\infty, \tag{4.2}$$

and functions $G_n, \tilde G_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ *and* $G_0, \tilde G_0 \in L^2(\mathbb{R}^d)$ *of the form*

$$G_n = \sum_{|\alpha|=0}^{\infty} \frac{P_{(r_p)}(2\alpha+1)}{P_{(\bar r_p)}(2\alpha+1)} c_{\alpha,n} h_\alpha, \quad \tilde G_n = \sum_{|\alpha|=0}^{\infty} \frac{P_{(\tilde r_p)}(2\alpha+1)}{P_{(\bar r_p)}(2\alpha+1)} \tilde c_{\alpha,n} h_\alpha$$

for $n \in \mathbb{N}_0$, *satisfying the conditions*

$$f_n = P_{(\bar r_p)}(H)G_n = P_{(\bar r_p)}(H)\tilde G_n \quad \text{on } \mathbb{R}^d \quad (n \in \mathbb{N})$$

and

$$G_n \xrightarrow{2} G_0, \quad \tilde G_n \xrightarrow{2} \tilde G_0 \quad \text{as } n \to \infty. \tag{4.3}$$

Moreover, $G_n = \tilde G_n$ *on* $\mathbb{R}^d$ *for* $n \in \mathbb{N}_0$.

The same holds in the Beurling case with an appropriate notation.

Proof. The existence of $P_{(\bar r_p)}(H)$ follows from (1.8). It is clear that $H^\beta h_\alpha = (2\alpha+1)^\beta h_\alpha$ for $\alpha, \beta \in \mathbb{N}_0^d$. By 4.2, we have

$$G_n = \sum_{|\alpha|=0}^{\infty} \frac{P_{(r_p)}(2\alpha+1)}{P_{(\bar r_p)}(2\alpha+1)} c_{\alpha,n} h_\alpha = P_{(r_p)}(H) \sum_{|\alpha|=0}^{\infty} \frac{c_{\alpha,n} h_\alpha}{P_{(\bar r_p)}(2\alpha+1)}$$

for $n \in \mathbb{N}_0$ and a similar representation holds for $\tilde G_n$ $(n \in \mathbb{N}_0)$. Hence

$$\begin{aligned} P_{(\bar r_p)}(H)G_n &= P_{(\bar r_p)}(H)P_{(r_p)}(H) \sum_{|\alpha|=0}^{\infty} \frac{c_{\alpha,n} h_\alpha}{P_{(\bar r_p)}(2\alpha+1)} \\ &= P_{(r_p)}(H)P_{(\bar r_p)}(H) \sum_{|\alpha|=0}^{\infty} \frac{c_{\alpha,n} h_\alpha}{P_{(\bar r_p)}(2\alpha+1)} = P_{(r_p)}(H)F_n = f_n \end{aligned}$$

and, similarly, $P_{(\bar r_p)}(H)\tilde G_n = f_n$ on $\mathbb{R}^d$ for$n \in \mathbb{N}$. We deduce from (4.2) that G_n and $\tilde G_n$ are smooth L^2 functions and (4.3) holds. By Proposition 4.1, we conclude that $G_n = \tilde G_n$ on $\mathbb{R}^d$ for $n \in \mathbb{N}$ and, consequently, $G_0 = \tilde G_0$. ☐

It is clear that the relation $\sim_1$ is reflexive and symmetric. We shall prove that $\sim_1$ is transitive.

Proposition 4.3. *Relation* $\sim_1$ *is transitive.*

PROOF. We prove the assertion in the Roumieu case; the proof in the Beurling case is similar. Let $(f_n) \sim_1 (g_n)$ and $(g_n) \sim_1 (h_n)$. Then there exist $P_{(r_p)}, P_{(\tilde{r}_p)} \in \mathcal{P}^{\{2t\}}$ and sequences (F_n), (G_n), (G_n^1), (H_n) of functions in $L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$, all convergent in $L^2(\mathbb{R}^d)$, such that

$$f_n = P_{(r_p)}(H)F_n, \quad g_n = P_{(r_p)}(H)G_n = P_{(\tilde{r}_p)}(H)G_n^1, \quad h_n = P_{(\tilde{r}_p)}(H)H_n$$

on $\mathbb{R}^d$ for $n \in \mathbb{N}$ and $F_n - G_n \xrightarrow{2} 0$, $G_n^1 - H_n \xrightarrow{2} 0$ as $n \to \infty$.

By Proposition 4.2, there is a $P_{(\bar{r}_p)} \in \mathcal{P}^{\{2t\}}$, where $(\bar{r}_p) \in \mathcal{R}$ with $r_p/\bar{r}_p \downarrow 0$ and $\tilde{r}_p/\bar{r}_p \downarrow 0$ as $p \to \infty$, and there are suitable functions $\bar{F}_n, \bar{G}_n, \bar{G}_n^1, \bar{H}_n$ on $\mathbb{R}^d$ such that $f_n = P_{(\bar{r}_p)}(H)\bar{F}_n$, $g_n = P_{(\bar{r}_p)}(H)\bar{G}_n = P_{(\bar{r}_p)}(H)\bar{G}_n^1$, $h_n = P_{(\bar{r}_p)}(H)\bar{H}_n$ on $\mathbb{R}^d$ for $n \in \mathbb{N}$ and $\bar{F}_n - \bar{G}_n \xrightarrow{2} 0$, $\bar{G}_n^1 - \bar{H}_n \xrightarrow{2} 0$ as $n \to \infty$. Putting $\Phi_n := \bar{G}_n - \bar{G}_n^1 + \bar{H}_n$, we have $h_n = P_{(\bar{r}_p)}(H)\Phi_n$ on $\mathbb{R}^d$ and, moreover, $\bar{F}_n - \Phi_n \xrightarrow{2} 0$ as $n \to \infty$. Hence $(f_n) \sim_1 (h_n)$, i.e., $\sim_1$ is transitive. □

DEFINITION 4.3. Let (f_n) be a t-fundamental sequence (of type $*$) in an open set $\Omega \subset \mathbb{R}^d$. The class of all t-fundamental sequences equivalent to (f_n) with respect to the relation $\sim_1$ is called a *t-tempered sequential ultradistributon* or, shortly, *t-ultradistribution* (of type $*$) and denoted by $f = [f_n]$. The set of all t-ultradistributions (of type $*$) in $\Omega \subset \mathbb{R}^d$ is denoted by $\mathcal{T}^* = \mathcal{T}^*(\mathbb{R}^d)$.

REMARK 4.1. As in the space $\mathcal{U}^*(\Omega)$ (see Section 2.2) we can consider appropriate operations in $\mathcal{T}^*$. But we do not go into details, remarking only that the operations of addition and multiplication by a constant are well defined in this set, i.e., $\mathcal{T}^*$ is a vector space.

EXAMPLE 4.1. Let $F_0 \in L^2(\mathbb{R}^d)$ and let (δ_n) be a delta-sequence in $\mathcal{D}^*(\mathbb{R}^d)$. Define $F_n := F_0 * \delta_n$ for $n \in \mathbb{N}$. Then (F_n) is a sequence of smooth functions in $L^2(\mathbb{R}^d)$ which is t-fundamental in both the Beurling and Roumieu cases.

EXAMPLE 4.2. Let $f = [f_n] \in \mathcal{T}^*$ be of the form $f_n = P(H)F_n$, where $P \in \mathcal{P}^{2*}$ and $F_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ for $n \in \mathbb{N}$. Moreover, assume that $F_n \xrightarrow{2} F_0$ as $n \to \infty$ for some function $F_0 \in L^2(\mathbb{R}^d)$. Define $\tilde{f}_n := P(H)(F_n * \delta_n)$ for $n \in \mathbb{N}$. Then $(\tilde{f}_n)$ is a t-fundamental sequence in $\mathbb{R}^d$ and $(f_n) \sim_1 (\tilde{f}_n)$.

DEFINITION 4.4. Let $f^m \in \mathcal{T}^*$ for $m \in \mathbb{N}_0$, i.e., $f^m = [(f_n^m)_n]$, where $(f_n^m)_n$ means a t-fundamental sequence representing f^m for $m \in \mathbb{N}_0$. We say that the sequence (f^m) converges to f^0 in $\mathcal{T}^*$ and write $f^m \xrightarrow{t} f^0$ as $m \to \infty$ or t-$\lim_{m\to\infty} f^m = f^0$ if there exist a $P \in \mathcal{P}^{2*}$, smooth functions $F_n^m \in L^2(\mathbb{R}^d)$ $(m \in \mathbb{N}_0, n \in \mathbb{N})$ and functions $F^m \in L^2(\mathbb{R}^d)$ $(m \in \mathbb{N}_0)$ such that

$$f_n^m = P(H)F_n^m \text{ on } \mathbb{R}^d \ (n \in \mathbb{N},\ m \in \mathbb{N}_0);$$

$$F_n^m \xrightarrow{2} F_n^0 \text{ as } m \to \infty,\ \text{uniformly in } n \in \mathbb{N};$$

$$F_n^m \xrightarrow{2} F^m \text{ as } n \to \infty \ (m \in \mathbb{N}_0) \text{ and } \ F^m \xrightarrow{2} F^0 \text{ as } m \to \infty.$$

Assumptions in the definition imply that

$$\lim_{m\to\infty} \lim_{n\to\infty} F_n^m = \lim_{n\to\infty} \lim_{m\to\infty} F_n^m \text{ in } \ L^2(\mathbb{R}^d).$$

THEOREM 4.1. *If the limit t-$\lim_{m\to\infty} f^m$ exists, then it is unique.*

PROOF. In the Roumieu case, let $f^m \xrightarrow{s} f$ and $f^m \xrightarrow{s} g$, where $f^m = [(f_n^m)_n] \in \mathcal{T}^{\{t\}}$ for $m \in \mathbb{N}$ and $f = [f_n], g = [g_n] \in \mathcal{T}^{\{t\}}$. We will show that $f = g$.

By Definition 4.4, there exist ultradifferential operators $P_{(r_p)}, P_{(\tilde{r}_p)} \in \mathcal{P}^{\{2t\}}$ with $(r_p), (\tilde{r}_p) \in \mathcal{R}$, smooth functions $F_n^m, F_n, G_n^m, G_n \in L^2(\mathbb{R}^d)$ and functions $F^m, F, G^m, G \in L^2(\mathbb{R}^d)$ $(n, m \in \mathbb{N})$ such that

$$f_n^m = P_{(r_p)}(H)F_n^m, \quad f_n = P_{(r_p)}(H)F_n \text{ on } \mathbb{R}^d \ (n, m \in \mathbb{N});$$

$$F_n^m \xrightarrow{2} F_n \quad \text{as} \quad m \to \infty \text{ uniformly in } n \in \mathbb{N};\ F_n \xrightarrow{2} F \text{ as } n \to \infty;$$

$$F_n^m \xrightarrow{2} F^m \quad \text{as} \quad n \to \infty \ (m \in \mathbb{N});\ F^m \xrightarrow{2} F \text{ as } m \to \infty$$

and, on the other hand,

$$f_n^m = P_{(\tilde{r}_p)}(H)G_n^m,\ g_n = P_{(\tilde{r}_p)}(H)G_n \text{ on } \mathbb{R}^d \ (n, m \in \mathbb{N});$$

$$G_n^m \xrightarrow{2} G_n \quad \text{as} \quad m \to \infty \text{ uniformly in } n \in \mathbb{N};\ G_n \xrightarrow{2} G \text{ as } n \to \infty;$$

$$G_n^m \xrightarrow{2} G^m \quad \text{as} \quad n \to \infty \ (m \in \mathbb{N});\ G^m \xrightarrow{2} G \text{ as } m \to \infty.$$

By Proposition 4.2, there exist a $P_{(\bar{r}_p)} \in \mathcal{P}^{\{2t\}}$, where $(\bar{r}_p) \in \mathcal{R}$, with $r_p/\bar{r}_p \downarrow 0$ and $\tilde{r}_p/\bar{r}_p \downarrow 0$ as $p \to \infty$, smooth functions $\bar{F}_n^m$, $\bar{G}_n^m$, $\bar{F}_n$, $\bar{G}_n$ in $L^2(\mathbb{R}^d)$ and functions $\bar{F}^m, \bar{G}^m$, $\bar{F}$, $\bar{G}$ in $L^2(\mathbb{R}^d)$ for $n, m \in \mathbb{N}$ such that the following conditions are satisfied

$$0 = P_{(\bar{r}_p)}(H)\Phi_n^m, \quad f_n - g_n = \bar{P}(D)(H)\Phi_n \text{ on } \mathbb{R}^d \quad (n, m \in \mathbb{N});$$

$$\Phi_n^m \xrightarrow{2} \Phi_n \quad \text{as} \quad m \to \infty \text{ uniformly in } n \in \mathbb{N}; \quad \Phi_n \xrightarrow{2} \Phi \text{ as } n \to \infty;$$

$$\Phi_n^m \xrightarrow{2} \Phi^m \quad \text{as} \quad n \to \infty \ (m \in \mathbb{N}) \text{ and } \Phi^m \xrightarrow{2} \Phi \text{ as } m \to \infty,$$

where $\Phi_n^m := \bar{F}_n^m - \bar{G}_n^m$, $\Phi_n := \bar{F}_n - \bar{G}_n$, $\Phi^m := \bar{F}^m - \bar{G}^m$ for $n, m \in \mathbb{N}$ and $\Phi := \bar{F} - \bar{G}$. Hence $P_{(\bar{r}_p)}(H)\Phi_n = \lim_{m\to\infty} P_{(\bar{r}_p)}(H)(\Phi_n^m - \Phi_n) = 0$ on $\mathbb{R}^d$ $(n \in \mathbb{N})$. As in Proposition 4.1, we conclude that $\bar{F} = \bar{G}$ on $\mathbb{R}^d$ and thus $f = g$.

The proof in the Beurling case is similar. □

We need the following assertion:

LEMMA 4.2. *Let φ be a function in $\mathcal{D}^*(\Omega)$ equal to 1 on $B(0, 1/2)$ and let $\varphi_m(x) := \varphi(x/m)$ for $x \in \mathbb{R}^d$ and $m \in \mathbb{N}$. If $f = [f_n]$ is a t-ultradistribution, then $f\varphi_m \xrightarrow{t} f$ as $m \to \infty$.*

PROOF. Let $P \in \mathcal{P}^{2*}$ (F_n) be a sequence of functions corresponding to (f_n) according to Definition 4.1. Put $F_n^m := \varphi_m F_n$ for $m, n \in \mathbb{N}$. Then $f^m = P(H)\varphi_m F_n \in \mathcal{T}^*$, because $\varphi_m F_n \xrightarrow{2} \varphi_m F_0$ as $n \to \infty$ for every fixed m. Since

$$\|F_n^m - F_n\|_2^2 \leqslant \int_{|x| > \frac{m}{2}} |\varphi^2(x/m) - 1|\, |F_n(x)|^2 dx, \qquad n, m \in \mathbb{N},$$

we have $F_n^m \xrightarrow{2} F_n$ as $m \to \infty$ uniformly in n. This completes the proof. □

COROLLARY 4.1. *For every t-ultradistribution f^0 there exists a sequence (f^m) of t-ultradistributions such that $f^m \xrightarrow{t} f^0$ as $m \to \infty$.*

REMARK 4.2. As in the case of s-fundamental sequences commented in Remark 2.4, every t-fundamental sequence (f_n) for which (4.1) holds can be identified with the formal representation $f = P(H)F_0$, since any other representation defines the same element of $\mathcal{T}^*$ (see (4.15) of Subsection 4.3).

4.2. $\tilde{t}$-Tempered sequential ultradistributions. In this subsection we develop a sequential theory of tempered ultradistributions closely related to the sequential approach of sections 2 and 3.

DEFINITION 4.5. A sequence (f_n) of smooth functions is called *$\tilde{t}$-fundamental* (of type $*$) in $\mathbb{R}^d$ if there exist an ultradifferential operator $P \in \mathcal{P}^*$, a function $P_1 \in \mathcal{P}_u^*$ and functions $F_0 \in L^2(\mathbb{R}^d)$ and $F_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ for $n \in \mathbb{N}$ such that

$$f_n = P(D)(P_1 F_n) \text{ on } \mathbb{R}^d \quad \text{and} \quad F_n \xrightarrow{2} F_0 \text{ as } n \to \infty. \tag{4.4}$$

The action of $P(D)$ on $P_1 F_n$ is understood as in Section 2; it is the limit of $\sum_{|\alpha|=0}^m a_\alpha D^\alpha (P_1 F_n)(x)$ as $m \to \infty$ for $x \in \mathbb{R}^d$. The following assertion will enable us to transfer one form of a fundamental sequence into another one.

For a given $h > 0$ and a subordinate function c denote for simplicity

$$E_{\pm h}(u) := e^{\pm h u^{1/t}} \quad \text{and} \quad E^{\pm c}(u) := e^{\pm c(u)^{1/t}} \quad \text{for } u \geqslant 0.$$

LEMMA 4.3. *Let P_1, F_n and F_0 be as in* (4.4). *For P_1 assume* (1.4) *in the Beurling case (resp.* (1.6) *in the Roumieu case) in the form*

$$|P_1(x)| \leqslant C E_{h_1}(|x|) \quad (\textit{resp. } |P_1(x)| \leqslant C E^{c_1}(|x|)), \qquad x \in \mathbb{R}^d,$$

where $h_1 > 0$ is a constant (resp. c_1 is a subordinate function). Then

(a) *for a given $h > 0$ (resp. for a given subordinate function c) there exists $r > 0$ (resp. $(r_p) \in \mathcal{R}$) such that*

$$\big|[E_h \mathcal{F}^{-1}(P_1/P_r)](x)\big| < \infty \quad (\textit{resp. } \big|[E^c \mathcal{F}^{-1}(P_1/P_{(r_p)})](x)\big| < \infty)$$

for every $x \in \mathbb{R}^d$.

(b) *there exists $r > 0$ (resp. $(r_p) \in \mathcal{R}$) such that*

$$E_{-2h_1}\big[(P_1 F_n - P_1 F_0) * \mathcal{F}^{-1}(P_1/P_r)\big] \xrightarrow{2} 0$$
$$\big(\textit{resp. } E^{-c}\big[(P_1 F_n - P_1 F_0) * \mathcal{F}^{-1}(P_1/P_{(r_p)})\big] \xrightarrow{2} 0\big)$$

as $n \to \infty$, where c is a subordinate function such that $2c_1^{1/t} \leqslant c^{1/t}$.

PROOF. We will prove the assertions only in the Roumieu case; the proof in the Beurling case is similar.

To prove (a) choose $(r_p^0) \in \mathcal{R}$ such that $E^c(|x|) \leqslant P_{(r_p^0)}(x)$ for $x \in \mathbb{R}^d$. The proof will be completed if we show that there exists $(r_p) \in \mathcal{R}$ such that

$P_{(r_p^0)}(D)(P_1/P_{(r_p)}) \in L^1(\mathbb{R}^d)$, since then the function $P_{(r_p^0)}\mathcal{F}^{-1}(P_1/P_{(r_p)})$ belongs to $L^\infty(\mathbb{R}^d)$. For all $x \in \mathbb{R}^d$, we have

$$P_{(r_p^0)}(D)(P_1/P_{(r_p)})(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha \sum_{0\leqslant\gamma\leqslant\alpha} \binom{\alpha}{\gamma}[D^{\alpha-\gamma}P_1 D^\gamma(1/P_{(r_p)})](x), \tag{4.5}$$

where

$$|a_\alpha| \leqslant \frac{C}{(\alpha!)^t R^0_{|\alpha|}} \quad \Big(R^0_{|\alpha|} := \prod_{i\leqslant|\alpha|} r_i^0\Big), \qquad \alpha \in \mathbb{N}_0^d \tag{4.6}$$

for some $C > 0$. By Lemma 1.2, there exist a subordinate function $\tilde{c}_1$ (related to P_1) and a subordinate function $\tilde{c}_{(r_p)}$ (suitably chosen to fulfill the inequality $\tilde{c}_1(|x|) \leqslant \tilde{c}_{(r_p)}(|x|)$ for $x \in \mathbb{R}^d$) such that

$$|D^{\alpha-\gamma}P_1(x)| \leqslant \frac{C(\alpha-\gamma)!}{\varepsilon^{|\alpha-\gamma|}}E^{\tilde{c}_1}(|x|), \qquad x \in \mathbb{R}^d,\ \alpha,\gamma \in \mathbb{N}_0^d,\ \gamma \leqslant \alpha; \tag{4.7}$$

$$|D^\gamma(1/P_{(r_p)})(x)| \leqslant \frac{C\gamma!}{\varepsilon^{|\gamma|}}E^{-\tilde{c}_{(r_p)}}, \qquad x \in \mathbb{R}^d,\ \gamma \in \mathbb{N}_0^d, \tag{4.8}$$

By (4.5), (4.6), (4.7) and (4.8), we get

$$\big|P_{(r_p^0)}(D)(P_1/P_{(r_p)})(x)\big| \leqslant C^2\bigg(\sum_{|\alpha|=0}^{\infty}\frac{(2/\varepsilon)^\alpha}{(\alpha!)^{t-1}R^0_{|\alpha|}}\bigg)E^{\tilde{c}_1}(|x|)E^{-\tilde{c}_{(r_p)}}(|x|).$$

This proves (a), since the sum on the right-hand side is finite.

To prove (b) note that, by Lemma 4.1, we can assume that (F_n) is a bounded sequence of smooth functions in $L^2(\mathbb{R}^d)$. By the assumption and (1.5), there exists a suitable subordinate function c_0 (depending on $(r_p) \in \mathcal{R}$) satisfying

$$C_0 := \int_{\mathbb{R}^d} E^{4c_1}(|s|)E^{-2c_0}(|s|)ds < \infty \tag{4.9}$$

and there is a constant $C > 0$ such that

$$\begin{aligned}\big|\big[(P_1F_n - P_1F_0)*\mathcal{F}^{-1}\big(P_1/P_{(r_p)}\big)\big](x)\big| &\\ &\leqslant C\|F_n - F_0\|_2\bigg(\int_{\mathbb{R}^d}E^{2c_1}(|x-s|)E^{-2c_0}(|s|)ds\bigg)^{1/2}\\ &\leqslant CC_0\|F_n - F_0\|_2E^{2c_1}(|x|)\end{aligned}$$

for all $x \in \mathbb{R}^d$, due to (4.9). Hence assertion (b) easily follows. □

Definition 4.6. Let (f_n) and (g_n) be $\tilde{t}$-fundamental sequences. We write $(f_n) \sim_2 (g_n)$, if there exist sequences (F_n), (G_n) of functions $F_n, G_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ $(n \in \mathbb{N})$, both convergent in $L^2(\mathbb{R}^d)$, an operator $P \in \mathcal{P}^*$ and a function $P_1 \in \mathcal{P}_u^*$ such that

$$f_n = P(D)(P_1F_n), \quad g_n = P(D)(P_1G_n) \quad \text{on } \mathbb{R}^d$$

and $F_n - G_n \xrightarrow{2} 0$ as $n \to \infty$.

PROPOSITION 4.4. *If the assumptions of Definition 4.6 are satisfied for* (f_n) *and if* $P(D)(P_1F_n) \xrightarrow{2} 0$*, then* $F_n \xrightarrow{2} 0$ *as* $n \to \infty$*. In particular, if* $P(D)(P_1F) = 0$*, then* $F = 0$*.*

PROOF. Using the Fourier transform, we have $P\widehat{P_1F_n} \xrightarrow{2} 0$. The same is true for $\widehat{P_1F_n}$, then for P_1F_n and, finally, for F_n. The particular case is clear. □

The key assertion in this subsection is related to the change of representative of some $\tilde{t}$-ultradistribution (see Definition 4.7 below). We consider the Roumieu case.

Assume that $P_{(r_p)}, P_{(\tilde{r}_p)} \in \mathcal{P}^{\{t\}}$, $P^1_{(r_p)}, P^2_{(\tilde{r}_p)} \in \mathcal{P}_u^{\{t\}}$ and $(F_n), (\tilde{F}_n)$ are sequences of functions in $L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$, both convergent in $L^2(\mathbb{R}^d)$, satisfying Definition 4.5, i.e.,

$$f_n = P_{(r_p)}(D)(P^1_{(r_p)}F_n) = P_{(\tilde{r}_p)}(D)(P^2_{(\tilde{r}_p)}\tilde{F}_n) \quad \text{on } \mathbb{R}^d \quad (n \in \mathbb{N}), \tag{4.10}$$

so that

$$\max\left\{|P_{(r_p)}(x)|, |P_{(\tilde{r}_p)}(x)|, |P^1_{(r_p)}(x)|, |P^2_{(\tilde{r}_p)}(x)|\right\} \leqslant Ce^{c(|x|)^{1/t}} \tag{4.11}$$

for a suitable subordinate function c and $x \in \mathbb{R}^d$. We may assume, without loosing generality, that $P^1_{(r_p)} = P^2_{(\tilde{r}_p)} = P_{(\bar{r}_p)}$. Actually one can use in (4.10) instead of $P^1_{(r_p)}(x)F_n(x)$ and $P^2_{(\tilde{r}_p)}(x)\tilde{F}_n(x)$, the following expressions

$$P_{(\bar{r}_p)}(x)\frac{P^1_{(r_p)}(x)F_n(x)}{P_{(\bar{r}_p)}(x)} \quad \text{and} \quad P_{(\bar{r}_p)}(x)\frac{P^2_{(\tilde{r}_p)}(x)\tilde{F}_n(x)}{P_{(\bar{r}_p)}(x)}, \qquad x \in \mathbb{R}^d,$$

respectively, where the sequence $(\bar{r}_p) \in \mathcal{R}$ is increasing slowly enough to guarantee L^2-convergence of the sequences

$$\left(\frac{P^1_{(r_p)}(x)F_n(x)}{P_{(\bar{r}_p)}(x)}\right), \quad \left(\frac{P^2_{(\tilde{r}_p)}(x)\tilde{F}_n(x)}{P_{(\bar{r}_p)}(x)}\right).$$

The above remarks concern the following proposition.

PROPOSITION 4.5. *Assume that* $P_{(r_p)}, P_{(\tilde{r}_p)} \in \mathcal{P}^{\{t\}}$, $P_{(\bar{r}_p)} \in \mathcal{P}_u^{\{t\}}$ *and functions* $F_n, \tilde{F}_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ *for* $n \in \mathbb{N}$*, and* $F_0, \tilde{F}_0 \in L^2(\mathbb{R}^d)$ *satisfy Definition* 4.5*, i.e.,* $f_n = P_{(r_p)}(D)(P_{(\bar{r}_p)}F_n) = P_{(\tilde{r}_p)}(D)(P_{(\bar{r}_p)}\tilde{F}_n)$ *on* $\mathbb{R}^d$ $(n \in \mathbb{N})$ *and* $F_n \xrightarrow{2} F_0$*,* $\tilde{F}_n \xrightarrow{2} \tilde{F}_0$ *as* $n \to \infty$*, so that* (4.11) *holds. Then there exist* $P_{(r^0_p)} \in \mathcal{P}^{\{t\}}$ *with* $(r^0_p) \in \mathcal{R}$ *and functions* $\bar{F}_n \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ *for* $n \in \mathbb{N}$ *and* $\bar{F}_0 \in L^2(\mathbb{R}^d)$ *such that*

$$f_n = P_{(r^0_p)}(D)(P_{(\bar{r}_p)}\bar{F}_n) \quad \textit{on } \mathbb{R}^d \quad \textit{and} \quad \bar{F}_n \xrightarrow{2} \bar{F}_0 \quad \textit{as} \quad n \to \infty.$$

PROOF. We know that there exists $(r^0_p) \in \mathcal{R}$ such that if we put

$$G_n := \mathcal{F}^{-1}(F_n/P_{(r^0_p)}) \quad \text{and} \quad \tilde{G}_n := \mathcal{F}^{-1}(\tilde{F}_n/P_{(r^0_p)}) \quad \text{on } \mathbb{R}^d \quad (n \in \mathbb{N}_0),$$

then $f_n = P_{(r^0_p)}(D)(P_{(\bar{r}_p)}G_n) = P_{(r^0_p)}(D)(P_{(\bar{r}_p)}\tilde{G}_n)$ on $\mathbb{R}^d$ $(n \in \mathbb{N})$ and, moreover, $G_n \xrightarrow{2} G_0$ and $\tilde{G}_n \xrightarrow{2} \tilde{G}_0$ as $n \to \infty$.

By Proposition 4.4, $G_n = \tilde{G}_n$ on $\mathbb{R}^d$ for $n \in \mathbb{N}_0$, so the assertion follows for the functions $F_n := G_n = \tilde{G}_n$ $(n \in \mathbb{N}_0)$. □

To prove that $\sim_2$, introduced in Definition 4.6, is an equivalence relation, it suffices to show that it is transitive.

PROPOSITION 4.6. *The relation $\sim_2$ is transitive.*

We omit the proof of the proposition, because it is similar to the proofs of Propositions 2.4 and 4.3. One has to use appropriate representations as it was demonstrated in those proofs.

DEFINITION 4.7. Let (f_n) be a $\tilde{t}$-fundamental sequence (of type $*$) in $\mathbb{R}^d$. The class of all $\tilde{t}$-fundamental sequences equivalent to (f_n) with respect to the relation $\sim_2$ is called a *$\tilde{t}$-tempered sequential ultradistributon* or, shortly, *$\tilde{t}$-ultradistribution* (of type $*$) and denoted by $f = [f_n]$. The set of all $\tilde{t}$-ultradistributions (of type $*$) in $\mathbb{R}^d$ is denoted by $\tilde{\mathcal{T}}^* = \tilde{\mathcal{T}}^*(\mathbb{R}^d)$.

We give the convergence structure in $\tilde{\mathcal{T}}^*$.

DEFINITION 4.8. Let $f^m \in \tilde{\mathcal{T}}^*$ for $m \in \mathbb{N}_0$, i.e., $f^m = [(f_n^m)_n]$, where $(f_n^m)_n$ means a $\tilde{t}$-fundamental sequence representing f^m for $m \in \mathbb{N}_0$. We say that the sequence (f^m) converges to f^0 in $\tilde{\mathcal{T}}^*$ and write $f^m \xrightarrow{\tilde{t}} f^0$ as $m \to \infty$ or $\tilde{t}$-$\lim_{m\to\infty} f^m = f^0$ if there exist $P \in \mathcal{P}^*$ and $P_1 \in \mathcal{P}_u^*$, smooth functions $F_n^m \in L^2(\mathbb{R}^d)$ $(m \in \mathbb{N}_0, n \in \mathbb{N})$ and functions $F^m \in L^2(\mathbb{R}^d)$ $(m \in \mathbb{N}_0)$ such that

$$f_n^m = P(D)(P_1 F_n^m) \text{ on } \mathbb{R}^d \quad (n \in \mathbb{N},\ m \in \mathbb{N}_0);$$

$$F_n^m \xrightarrow{2} F_n^0 \text{ as } m \to \infty, \text{ uniformly in } n \in \mathbb{N};$$

$$F_n^m \xrightarrow{2} F^m \text{ as } n \to \infty\ (m \in \mathbb{N}_0) \quad \text{and} \quad F^m \xrightarrow{2} F^0 \text{ as } m \to \infty.$$

The assumptions of the definition imply that

$$\lim_{m\to\infty} \lim_{n\to\infty} F_n^m = \lim_{n\to\infty} \lim_{m\to\infty} F_n^m \text{ in } L^2(\mathbb{R}^d).$$

By suitable modifications of the proofs of Proposition 4.4 and Theorem 4.1, one can prove the following theorem

THEOREM 4.2. *If the limit $\tilde{t}$-$\lim_{n\to\infty} f^n$ exists, then it is unique.*

By [**25**] we know that every $f \in \mathcal{S}'^*(\mathbb{R}^d)$ can be identified with the formal representation $f = P(D)(P_1F_0)$, where $P \in \mathcal{P}^*$, $P_1 \in \mathcal{P}_u^*$ and $F_0 \in L^2(\mathbb{R}^d)$ is a function of the form $F_0 = \sum_{|\alpha|=0}^{\infty} c_{\alpha,0} h_\alpha \in L^2(\mathbb{R}^d)$ with $(c_{\alpha,0})_{\alpha\in\mathbb{N}_0^d} \in l^2$.

Actually, we need the following assertion:

LEMMA 4.4. *An f is an element of the space $\mathcal{S}'^*(\mathbb{R}^d)$ if and only if*

$$f = P(D)(P_1F_0) \tag{4.12}$$

for some $P \in \mathcal{P}^*$, $P_1 \in \mathcal{P}_u^*$ and $F_0 \in L^2(\mathbb{R}^d)$ of the form

$$F_0 = \sum_{|\alpha|=0}^{\infty} c_{\alpha,0} h_\alpha \quad \textit{with} \quad (c_{\alpha,0}) \in l^2, \tag{4.13}$$

that is

$$(f,\varphi)_{\mathcal{S}'^*} = \int_{\mathbb{R}^d} F_0(x)P_1(x)P(-D)\varphi(x)dx, \qquad \varphi \in \mathcal{S}^*(\mathbb{R}^d).$$

The proof is a consequence of the well known representation theorem based on the Hahn–Banach theorem and assertions (a) and (b) of Lemma 1.3.

We may formulate the above assertion in the form of the proposition which will be needed in Section 5.

PROPOSITION 4.7. *Let* $f \in \mathcal{S}'^*(\mathbb{R}^d)$ *be of the form* (4.12)–(4.13). *Then the sequence* (f_n)*, where* $f_n := (P(D)(P_1F_n)$ *and* $F_n := \sum_{|\alpha|=0}^n c_{\alpha,0}h_\alpha$ *for* $n \in \mathbb{N}$*, is* $\tilde{t}$*-fundamental and determines* $\tilde{f} = [f_n] \in \tilde{\mathcal{T}}^*$*.*

Conversely, if $\tilde{f} = [f_n] \in \tilde{\mathcal{T}}^*$*, where* (f_n) *is a* $\tilde{t}$*-fundamental sequence of the form* (4.4) *in Definition 4.5, then the corresponding* $f = P(D)(P_1F_0)$*, where* F_0 *is the* L^2*-limit of the sequence* (F_n)*, is an element of* $\mathcal{S}'^*(\mathbb{R}^d)$*.*

The above correspondence between $\mathcal{S}'^*(\mathbb{R}^d)$ *and* $\tilde{\mathcal{T}}^*$ *defines a linear bijection between these spaces.*

4.3. Tempered ultradistributions as functionals. Let $f = [f_n]$ be an element of $\mathcal{T}^*$, where the functions f_n are of the form $f_n = P(H)F_n$ on $\mathbb{R}^d$ with $P \in \mathcal{P}^{2*}$ and $F_n \in L^2(\mathbb{R}^d)$ for $n \in \mathbb{N}$ such that $F_n \xrightarrow{2} F_0$ as $n \to \infty$ for some $F_0 \in L^2(\mathbb{R}^d)$.

We define the action of $f = [f_n]$ on $\mathcal{S}^*(\mathbb{R}^d)$ as the mapping

$$\mathcal{S}^*(\mathbb{R}^d) \ni \varphi \mapsto f(\varphi) := (f,\varphi)_{\mathcal{T}^*} \in \mathbb{R}, \tag{4.14}$$

where

$$(f,\varphi)_{\mathcal{T}^*} := \lim_{n\to\infty}(f_n,\varphi) = \int_{\mathbb{R}^d}(F_0P(H)\varphi)(x)dx = (F_0, P(H)\varphi)_{L^2}. \tag{4.15}$$

As in the case of s-ultradistributions, if there is another representation of f_n in the form $f_n = \tilde{P}(H)\tilde{F}_n$ on $\mathbb{R}^d$ for $n \in \mathbb{N}$, where $\tilde{P} \in \mathcal{P}^{2*}$ and $\tilde{F}_n \xrightarrow{2} \tilde{F}_0$ as $n \to \infty$, then we have

$$\lim_{n\to\infty}(f_n,\varphi) = \int_{\mathbb{R}^d}(\tilde{F}_0\tilde{P}(H)\varphi)(x)dx = \int_{\mathbb{R}^d}(F_0P(H)\varphi)(x)dx,$$

i.e., the definition of $(f,\varphi)_{\mathcal{T}^*}$ in (4.15) is consistent. Lemma 1.3 implies that the mapping in (4.14) is linear.

We prove now the same for $f = [f_n] \in \tilde{\mathcal{T}}^*$, where the functions f_n are of the form $f_n = P(D)(P_1F_n)$ for $P \in \mathcal{P}^*$, $P_1 \in \mathcal{P}^*_u$, $F_n \in L^2(\mathbb{R}^d)$ $(n \in \mathbb{N})$ and $F_n \xrightarrow{2} F$ as $n \to \infty$ for some $F \in L^2(\mathbb{R}^d)$.

The action of $f = [f_n] \in \tilde{\mathcal{T}}^*$ on $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$ is defined as the mapping

$$\mathcal{S}^*(\mathbb{R}^d) \ni \varphi \mapsto (f,\varphi)_{\tilde{\mathcal{T}}^*} \in \mathbb{R}, \tag{4.16}$$

where

$$(f,\varphi)_{\tilde{\mathcal{T}}^*} := \lim_{n\to\infty}(F_n, P_1P(-D)\varphi) = \int_{\mathbb{R}^d}(F_0P_1P(-D)\varphi)(x)dx. \tag{4.17}$$

Note that the limit in (4.16) exists, because $P_1P(-D)\varphi \in \mathcal{S}^*(\mathbb{R}^d)$, in view of part (b) of Lemma 1.3.

If f_n is represented in another form: $f_n = \tilde{P}(D)(\tilde{P}_1\tilde{F}_n)$ on $\mathbb{R}^d$ with $\tilde{P}(D) \in \mathcal{P}^*$, $\tilde{P}_1 \in \mathcal{P}^*_u$, $\tilde{F}_n \in L^2(\mathbb{R}^d)$ for $n \in \mathbb{N}$ and $\tilde{F}_n \xrightarrow{2} \tilde{F}_0$ as $n \to \infty$ for some $\tilde{F}_0 \in L^2(\mathbb{R}^d)$, then

$$\lim_{n\to\infty}(\tilde{F}_n, \tilde{P}_1\tilde{P}(-D)\varphi) = \lim_{n\to\infty}(F_n, P_1P(-D)\varphi), \qquad \varphi \in \mathcal{S}^*(\mathbb{R}^d),$$

i.e., the definition in (4.17) is consistent. The linearity of the mapping (4.16) follows by Lemma 1.3.

The continuity of the mappings (4.14) and (4.16) follows from the following assertion.

PROPOSITION 4.8. *Let $f \in \mathcal{T}^*$ (resp. $f \in \tilde{\mathcal{T}}^*$) and let $\varphi_n \in \mathcal{S}^*(\mathbb{R}^d)$ for $n \in \mathbb{N}_0$ be functions such that $\varphi_n \xrightarrow{\mathcal{S}^*} \varphi_0$ as $n \to \infty$. Then*

$$(f, \varphi_n)_{\mathcal{T}^*} \to (f, \varphi_0)_{\mathcal{T}^*} \quad (\textit{resp. } (f, \varphi_n)_{\tilde{\mathcal{T}}^*} \to (f, \varphi_0)_{\tilde{\mathcal{T}}^*}) \quad \textit{as } n \to \infty.$$

PROOF. If $f \in \mathcal{T}^*$, then we have $(f, \varphi_n)_{\mathcal{T}^*} = (F_0, P(H)\varphi_n)_{L^2}$ for $n \in \mathbb{N}_0$, according to (4.1) and (4.15). Hence, by the Schwarz inequality, we get

$$\begin{aligned}|(f, \varphi_n)_{\mathcal{T}^*} - (f, \varphi_0)_{\mathcal{T}^*}| &= |(F_0, P(H)(\varphi_n - \varphi_0))_{L^2}| \\ &\leqslant \|F_0\|_2 \cdot \|P(H)(\varphi_n - \varphi_0)\|_2\end{aligned}$$

and the assertion follows, in view of part (c) of Lemma 1.3. The proof in the case $f \in \tilde{\mathcal{T}}^*$ is analogous. □

The above result can be generalized in the following way:

PROPOSITION 4.9. *Let $f^m \in \mathcal{T}^*$ (resp. $f^m \in \tilde{\mathcal{T}}^*$) and $\varphi_m \in \mathcal{S}^*(\mathbb{R}^d)$ for $m \in \mathbb{N}_0$. If $f^m \xrightarrow{t} f^0$ (resp. $f^m \xrightarrow{\tilde{t}} f^0$) and $\varphi_m \xrightarrow{\mathcal{S}^*} \varphi_0$ as $m \to \infty$, then $(f^m, \varphi_m)_{\mathcal{T}^*} \to (f^0, \varphi_0)_{\mathcal{T}^*}$ (resp. $(f^m, \varphi_m)_{\tilde{\mathcal{T}}^*} \to (f^0, \varphi_0)_{\tilde{\mathcal{T}}^*}$) as $m \to \infty$.*

PROOF. We give the proof only in the $\mathcal{T}^*$ case. By Definition 4.4, we have

$$\lim_{m\to\infty}(f^m, \varphi_m)_{\mathcal{T}^*} = \lim_{m\to\infty}\lim_{n\to\infty}(F^m_n, P(H)\varphi_m)_{L^2},$$

where $F^m_n \xrightarrow{2} F^m$ as $n \to \infty$ for every $m \in \mathbb{N}$ and $F^m \xrightarrow{2} F^0$ as $m \to \infty$. Hence, using the Schwarz inequality, we have

$$\begin{aligned}|(F^m, P(H)\varphi_m)_{L^2} &- (F^0, P(H)\varphi_0)_{L^2}| \\ &\leqslant |(F^m, P(H)(\varphi_m - \varphi_0))_{L^2}| + |(F^m - F^0, P(H)\varphi_0)_{L^2}| \\ &\leqslant \|F^m\|_2 \cdot \|P(H)(\varphi_m - \varphi_0)\|_2 + \|F^m - F^0\|_2 \cdot \|P(H)\varphi_0\|_2.\end{aligned}$$

It suffices now to use again part (c) of Lemma 1.3 to complete the proof. □

5. Relations between spaces of tempered ultradistributions

In connection with the spaces $\mathcal{S}^*(\mathbb{R}^d)$ and $\mathcal{S}'^*(\mathbb{R}^d)$, where $* = (t)$ in the Beurling case (resp. $* = \{t\}$ in the Roumieu case) consider the following spaces of numerical sequences

$$\mathbf{s}^* = \Big\{(a_\alpha)_{\alpha\in\mathbb{N}_0^d} : \forall h > 0 \text{ (resp. } \exists h > 0) \sum_{|\alpha|=0}^{\infty} |a_\alpha|^2 e^{h|\alpha|^{1/(2t)}} < \infty\Big\}$$
$$\mathbf{s}'^* = \Big\{(b_\alpha)_{\alpha\in\mathbb{N}_0^d} : \exists k > 0 \text{ (resp. } \forall k > 0) \sum_{|\alpha|=0}^{\infty} |b_\alpha|^2 e^{-k|\alpha|^{1/(2t)}} < \infty\Big\}.$$

By the Köthe theory of echelon and co-echelon spaces (see [**11**]) the spaces $\mathbf{s}^*$ and $\mathbf{s}'^*$ with their natural convergence structure constitute a dual pair.

It is well known that the mapping

$$\mathbf{s}^* \ni (a_\alpha)_{\alpha\in\mathbb{N}_0^d} \mapsto \sum_{|\alpha|=0}^{\infty} a_\alpha h_\alpha \in \mathcal{S}^*(\mathbb{R}^d) \tag{5.1}$$

is a bijective isomorphism between the spaces $\mathbf{s}^*$ and $\mathcal{S}^*(\mathbb{R}^d)$.

On the other hand, to every $f \in \mathcal{T}^*$ we can assign a unique $(b_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbf{s}'^*$.

In fact, assume that $f = [f_n] \in \mathcal{T}^*$ satisfies (4.1), i.e.,

$$f_n = P(H)F_n \text{ on } \mathbb{R}^d \ (n \in \mathbb{N}) \quad \text{and} \quad F_n \xrightarrow{2} F_0 \text{ as } n \to \infty, \tag{5.2}$$

where

$$F_n = \sum_{|\alpha|=0}^{\infty} c_{\alpha,n} h_\alpha \quad \text{with} \quad (c_{\alpha,n})_{\alpha\in\mathbb{N}_0^d} \in l^2 \qquad (n \in \mathbb{N}_0). \tag{5.3}$$

Morepover, let $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$ be of the form

$$\varphi = \sum_{|\alpha|=0}^{\infty} r_\alpha h_\alpha. \tag{5.4}$$

We know that $(r_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \mathbf{s}^*$ (see 5.1).

By (4.15), (5.2), (5.3) and (5.4), we have

$$(f,\varphi)_{\mathcal{T}^*} = \lim_{n\to\infty} \sum_{|\alpha|=0}^{\infty} P(2\alpha+1) c_{\alpha,n} r_\alpha = \sum_{|\alpha|=0}^{\infty} b_\alpha r_\alpha, \tag{5.5}$$

where

$$b_\alpha := P(2\alpha+1)c_{\alpha,0}, \qquad \alpha \in \mathbb{N}_0^d, \tag{5.6}$$

because $c_{\alpha,n} \to c_{\alpha,0}$ in l^2 as $n \to \infty$ for $\alpha \in \mathbb{N}_0^d$.

Assign to $f = [f_n] \in \mathcal{T}^*$ the sequence $(b_\alpha)_{\alpha\in\mathbb{N}_0^d} \in s'^*$ defined in (5.6) which does not depend on a representation (f_n) of f, by Proposition 4.2.

The described mapping is a bijective isomorphism between $\mathcal{T}^*$ and $\mathbf{s}'^*$.

Let us recall the following well known assertion (see [**5**]- [**10**]):

Proposition 5.1. *The bijective isomorphism* (5.1) *induces the isomorphism of* $\mathbf{s}'^*$ *onto* $\mathcal{S}'^*(\mathbb{R}^d)$ *given by*

$$\mathbf{s}'^* \ni (b_\alpha)_{\alpha\in\mathbb{N}_0^d} \mapsto \sum_{|\alpha|=0}^{\infty} b_\alpha h_\alpha \in \mathcal{S}'^*(\mathbb{R}^d).$$

According to the preceding remarks and Proposition 5.1 to each $f = [f_n] \in \mathcal{T}^*$ one can uniquely assign an element of $\mathcal{S}'^*(\mathbb{R}^d)$ of the form $\sum_{|\alpha|=0}^{\infty} b_\alpha h_\alpha$. On the other hand, we can assign to $f = [f_n] \in \mathcal{T}^*$ the functional T on $\mathcal{S}^*(\mathbb{R}^d)$ given by $T(\varphi) := (f,\varphi)_{\mathcal{T}^*}$, $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$, where $(f,\varphi)_{\mathcal{T}^*}$ is defined in (5.5). Clearly, the functional T is linear and continuous on $\mathcal{S}^*(\mathbb{R}^d)$, i.e., $T \in \mathcal{S}'^*(\mathbb{R}^d)$.

Conversely, if $T \in \mathcal{S}'^*(\mathbb{R}^d)$ is of the form $T = \sum_{|\alpha|=0}^{\infty} b_\alpha h_\alpha$, then the sequence (f_n) of functions f_n given by (5.2), where

$$F_n = \sum_{|\alpha|=0}^{n} \frac{b_\alpha h_\alpha}{P(2\alpha+1)} \quad (n\in\mathbb{N}) \qquad \text{and} \qquad F_0 = \sum_{|\alpha|=0}^{\infty} \frac{b_\alpha h_\alpha}{P(2\alpha+1)},$$

is t-fundamental and $f = [f_n]$ is the element of $\mathcal{T}^*$ corresponding to T.

Thus we have

Proposition 5.2. *The mapping* $B : \mathcal{T}^* \to \mathcal{S}'^*(\mathbb{R}^d)$ *given by*

$$\mathcal{T}^* \ni f \;\mapsto\; T = B(f) \in \mathcal{S}'^*(\mathbb{R}^d),$$

where $T(\varphi) := (f,\varphi)_{\mathcal{T}^*}$ *for* $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$, *is a linear and sequentially continuous bijection.*

We formulate now the concluding theorem of this section.

Theorem 5.1. (i) *For each continuous linear functional* T *on* $\mathcal{S}^*(\mathbb{R}^d)$ *there exists a unique* t*-ultradistribution* $f \in \mathcal{T}^*$ *such that*

$$T(\varphi) = (f,\varphi)_{\mathcal{T}^*}, \quad \varphi \in \mathcal{S}^*(\mathbb{R}^d). \tag{5.7}$$

Conversely, for each t*-ultradistribution* f, *formula* (5.7) *defines a sequentially continuous linear functional on* $\mathcal{S}^*(\mathbb{R}^d)$.

The correspondence between continuous linear functionals on $\mathcal{S}^*(\mathbb{R}^d)$ *and* t*-ultradistributions in* $\mathcal{T}^*$, *described by* (5.7), *is bijective.*

(ii) *A sequence* (f^m) *of* t*-ultradistributions* $f^m \in \mathcal{T}^*$, *represented by* t*-fundamental sequences* $(f_n^m)_n$ *for* $m \in \mathbb{N}$, *converges to* $f^0 \in \mathcal{T}^*$ *if and only if*

$$\lim_{m\to\infty}\lim_{n\to\infty} (f_n^m,\varphi)_{\mathcal{T}^*} = (f^0,\varphi)_{\mathcal{T}^*}, \qquad \varphi \in \mathcal{S}^*(\mathbb{R}^d). \tag{5.8}$$

Proof. Assertion (i) is already proved above.

In order to show (ii) it suffices to prove that (5.8) implies $f^m \xrightarrow{t} f^0$ as $m \to \infty$. We apply the notation from Definition 4.4. Assume that the Hermite expansions of the functions $F_n^m \in L^2(\mathbb{R}^d)$ $(m \in \mathbb{N}_0, n \in \mathbb{N})$, $F^m \in L^2(\mathbb{R}^d)$ $(m \in \mathbb{N}_0)$ in Definition 4.4, and of a given function $\varphi \in \mathcal{S}^*(\mathbb{R}^d)$ are of the form

$$F_n^m = \sum_{|\alpha|=0}^{\infty} a_{\alpha,n}^m h_\alpha, \quad F^m = \sum_{|\alpha|=0}^{\infty} a_\alpha^m h_\alpha \quad (m\in\mathbb{N}_0, n\in\mathbb{N}), \quad \varphi = \sum_{|\alpha|=0}^{\infty} r_\alpha h_\alpha.$$

By the duality of $\mathbf{s}^*$ and $\mathbf{s}'^*$, we have

$$\sum_{|\alpha|=0}^{\infty} a_{\alpha,n}^m r_\alpha \to \sum_{|\alpha|=0}^{\infty} a_{\alpha,n}^0 r_\alpha \quad \text{as } m \to \infty, \text{ uniformly in } n \in \mathbb{N}.$$

Moreover, $A_n \to A_0$ as $n \to \infty$ in $\mathbf{s}'^*$, where $A_n := (a_{\alpha,n}^0)_{\alpha \in \mathbb{N}_0^d} \in \mathbf{s}'^*$ for $n \in \mathbb{N}_0$. This implies the assertion. □

Remark 5.1. We have shown in Propositions 4.7 and 4.9 that there exists a linear continuous bijection between the spaces $\mathcal{S}'^*(\mathbb{R}^d)$ and $\tilde{\mathcal{T}}^*$, i.e., the spaces are isomorphic: $\mathcal{S}'^*(\mathbb{R}^d) \cong \tilde{\mathcal{T}}^*$.

Remark 5.1, together with Theorem 5.1, leads to the following conclusion.

Theorem 5.2. *The spaces $\mathcal{T}^*$ and $\tilde{\mathcal{T}}^*$ are isomorphic: $\mathcal{T}^* \cong \tilde{\mathcal{T}}^*$. This means that every $f \in \mathcal{T}^*$ can be represented as an equivalence class of $\tilde{t}$-fundamental sequences in the sense of Definition* 4.5. *Conversely, every $f \in \tilde{\mathcal{T}}^*$ can be represented as an equivalence class of t-fundamental sequence in the sense of Definition* 4.1. *The convergence structures described in Definitions* 4.4 *and* 4.8 *are equivalent.*

6. *s*-Ultradistributions as continuous linear functionals

We recall that a linear functional f on the corresponding space of test functions is an ultradistiribution or tempered ultradistribution if it is sequentially continuous.

Using our approach to the $\tilde{t}$-ultradistributions we prove:

Theorem 6.1. *If $f \in \mathcal{T}^* \cong \tilde{\mathcal{T}}^*$, then $f \in \mathcal{U}^*(\Omega)$.*

Proof. We know that there exist $P \in \mathcal{P}^*$ and $P_1 \in \mathcal{P}_u^*$ and there are functions $F_n \in L^2(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d)$ for $n \in \mathbb{N}$ and $F_0 \in L^2(\mathbb{R}^d)$ such that $F_n \xrightarrow{2} F_0$ as $n \to \infty$ and $f_n = P(D)(P_1 F_n)$ on $\mathbb{R}^d$ for $n \in \mathbb{N}$.

Fix $K, K_1 \Subset \Omega$ such that $K \Subset K_1^\circ$ and a function $\kappa_K \in \mathcal{D}^*(\Omega)$ as in (2.5). We have $P_1 F_n = \kappa_K P_1 F_n$ on K for $n \in \mathbb{N}$ and the inequality

$$\left(\int_{\mathbb{R}^d} \left| \left[\kappa_K P_1 (F_n - F_0) \right](x) \right|^2 dx \right)^{1/2} \leqslant \sup_{x \in K} |P_1(x)| \cdot \|F_n - F_0\|_2$$

implies $\kappa_K P_1 F_n \xrightarrow{2} \kappa_K P_1 F_0$ as $n \to \infty$, so

$$\mathcal{F}(\kappa_K P_1 F_n) \xrightarrow{2} \mathcal{F}(\kappa_K P_1 F_0) \qquad \text{as } n \to \infty.$$

Moreover, by Proposition 4.1, we have

$$\tilde{F}_n \xrightarrow{\mathcal{C}(K)} \tilde{F}_0 \quad \text{as } n \to \infty \quad \text{and} \quad \operatorname{supp} \tilde{F}_n \subset K_1 \quad (n \in \mathbb{N}_0),$$

where

$$\tilde{F}_n := \kappa_K \mathcal{F}^{-1}\big(\langle \cdot \rangle^{-d} \widehat{\kappa_K P_1 F_n} \big) \quad \text{for } n \in \mathbb{N}_0$$

with $\langle \cdot \rangle^{-d}$ meaning the function: $\langle \xi \rangle^{-d} = (1 + |\xi|^2)^{-d/2}$ for $\xi \in \mathbb{R}^d$.

If $\tilde{P}(D) := (1 + D_1^2 + \ldots + D_d^2)^{d/2} P(D)$, then $\tilde{P} \in \mathcal{P}^*$ and $f_n = \tilde{P}(D) \tilde{F}_n$ on K, so (f_n) is an s-fundamental sequence in the sense od Definition 2.1. □

The following assertion follows from the proof of Theorem 6.1.

Corollary 6.1. *Let $f \in \mathcal{T}^* \cong \tilde{\mathcal{T}}^*$, let Ω be an open set in $\mathbb{R}^d$ and let $\theta \in \mathcal{D}^*(\mathbb{R}^d)$ be a function such that* $\operatorname{supp}\theta \subset \Omega$. *Then $\theta f = [f_n] \in \mathcal{U}^*(\mathbb{R}^d)$ for some s-fundamental sequence (f_n) having the following properties: for every $K \Subset \Omega$ there exist $P \in \mathcal{P}^*$ and functions F_n such that*

$$f_n = P(D)F_n \text{ on } K \quad (n \in \mathbb{N}) \quad \text{and} \quad F_n \xrightarrow{\mathcal{C}(\mathbb{R}^d)} F_0 \quad \text{as } n \to \infty.$$

Moreover

$$\operatorname{supp} F_n \subset \Omega \quad \text{for} \quad n \in \mathbb{N}_0.$$

Now, we are able to prove the main result of this section.

Main Theorem 6.2. *(i) For every continuous linear functional T on $\mathcal{D}^*(\Omega)$ there exists a unique s-ultradistribution $f \in \mathcal{U}^*(\Omega)$ such that*

$$T(\varphi) = (f, \varphi)_{\mathcal{U}^*(\Omega)}, \quad \varphi \in \mathcal{D}^*(\Omega), \tag{6.1}$$

where $(f, \varphi)_{\mathcal{U}^*(\Omega)}$ is defined by (3.3).

Conversely, for every s-ultradistribution $f \in \mathcal{U}^*(\Omega)$, formula (6.1) *defines a continuous linear functional T on $\mathcal{D}^*(\Omega)$.*

The correspondence between continuous linear functionals on $\mathcal{D}^*(\Omega)$ and s-ultradistributions in $\mathcal{U}^*(\Omega)$, described by (6.1)*, is bijective.*

(ii) A sequence of s-ultradistributions $f^m \in \mathcal{U}^*(\Omega)$ converges to an s-ultradistribution $f^0 \in \mathcal{U}^*(\Omega)$ if and only if

$$\lim_{m\to\infty} (f^m, \varphi)_{\mathcal{U}^*(\Omega)} = (f^0, \varphi)_{\mathcal{U}^*(\Omega)} \quad \text{for all } \varphi \in \mathcal{D}^*(\Omega).$$

Proof. It suffices to prove only the first part of assertion (i).

Consider a locally finite covering of Ω consisting of bounded open subsets Ω_i and $\tilde{\Omega}_i$ of Ω such that $\overline{\Omega}_i \Subset \tilde{\Omega}_i$ and let functions $\varphi_i \in \mathcal{D}^*(\Omega)$ form a partition of unity for $i \in \mathbb{N}$, i.e., $\varphi_i(x) = 1$ if $x \in \Omega_i$ and $\operatorname{supp}\varphi_i \subset \tilde{\Omega}_i$ for $i \in \mathbb{N}$. If T is a continuous linear functional on $\mathcal{D}^*(\Omega)$, then T_i, defined by $T_i(\psi) = T(\varphi_i \psi)$ for $\psi \in \mathcal{S}^*(\mathbb{R}^d)$, is a continuous linear functional on $\mathcal{S}^*(\mathbb{R}^d)$.

By Lemma 4.4 and Theorem 5.1, there is a sequence of $f^i \in \mathcal{T}^* \cong \tilde{\mathcal{T}}^*$ such that $T_i(\psi) = (f^i, \psi)_{\mathcal{T}^*}$ in the sense of (4.15) and $T_i(\psi) = (f^i, \psi)_{\tilde{\mathcal{T}}^*}$ in the sense of (4.16) for $\psi \in \mathcal{S}^*(\mathbb{R}^d)$ and $i \in \mathbb{N}$. By Corollary 6.1, each f^i can be represented in the form $f^i = [(f^i_n)_n] \in \mathcal{U}^*(\mathbb{R}^d)$, where

$$f^i_n = P_i(D)F^i_n \quad (n \in \mathbb{N}), \quad F^i_n \xrightarrow{\mathcal{C}(\mathbb{R}^d)} F^i \quad \text{as } n \to \infty, \quad \operatorname{supp} F^i_n \subset \tilde{\Omega}_i \quad (n \in \mathbb{N}_0)$$

for some $P_i \in \mathcal{P}^*$ and functions F^i_n $(n \in \mathbb{N}_0)$ with suitable properties.

Fix a pair of different i, j such that $\Omega_i \cap \Omega_j \neq \emptyset$. We have

$$T(\varphi) = (f^i, \varphi)_{\mathcal{U}^*(\Omega)} = (f^j, \varphi)_{\mathcal{U}^*(\Omega)} \quad \text{for } \varphi \in \mathcal{D}^*(\Omega_i \cap \Omega_j).$$

Consider $f^i - f^j \in \mathcal{U}^*(\mathbb{R}^d)$ as an s-ultradistribution restricted to $\Omega_i \cap \Omega_j$. There exists an s-fundamental sequence of $r^{i,j}_n$ such that $f^i - f^j = [(r^{i,j}_n)_n]$ on $\Omega_i \cap \Omega_j$. This means $f^i - f^j = [(r^{i,j}_n)_n]$, where $r^{i,j}_n$ are smooth functions with $\operatorname{supp} r^{i,j}_n \subset \tilde{\Omega}_i \cap \tilde{\Omega}_j$ such that for every $K \Subset \Omega_i \cap \Omega_j$ we have $r^{i,j}_n = P_{i,j}(D)R^{i,j}_n$ on K for some $P_{i,j} \in \mathcal{P}^*$ and functions $R^{i,j}_n$ for $n \in \mathbb{N}$; moreover, $R^{i,j}_n \to 0$ as $n \to \infty$ uniformly on K. By

Proposition 2.2, we have $f^i = f^j$ on $\Omega_i \cap \Omega_j$. Since our covering of Ω is locally finite, we may define

$$f := \sum_{i\in\mathbb{N}} f^i \varphi_i \in \mathcal{U}^*(\Omega) \quad \text{with} \quad f|\Omega_i = f^i \quad (i \in \mathbb{N}). \tag{6.2}$$

Consequently, $T(\varphi) = (f, \varphi)_{\mathcal{U}^*(\Omega)}$ for $\varphi \in \mathcal{D}^*(\Omega)$ (see (3.3)).

Suppose $T(\varphi) = (f^1, \varphi)_{\mathcal{U}^*(\Omega)}$ and $T(\varphi) = (f^2, \varphi)_{\mathcal{U}^*(\Omega)}$ for $f^1, f^2 \in \mathcal{U}^*(\Omega)$ and for $\varphi \in \mathcal{D}^*(\Omega)$. Let $f^1 - f^2 = [g_n]$, where (g_n) is an s-fundamental sequence of the form $g_n = P(D)(G_n)$ on $K \Subset \Omega$ for suitable P and G_n. By (3.3), $\lim_{n\to\infty} g_n = 0$ on Ω. Hence, by Proposition 2.2, $f^1 - f^2 = [g_n] = 0$, so the s-ultradistribution f defined in (6.2) is unique. $\square$

Acknowledgements. The first, third and fourth authors were supported by the Project 174024 of the Serbian Ministry of Science. The second author was partly supported by the Center for Innovation and Transfer of Natural Sciences and Engineering Knowledge.

References

1. P. Antosik, J. Mikusiński, R. Sikorski, *Theory of distributions. The sequential aproach*, Elsevier–PWN, Amsterdam–Warszawa, 1973.
2. A. Beurling, *Quasi-analyticity and general distributions*, Lectures 4 and 5, Am. Math. Soc. Summer Inst., Stanford, 1961.
3. J. Bonet, P. Domański, *Köthe coechelon spaces as locally convex algebras*, Stud. Math. **199** (2010), 241–265.
4. R. W. Braun, R. Meise, B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Result. Math. **17** (1990), 206–237.
5. R. Carmichael, A. Kamiński, S. Pilipović, *Boundary Values and Convolution in Ultradistribution Spaces*, World Scientific, Singapore, 2007.
6. T. Gramchev, S. Pilipović, L. Rodino, *Classes of degenerate elliptic operators in Gelfand-Shilov spaces*, New developments in pseudo-differential operators, Oper. Theory, Adv. Appl. **189** (2009), 15–31.
7. ———, *Global regularity and stability in S-spaces for classes of degenerate Shubin operators*, Pseudo-differential operators: complex analysis and partial differential equations, Oper. Theory, Adv. Appl. **205** (2010), 81–90.
8. T. Gramchev, A. Lecke, S. Pilipović, L. Rodino, *Gelfand-Shilov type space through Hermite expansions*; in: S. Pilipović (Ed.), *Pseudo-Differential Operators and Generalized Functions, Selected papers of the 9th ISAAC congress, Cracow, Poland, August 5Ű9, 2013* Oper. Theory, Adv. Appl. **245**, Birkhäuser/Springer, 2015, pp. 95–105.
9. M. Langenbruch, *Hermite functions and weighted spaces of generalized functions*, Manuscr. Math. **119** (2006), 269–285.
10. A. Kamiński, D. Perišić, S. Pilipović, *On various integral transformations of tempered ultradistributions*, Demonstr. Math. **33** (2000), 641–655.
11. G. Köthe, *Topological Vector Spaces II*, Vol. II, Springer–Verlag, New York Inc., 1979.
12. H. Komatsu, *Ultradistributions, I: Structure theorems and a characterization*, J. Fac. Sci., Univ. Tokyo, Sect. I A **20** (1973), 25–105.
13. ———, *Ultradistributions, II: The kernel theorem and ultradistributions with support in a submanifold* J. Fac. Sci., Univ. Tokyo, Sect. I A **24** (1977), 607–628.
14. ———, *Ultradistributions III: Vector valued distributions and the theory of kernels*, J. Fac. Sci., Univ. Tokyo, Sect. I A **29** (1982), 653–718.
15. ———, *Microlocal Analysis in Gevrey Classes and in Complex Domains*, Lect. Notes Math. **1726** (1989), 426–493.

16. ______, *An Introduction to the Theory of Generalized Functions*, Lecture Notes, Tokyo, 1999.
17. D. Kovačević, S. Pilipović, *Structural properties of the space of tempered ultradistributions*, Proc. Conf. on Complex Analysis and Generalized Functions, Varna 1991, Publ. House of the Bulgar. Acad. Sci., Sofia (1993), 206–237.
18. J. Mikusiński *Sur la méthode de généralisation de M. Laurent Schwartz et sur la convergence faible*, Fundam. Math. **35** (1948).
19. ______, *Une définition de distribution*, Bull. Acad. Pol. Sci., Cl. III **4** (1955), 589–591.
20. J. Mikusiński, R. Sikorski, *The elementary theory of distributions, I*, Diss. Math. **12** (1957).
21. ______, *The elementary theory of distributions, II*, Diss. Math. **25** (1961).
22. S. Mincheva-Kamińska, *A sequential approach to the convolution of Roumieu ultradistributions*, submitted.
23. H. J. Petzsche, D. Vogt, *Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions*, Math. Ann. 267 (1984), 17–35.
24. S. Pilipović, *Tempered ultradistributions*, Boll. Unione Mat. Ital., VII. Ser., B **2**(2) (1988), 235–251.
25. ______, *Characterization of bounded sets in spaces of ultradistributions*, Proc. Am. Math. Soc. **120** (1994), 1191–1206.
26. S. Pilipović, B. Prangoski, J. Vindas, *On Quasianalytic Classes of Gelfand-Shilov Type. Parametrix and Convolution*, preprint, arXiv:1507.08331 [math.FA].
27. C. Roumieu, *Sur quelques extensions de la notion de distribution*, Ann. Sci. Éc. Norm. Supér. (3) **77** (1960), 41–121.
28. L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.
29. S. L. Sobolev, *Méthode nouvelle á résoudre le probleme de Cauchy pour les équations hyporboliques normales*, Rec. Math. Moscou, n. Ser. **1** (1936), 39–71.
30. D. Vogt, *Sequence space representations of spaces of test functions and distributions. Functional analysis, holomorphy, and approximation theory* (Rio de Janeiro, 1979), pp. 405–443, Lect. Notes Pure Appl. Math. **83**, Dekker, New York, 1983.

Faculty of Electrical Engineering
University of Banja Luka
Banja Luka
Bosnia and Herzegovina
snjezana.maksimovic@etfbl.net

(Received 18 02 2016)
(Revised 19 04 2016)

Faculty of Mathematics and Natural Sciences
University of Rzeszow
Rzeszow
Poland
svetlana.kaminska@gmail.com

Faculty of Sciences and Mathematics
University of Novi Sad
Novi Sad
Serbia
pilipovic@dmi.uns.ac.rs

Faculty of Mathematics and Natural Sciences
University of Sts. Cyril and Methodius
Skopje
Macedonia
petar@pmf.ukim.mk