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# CONFORMAL AND GEODESIC MAPPINGS OF GENERALIZED EQUIDISTANT SPACES

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Abstract. We consider conformal and geodesic mappings of generalized equidistant spaces. We prove the existence of mentioned nontrivial mappings and construct examples of conformal and geodesic mapping of a 3-dimensional generalized equidistant space. Also, we find some invariant objects (three tensors and four which are not tensors) of special geodesic mapping of generalized equidistant space.

## 1. Introduction

Equidistant spaces are defined by the existence of concincular vector fields which are characterized by the property that their covariant derivative is proportional to the unity tensor. Examples of Riemannian spaces with concincular vector fields are the well known spatially homogeneous and isotropic cosmological models of space-time (pseudo-Riemannian manifolds with Friedmann–Lemaitre–Robertson–Walker metric) [**7**]. Equidistant spaces were studied in [**1**, **2**, **6**–**11**, **20**, **21**, **23**, **24**], etc.

The investigation of conformal and geodesic mapping theory for special spaces is an important and active research topic. Conformal mappings of Riemannian spaces with concincular vector fields were studied in the works of Brinkmann, Fialkov, Yano, de Vries. On the other hand, geodesic mappings of equidistant Riemannian spaces appeared in the papers of Sinjukov, Solodovnikov, Rosenfeld, Mikeš, Kiosak, Hall and many others.

In recent times, it has become very interesting to investigate spaces with nonsymmetric affine connection. The beginning of the study of general (nonsymmetric) affine connection spaces is especially related to the works [**3**,**4**] of Einstein on Unified

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Field Theory. Many new and interesting results related to generalized Riemannian spaces and, in general, nonsymmetric affine connection spaces, appeared in the papers of Eisenhart, Minčić, Nitescu, Prvanović, Stanković, Bohner, Yano, etc.

For the first time, equidistant generalized Riemannian space was defined in [**1**] where geodesic mappings of such defined spaces were discussed. In the present paper we continue our previous investigations, primarily from [**1**] and [**6**], studying conformal and geodesic mappings of generalized equidistant spaces. Note that the study is of a local character. All functions considered are assumed to be sufficiently smooth.

**1.1. Generalized Riemannian spaces.** A *generalized Riemannian space* $\mathbb{GR}_\mathbb{N}$ in the sense of Eisenhart's definition [**5**] is a differentiable $N$-dimensional manifold, equipped with a nonsymmetric basic tensor $g_{ij}(x)$, $x = (x^1, \dots, x^N)$, where $\det(g_{ij}) \neq 0$. We can write $g_{ij} = g_{\underline{ij}} + g_{\underset{\vee}{ij}}$ where $\underline{ij}$ denotes symmetrization and $\underset{\vee}{ij}$ antisymmetrization with division by indices $i$ and $j$. The Riemannian space $\mathbb{R}_\mathbb{N}$ determined by the symmetric part of the metric tensor of generalized Riemannian space $\mathbb{GR}_\mathbb{N}$, is *adjoint space of* the space $\mathbb{GR}_\mathbb{N}$.

*Generalized Christoffel symbols* of the first kind of the space $\mathbb{GR}_\mathbb{N}$ are given by

$$\Gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}), i, j, k = 1, \dots, N,$$

where $g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}$. The connection coefficients of this space are generalized Christoffel symbols of the second kind $\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{p.jk}$, where $(g^{\underline{ij}}) = (g_{\underline{ij}})^{-1}$, supposing $\det(g_{\underline{ij}}) \neq 0$. Generally, it is $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. Therefore, one can define the symmetric and anti-symmetric part of $\Gamma^i_{jk}$, respectively

$$\Gamma^i_{\underline{jk}} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}), \quad \Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}).$$

The magnitude $\Gamma^i_{\underset{\vee}{jk}}$ is *torsion tensor* of the spaces $\mathbb{GR}_\mathbb{N}$. Obviously, $\Gamma^i_{jk} = \Gamma^i_{\underline{jk}} + \Gamma^i_{\underset{\vee}{jk}}$. Notice that in $\mathbb{GR}_\mathbb{N}$ we have $\Gamma^p_{\underset{\vee}{ip}} = 0$ (eq. (2.10) in [**15**]).

Using the nonsymmetry of the connection $\Gamma^i_{jk}$, in the generalized Riemannian space, one can define four kinds of covariant derivatives (see [**12**–**19**]). For example, for a tensor $a^i_j$, we have

$$\begin{aligned} a^i_{j\underset{1}{|}m} &= a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p\,, \quad & a^i_{j\underset{2}{|}m} &= a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p, \\ a^i_{j\underset{3}{|}m} &= a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p\,, \quad & a^i_{j\underset{4}{|}m} &= a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p. \end{aligned} \tag{1.1}$$

Also, we can consider covariant derivative in $\mathbb{GR}_\mathbb{N}$ with respect to the symmetric part of the connection $\Gamma^i_{jk}$. Thus,

$$a^i_{j;m} = a^i_{j,m} + \Gamma^i_{\underline{pm}}a^p_j - \Gamma^p_{\underline{jm}}a^i_p. \tag{1.2}$$

In the Riemannian space ($\underset{\vee}{\Gamma}{}^i_{jk} = 0$) all types of covariant derivates at (1.1) reduce to (1.2).

In the case of the space $\mathbb{GR_N}$, we have five independent *curvature tensors* [**14**] (in [**14**] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\begin{aligned}
\underset{1}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^p_{jm}\Gamma^i_{pn} - \Gamma^p_{jn}\Gamma^i_{pm},\\
\underset{2}{R}{}^i{}_{jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^p_{mj}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{mp},\\
\underset{3}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{nm}(\Gamma^i_{pj} - \Gamma^i_{jp}),\\
\underset{4}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{np} - \Gamma^p_{nj}\Gamma^i_{pm} + \Gamma^p_{mn}(\Gamma^i_{pj} - \Gamma^i_{jp}),\\
\underset{5}{R}{}^i{}_{jmn} &= \frac{1}{2}\big(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^p_{jm}\Gamma^i_{pn} + \Gamma^p_{mj}\Gamma^i_{np}\\
&\qquad\qquad\qquad\qquad\qquad - \Gamma^p_{jn}\Gamma^i_{mp} - \Gamma^p_{nj}\Gamma^i_{pm}\big)
\end{aligned}$$

These curvature tensors produce *Ricci tensors of $\theta$-kind*, i.e., $\underset{\theta}{R}{}^\alpha_{jm\alpha} = \underset{\theta}{R}{}_{jm}$, $\theta \in \{1, \dots, 5\}$.

## 2. Generalized equidistant spaces

Let $\mathbb{GR_N}$ be a generalized Riemannian space with a nonsymmetric metric tensor $g_{ij}$.

Definition 2.1. A vector field $\varphi$ is called *concircular* if

$$\varphi^i_{;j} = \rho\delta^i_j. \tag{2.1}$$

where $\rho$ is a function, $\delta^i_j$ is the Kronecker delta, $(;)$ denotes covariant derivative with respect to the symmetric part of the connection $\Gamma^i_{jk}$.

If $\rho = const$, $\varphi$ is called *convergent*. A generalized Riemannian space $\mathbb{GR_N}$ with concircular vector field is called *generalized equidistant space.*

Condition (2.1) can be presented as $\varphi_{i;j} = \rho g_{\underline{ij}}$ which means that the covariant derivative of $\varphi_i$ (denoted by (;), (1.2)) is proportional to the symmetric part of the metric tensor of the space $\mathbb{GR_N}$. Also, the previous condition can be presented as

$$\varphi_{i\underset{1}{|}j} = \rho g_{\underline{ij}} - \underset{\vee}{\Gamma}{}^p_{ij}\varphi_p, \quad \varphi_{i\underset{2}{|}j} = \rho g_{\underline{ij}} - \underset{\vee}{\Gamma}{}^p_{ji}\varphi_p$$

where $(|)$ denotes covariant derivative of the corresponding kind in the space $\mathbb{GR_N}$ and $\underset{\vee}{\Gamma}{}^i_{pj}$ is the torsion tensor of $\mathbb{GR_N}$.

It is known that in equidistant space $\mathbb{R_N}$ with symmetric metric tensor $g_{ij}$, where the concircular vector fields are nonisotropic (i.e., $g_{ij}\varphi^i\varphi^j \neq 0$), we can introduce a system of the so-called canonical coordinates $(x^i)$, where the metric is of the form

$$ds^2 = a(x^1)(dx^1)^2 + b(x^1)d\tilde{s}^2, \tag{2.2}$$

$a, b \in C^1$ are nonzero functions, and $d\tilde{s}^2 = \tilde{g}_{\sigma\mu}(x^2, \ldots, x^N) dx^\sigma\, dx^\mu$ is the metric form of certain Riemannian spaces $\tilde{\mathbb{R}}_{\mathbb{N}-1}$ (see [**6**]). Here, and in what follows, the indices $\sigma, \mu, \theta \ldots$ take values from 2 to $N$.

Let us look at the metric form of the space $\mathbb{GR}_\mathbb{N}$:

$$ds^2 = g_{ij} dx^i dx^j = (g_{\underline{ij}} + g_{\underset{\vee}{ij}}) dx^i dx^j. \tag{2.3}$$

As it holds

$$g_{ij} dx^i dx^j = g_{ji} dx^j dx^i \Leftrightarrow (g_{ij} - g_{ji}) dx^i dx^j = 0 \Leftrightarrow g_{\underset{\vee}{ij}} dx^i dx^j = 0,$$

we get that (2.3) becomes $ds^2 = g_{ij} dx^i dx^j = g_{\underline{ij}} dx^i dx^j$. So, we conclude that the basic metric form of $\mathbb{GR}_\mathbb{N}$ can also be presented as (2.2). The symmetric parts of Christoffel symbols of the second kind satisfy:

$$\Gamma^1_{11} = \frac{1}{2}\frac{a'}{a}, \quad \Gamma^1_{\underline{1\sigma}} = \Gamma^\sigma_{11} = 0, \quad \Gamma^1_{\underline{\mu\sigma}} = -\frac{1}{2}\frac{b'}{a}\tilde{g}_{\sigma\mu},$$
$$\Gamma^\mu_{\underline{\sigma 1}} = \frac{1}{2}\frac{b'}{b}\delta^\mu_\sigma, \quad \Gamma^\nu_{\underline{\sigma\mu}} = \tilde{\Gamma}^\nu_{\sigma\mu}, \quad (\sigma, \mu, \nu > 1),$$

where $\tilde{g}_{\sigma\mu}$ are arbitrary symmetric functions of $x^2, \ldots, x^N$, $\det(\tilde{g}_{\sigma\mu}) \neq 0$, $\tilde{\Gamma}^\nu_{\sigma\mu}$ is Christoffel symbols of the second kind derived from $\tilde{g}_{\sigma\mu}(x^\nu)$.

Consider two generalized equidistant spaces $\mathbb{GR}_\mathbb{N}$ and $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$, where the space $\mathbb{GR}_\mathbb{N}$ has a metric form (2.2), and the space $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ has an analogous metric

$$d\bar{s}^2 = A(x^1)(dx^1)^2 + B(x^1)\hat{g}_{\sigma\mu} dx^\sigma dx^\mu, \tag{2.4}$$

where $A, B \in C^1$ are nonzero functions, and $\hat{g}_{\sigma\mu}$ are arbitrary symmetric functions of $x^2, \ldots, x^N$, $\det(\hat{g}_{\sigma\mu}) \neq 0$.

Let $f : \mathbb{GR}_\mathbb{N} \to \mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ be a mapping of two equidistant spaces. Consider the map in a common coordinate system $x$, i.e. the point $M \in \mathbb{GR}_\mathbb{N}$ and its image $f(M) \in \mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ have the same coordinates $x = (x^1, x^2, \ldots, x^N)$. The corresponding geometric objects in $\mathbb{GR}_\mathbb{N}$ will be marked with a bar. Then the symmetric part of the deformation tensor $P^i_{jk} = \overline{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of that mapping has the form

$$\begin{aligned} P^1_{11} &= \frac{1}{2}\left(\frac{A'}{A} - \frac{a'}{a}\right), \quad P^1_{\underline{1\sigma}} = P^\sigma_{11} = 0, \quad P^\sigma_{\underline{1\mu}} = \frac{1}{2}\left(\frac{B'}{B} - \frac{b'}{b}\right)\delta^\sigma_\mu, \\ P^1_{\underline{\sigma\mu}} &= -\frac{1}{2}\left(\frac{B'}{A}\hat{g}_{\sigma\mu} - \frac{b'}{a}\tilde{g}_{\sigma\mu}\right), \quad P^\sigma_{\underline{\mu\theta}} = \hat{\Gamma}^\sigma_{\mu\theta} - \tilde{\Gamma}^\sigma_{\mu\theta} \end{aligned} \tag{2.5}$$

The antisymmetric part of the deformation tensor will be denoted by $\xi^i_{jk}$, i.e., $\xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \overline{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}}$.

The following lemma holds for an arbitrary mapping of an arbitrary generalized Riemannian space.

Lemma 2.1. *Under a mapping $f$ of generalized Riemannian space $\mathbb{GR}_\mathbb{N}$ onto generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$, in the common coordinate system $x$ with respect* *to the mapping, antisymmetric tensor* $\xi^i_{jk}$ *satisfies*

$$\xi^p_{ik}\overline{g}_{\underline{pj}} + \xi^p_{jk}\overline{g}_{\underline{ip}} = -\underset{\vee}{\Gamma}{}^p_{ik}\overline{g}_{\underline{pj}} - \underset{\vee}{\Gamma}{}^p_{jk}\overline{g}_{\underline{ip}}, \tag{2.6}$$

*where* $\overline{g}_{\underline{ij}}$ *is the symmetric part of the metric tensor of* $\mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ *and* $\underset{\vee}{\Gamma}{}^p_{ij}$ *is the torsion tensor of* $\mathbb{G}\mathbb{R}_{\mathbb{N}}$.

Proof. It is known that in the generalized Riemannian spaces holds $\overline{g}_{\underline{ij}\underset{1}{||}k} \equiv 0$, where we denote by $\underset{1}{||}$ covariant derivative of the first kind in $\mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ (see [**12**]). Using the definition of covariant derivative, we get

$$\begin{aligned} &\frac{\partial\overline{g}_{\underline{ij}}}{\partial x^k} - \overline{\Gamma}{}^p_{ik}\overline{g}_{\underline{pj}} - \overline{\Gamma}{}^p_{jk}\overline{g}_{\underline{ip}} \equiv 0 \Leftrightarrow \frac{\partial\overline{g}_{\underline{ij}}}{\partial x^k} - (\overline{\Gamma}{}^p_{\underline{ik}} + \underset{\vee}{\overline{\Gamma}}{}^p_{ik})\overline{g}_{\underline{pj}} - (\overline{\Gamma}{}^p_{\underline{jk}} + \underset{\vee}{\overline{\Gamma}}{}^p_{jk})\overline{g}_{\underline{ip}} \equiv 0 \\ &\qquad\qquad\qquad\qquad \Leftrightarrow \underset{\vee}{\overline{\Gamma}}{}^p_{ik}\overline{g}_{\underline{pj}} + \underset{\vee}{\overline{\Gamma}}{}^p_{jk}\overline{g}_{\underline{ip}} \equiv 0, \end{aligned} \tag{2.7}$$

where we used $\overline{g}_{\underline{ij};k} \equiv 0$, $(;)$ is covariant derivative in the adjoint Riemannian space $\mathbb{R}_{\mathbb{N}}$. As it is $\xi^i_{jk} = \underset{\vee}{\overline{\Gamma}}{}^i_{jk} - \underset{\vee}{\Gamma}{}^i_{jk}$, we get (2.6) from the last equation in (2.7). □

## 3. Conformal mappings of generalized equidistant spaces

Let $\mathbb{G}\mathbb{R}_{\mathbb{N}}$ and $\mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ be two generalized Riemannian spaces.

Definition 3.1. [**22**] The mapping $f : \mathbb{G}\mathbb{R}_{\mathbb{N}} \to \mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ is *conformal* if in the common coordinate system $x$ with respect to the mapping, the metric tensors $g_{ij}$ and $\overline{g}_{ij}$ of this spaces satisfy $\overline{g}_{ij} = e^{2\psi}\, g_{ij}$, where $\psi$ is a function on $\mathbb{G}\mathbb{R}_{\mathbb{N}}$.

For the Christoffel symbols of the first kind of $\mathbb{G}\mathbb{R}_{\mathbb{N}}$ and $\mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ the following relation is valid

$$\overline{\Gamma}_{i.jk} = e^{2\psi}(\Gamma_{i.jk} + g_{ji}\psi_{,k} - g_{jk}\psi_{,i} + g_{ik}\psi_{,j})$$

and for the Christoffel symbols of the second kind

$$\overline{\Gamma}{}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{jp}\psi_{,k} - g_{jk}\psi_{,p} + g_{pk}\psi_{,j}). \tag{3.1}$$

Let us introduce the notation $\psi_k = \psi_{,k} = \partial\psi/\partial x^k$ and $\psi^i = g^{\underline{ip}}\psi_p$. From (3.1) we obtain

$$\overline{\Gamma}{}^i_{jk} = \Gamma^i_{jk} + g^{\underline{ip}}(g_{\underline{jp}}\psi_k - g_{\underline{jk}}\psi_p + g_{\underline{pk}}\psi_j) + g^{\underline{ip}}(\underset{\vee}{g}_{jp}\psi_k - \underset{\vee}{g}_{jk}\psi_p + \underset{\vee}{g}_{pk}\psi_j),$$

i.e.,

$$\overline{\Gamma}{}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\,\psi_k + \delta^i_k\,\psi_j - \psi^i g_{\underline{jk}} + \xi^i_{jk}, \quad (\xi^i_{jk} = \underset{\vee}{\overline{\Gamma}}{}^i_{jk} - \underset{\vee}{\Gamma}{}^i_{jk})$$

where

$$\xi^i_{jk} = g^{\underline{ip}}(\underset{\vee}{g}_{jp}\psi_k - \underset{\vee}{g}_{jk}\psi_p + \underset{\vee}{g}_{pk}\psi_j) = -\xi^i_{kj}.$$

So, under conformal mappings, the deformation tensor satisfies

$$P^i_{\underline{jk}} = \delta^i_j\,\psi_k + \delta^i_k\,\psi_j - \psi^i, g_{\underline{jk}}, \quad \underset{\vee}{P^i_{jk}} = \xi^i_{jk} = g^{\underline{ip}}\,(\underset{\vee}{g_{jp}}\,\psi_k - \underset{\vee}{g_{jk}}\,\psi_p + \underset{\vee}{g_{pk}}\,\psi_j). \tag{3.2}$$

By comparing (2.5) with the first equation in (3.2) for all cases of concrete values of indices $i, j, k$, we get

$$\psi_1 = \frac{1}{2}\left(\frac{A'}{A} - \frac{a'}{a}\right) = \frac{1}{2}\left(\frac{B'}{B} - \frac{b'}{b}\right) = \frac{a}{2b}\left(\frac{B'}{A} - \frac{b'}{a}\right), \quad \psi_\sigma = 0, \quad \sigma = 2, \dots, N,$$

wherefrom we obtain $A(x^1) = \rho(x^1)a(x^1)$, $B(x^1) = \rho(x^1)b(x^1)$ and

$$\psi = \frac{1}{2}\ln|\rho| + c, \tag{3.3}$$

where $\rho$ is an arbitrary function of $x^1$, $\rho' \neq 0$, and $c$ is a constant. And also, according to (3.2), right, and (3.3), we get

$$\xi^1_{1\sigma} = \xi^\sigma_{\mu\theta} = 0, \quad \xi^1_{\sigma\mu} = -\frac{\rho'}{2a\rho}\underset{\vee}{g_{\sigma\mu}}, \quad \xi^\sigma_{1\mu} = \frac{1}{2}\frac{\rho'}{\rho}g^{\underline{\sigma\rho}}\underset{\vee}{g_{\rho\mu}}. \tag{3.4}$$

The basic metric form of the space $\mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ is

$$d\bar{s}^2 = \rho(x^1)\Big(a(x^1)(dx^1)^2 + b(x^1)\tilde{g}_{\sigma\mu}dx^\sigma dx^\mu\Big). \tag{3.5}$$

Thus, the following theorem holds

THEOREM 3.1. *Generalized equidistant space $\mathbb{G}\mathbb{R}_{\mathbb{N}}$ with fundament metric form* (2.2) *admits conformal mapping $f$ on the generalized equidistant space $\mathbb{G}\overline{\mathbb{R}}_{\mathbb{N}}$ with fundament metric form* (3.5)*, which is nontrivial for $\rho' \neq 0$, determined by a non-constant $\psi$ given in* (3.3)*, and by the anti-symmetric tensor $\xi^i_{jk}$ given by* (3.4).

EXAMPLE 3.1. Let the generalized Riemannian space $\mathbb{G}\mathbb{R}_3$ be given by the nonsymmetric matrix

$$(g_{ij}) = \begin{bmatrix} 1 & (x^1)^2 + (x^3)^2 & (x^2)^2 \\ -(x^1)^2 - (x^3)^2 & e^{x^1}x^3 & e^{x^1}x^2 + 1 \\ -(x^2)^2 & e^{x^1}x^2 - 1 & e^{x^1}(x^2 + x^3) \end{bmatrix}$$

Suppose that $x^2x^3 + (x^3)^2 - (x^2)^2 \neq 0$. Consider the symmetric and anti-symmetric part of the basic matrix, respectively

$$(g_{\underline{ij}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{x^1}x^3 & e^{x^1}x^2 \\ 0 & e^{x^1}x^2 & e^{x^1}(x^2 + x^3) \end{bmatrix}$$

$$(\underset{\vee}{g_{ij}}) = \begin{bmatrix} 0 & (x^1)^2 + (x^3)^2 & (x^2)^2 \\ -(x^1)^2 - (x^3)^2 & 0 & 1 \\ -(x^2)^2 & -1 & 0 \end{bmatrix}$$

Obviously, this space is equidistant and has the metric form (2.2) for $a(x^1) = 1$ and $b(x^1) = e^{x^1}$. The inverse matrix $(g^{\underline{ij}}) = (g_{\underline{ij}})^{-1}$ is in the form

$$(g^{\underline{ij}}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{x^2+x^3}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)} & \dfrac{-x^2}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)} \\ 0 & \dfrac{-x^2}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)} & \dfrac{x^3}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)} \end{bmatrix}$$

Let us construct a conformal mapping of the space $\mathbb{GR}_3$. According to the previous theorem, we can take $\rho(x^1) = e^{x^1}$, $\rho' \neq 0$, wherefrom $\psi = \frac{1}{2}x^1$ for $c = 0$. From (3.4) we obtain

$$\begin{aligned} &\xi^1_{1\sigma} = \xi^\sigma_{\mu\theta} = 0, \quad \xi^1_{23} = -\frac{1}{2}, \quad \xi^2_{13} = \frac{x^2+x^3}{2e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)}, \\ &\xi^3_{12} = \frac{-x^3}{2e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)}. \end{aligned} \tag{3.6}$$

Thus, the conformal mapping is determined by the deformation tensor

$$P^i_{jk} = \delta^i_j\,\psi_k + \delta^i_k\,\psi_j - \psi^i g_{\underline{jk}} + \xi^i_{jk},$$

where $\psi_1 = \frac{1}{2}$, $\psi_2 = \psi_3 = 0$, and $\xi^i_{jk}$ is given by (3.6).

## 4. Geodesic mappings of generalized equidistant spaces

Let $\mathbb{GR}_\mathbb{N}$ and $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ be two generalized Riemannian spaces.

Definition 4.1. [**15**] A diffeomorphism $f : \mathbb{GR}_\mathbb{N} \to \mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ is called *geodesic mapping* of $\mathbb{GR}_\mathbb{N}$ onto $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ if $f$ maps any geodesic curve in $\mathbb{GR}_\mathbb{N}$ onto a geodesic curve in $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$.

According to [**15, 16**], a necessary and sufficient condition that the mapping $f$ is geodesic is that the deformation tensor has the form

$$P^i_{jk} = \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}, \tag{4.1}$$

where

$$\psi_i = \frac{1}{1+N}P^p_{\underline{ip}} = \frac{1}{1+N}(\overline{\Gamma}^p_{\underline{ip}} - \Gamma^p_{\underline{ip}}), \tag{4.2}$$

$$\xi^i_{jk} = \underset{\vee}{P^i_{jk}} = \underset{\vee}{\overline{\Gamma}^i_{jk}} - \underset{\vee}{\Gamma^i_{jk}}. \tag{4.3}$$

Equation (4.1) can be written as $\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}$.

Definition 4.2. [**15**] A geodesic mapping $f : \mathbb{GR}_\mathbb{N} \to \mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ is *equitorsion* if the torsion tensors of the spaces $\mathbb{GR}_\mathbb{N}$ and $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ are equal in the corresponding points.

According to (4.3), it means that $\underset{\vee}{\overline{\Gamma}}{}^{i}_{jk} - \underset{\vee}{\Gamma}{}^{i}_{jk} = \xi^{i}_{jk} = 0$.

Let us construct a geodesic mapping of generalized equidistant space $\mathbb{GR_N}$ with metric form (2.2) onto generalized equidistant space $\mathbb{G\overline{R}_N}$ with metric form (2.4). By comparing equations which describe symmetric part of deformation tensor (2.5) with necessary and sufficient condition (4.1) of the geodesic mapping we obtain

$$P^{1}_{11} = \frac{1}{2}\Big(\frac{A'}{A} - \frac{a'}{a}\Big) = \psi_1\delta^1_1 + \psi_1\delta^1_1 = 2\psi_1 \Rightarrow \psi_1 = \frac{1}{4}\Big(\frac{A'}{A} - \frac{a'}{a}\Big)$$
$$P^{1}_{\underline{1\sigma}} = 0 = \psi_1\delta^1_\sigma + \psi_\sigma\delta^1_1 \Rightarrow \psi_\sigma = 0$$
$$P^{\sigma}_{\underline{1\mu}} = \frac{1}{2}\Big(\frac{B'}{B} - \frac{b'}{b}\Big)\delta^\sigma_\mu = \psi_1\delta^\sigma_\mu + \psi_\mu\delta^\sigma_1 = \psi_1\delta^\sigma_\mu \Rightarrow \psi_1 = \frac{1}{2}\Big(\frac{B'}{B} - \frac{b'}{b}\Big)$$
$$P^{1}_{\underline{\sigma\mu}} = -\frac{1}{2}\Big(\frac{B'}{A}\hat{g}_{\sigma\mu} - \frac{b'}{a}\tilde{g}_{\sigma\mu}\Big) = \psi_\sigma\delta^1_\mu + \psi_\mu\delta^1_\sigma = 0 \Rightarrow \frac{B'}{A} - \frac{b'}{a} = 0$$

From here, after some calculation, we obtain

$$A = \frac{pa(x^1)}{(1+qb(x^1)^2}, \quad B = \frac{pb(x^1)}{1+qb(x^1)},$$

where $p, q$ are constants such that $p \neq 0$, $1 + qb(x^1) \neq 0$ and $qb'(x^1)$ is not zero identically. The metric of $\mathbb{G\overline{R}_N}$ has the form

$$d\overline{s}^2 = \frac{pa(x^1)}{(1+qb(x^1)^2}(dx^1)^2 + \frac{pb(x^1)}{1+qb(x^1)}\tilde{g}_{\sigma\mu}dx^\sigma dx^\mu. \tag{4.4}$$

Also,

$$\psi = -\frac{1}{2}\ln|1+qb(x^1)| + c, \tag{4.5}$$

where $c$ is a constant. According to (2.6) and (4.4) for all cases of concrete values of indices $i$, $j$, $k$, we get the following system of equations

$$\xi^1_{\sigma 1} = -\underset{\vee}{\Gamma}{}^1_{\sigma 1}, \quad (\xi^\rho_{1\mu} + \underset{\vee}{\Gamma}{}^\rho_{1\mu})\overline{g}_{\underline{\rho\sigma}} + (\xi^1_{\sigma\mu} + \underset{\vee}{\Gamma}{}^1_{\sigma\mu})\overline{g}_{11} = 0,$$
$$(\xi^\rho_{\sigma\theta} + \underset{\vee}{\Gamma}{}^\rho_{\sigma\theta})\overline{g}_{\underline{\rho\mu}} + (\xi^\rho_{\mu\theta} + \underset{\vee}{\Gamma}{}^\rho_{\mu\theta})\overline{g}_{\underline{\sigma\rho}} = 0,$$

i.e.,

$$\begin{aligned} &\xi^1_{\sigma 1} = -\underset{\vee}{\Gamma}{}^1_{\sigma 1}, \quad pb(x^1)(1+qb(x^1))(\xi^\rho_{1\mu} + \underset{\vee}{\Gamma}{}^\rho_{1\mu})\tilde{g}_{\rho\sigma} + pa(x^1)(\xi^1_{\sigma\mu} + \underset{\vee}{\Gamma}{}^1_{\sigma\mu}) = 0, \\ &\qquad (\xi^\rho_{\sigma\theta} + \underset{\vee}{\Gamma}{}^\rho_{\sigma\theta})\tilde{g}_{\rho\mu} + (\xi^\rho_{\mu\theta} + \underset{\vee}{\Gamma}{}^\rho_{\mu\theta})\tilde{g}_{\sigma\rho} = 0, \end{aligned} \tag{4.6}$$

wherefrom we can determine anti-symmetric tensor $\xi^i_{jk}$.

Thus, the following theorem holds

THEOREM 4.1. *Generalized equidistant space* $\mathbb{GR_N}$ *with fundament metric form* (2.2) *admits geodesic mapping* $f$ *on the generalized equidistant space* $\mathbb{G\overline{R}_N}$ *with fundament metric form* (4.4)*, which is nontrivial for* $p \neq 0$, $1+qb(x^1) \neq 0$ *and nonzero* $qb'(x^1)$*, determined by nonconstant* $\psi$ *given in* (4.5)*, and by the antisymmetric tensor* $\xi^i_{jk}$ *given by* (4.6).

EXAMPLE 4.1. By the theorem given above, the function $\psi = -\frac{1}{2}\ln|1+x^1|$ and the antisymmetric tensor

$$\xi^1_{21} = \xi^1_{31} = \xi^2_{23} = \xi^3_{23} = 0, \quad \xi^1_{23} = x^3 - x^2,$$
$$\xi^2_{12} = -\xi^3_{13} = \frac{x^2(x^2-x^3)}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)},$$
$$\xi^2_{13} = \frac{(x^2)^2-(x^3)^2}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)}, \quad \xi^3_{12} = \frac{x^3(x^3-x^2)}{e^{x^1}(x^2x^3+(x^3)^2-(x^2)^2)},$$

determine a geodesic mapping of generalized equidistant space $\mathbb{GR}_3$ given in the Example 3.1 onto equidistant space $\mathbb{R}_3$.

**4.1. Invariant objects of equitorsion geodesic mapping.** Invariant geometrical objects (invariants) are objects that do not change structure according to the corresponding mappings. In the case of geodesic mapping between two Riemannian spaces we have invariant geometric objects: the Tomas projective parameter and the Weyl projective tensor (see, for example, [**10**]). In [**25**] we found some new invariants according to the equitiorsion geodesic mappings $f : \mathbb{GR}_\mathbb{N} \to \mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$. All these objects exist in the space $\mathbb{GR}_\mathbb{N}$ and they are generalization of the Weyl projective tensor. Among five invariants, three of them are tensors and we called them "equitorsion projective tensors", and two of them are not tensors and we called them "equitorsion projective parameters". Using the condition of the equidistant spaces, we can find some interesting invariant geometrical objects which appear under equitorsion geodesic mapping of generalized equidistant spaces.

Let $f : \mathbb{GR}_\mathbb{N} \to \mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ be a geodesic mapping of the generalized Riemannian space which satisfies the condition

$$\psi_{ij} = \omega g_{\underline{ij}}, \tag{4.7}$$

where $\omega$ is an invariant and $\psi_{ij} = \psi_{i;j} - \psi_i\psi_j$, or in the terms of covariant derivatives of the first and the second kinds

$$\underset{1}{\psi}_{ij} = \omega g_{\underline{ij}} - \Gamma^p_{\underset{\vee}{ij}}\psi_p, \quad \text{i.e.} \quad \underset{2}{\psi}_{ij} = \omega g_{\underline{ij}} + \Gamma^p_{\underset{\vee}{ij}}\psi_p, \tag{4.8}$$

$\underset{k}{\psi}_{ij} = \psi_{i\underset{k}{|}j} - \psi_i\psi_j$, $k = 1,2$. Then the space $\mathbb{GR}_\mathbb{N}$ is generalized equidistant whose equidistant congruence is generated by the vector $\psi_i$. Indeed, conditions (4.7) and (4.8) are equivalent to $\varphi_{i;j} = \rho g_{\underline{ij}}$ in the case $\varphi = e^{-\psi}$, $\rho = -\omega e^{-\psi}$ and the last one presents the equation of generalized equidistant space. Further, suppose that the mapping $f$ is equitorsion, i.e., the condition $\xi^i_{jk} = 0$ is in force. Let us find the invariant objects of this mapping of generalized equidistant space $\mathbb{GR}_\mathbb{N}$.

As it is known, the relations between the corresponding kinds of curvatures tensors of the spaces $\mathbb{GR}_\mathbb{N}$ and $\mathbb{G}\overline{\mathbb{R}}_\mathbb{N}$ under equitorsion geodesic mapping are [**22**]

$$
(4.9)\quad
\begin{aligned}
\overline{\underset{1}{R}}{}^i_{jmn} &= \underset{1}{R}{}^i{}_{jmn} + \delta^i_j(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m \underset{1}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^p_{\underset{\vee}{mn}}\psi_p\delta^i_j\\
\overline{\underset{2}{R}}{}^i_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{2}{\psi}_{nm}) + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{2}{\psi}_{jm} + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j + 2\Gamma^p_{\underset{\vee}{nm}}\psi_p\delta^i_j\\
\overline{\underset{3}{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m\\
\overline{\underset{4}{R}}{}^i_{jmn} &= \underset{4}{R}{}^i{}_{jmn} + \delta^i_j(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m\\
\overline{\underset{5}{R}}{}^i_{jmn} &= \underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}\delta^i_j(\underset{1}{\psi}_{mn} - \underset{2}{\psi}_{nm} + \underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \frac{1}{2}\delta^i_m(\underset{1}{\psi}_{jn} + \underset{2}{\psi}_{jn})\\
&\qquad - \frac{1}{2}\delta^i_n(\underset{1}{\psi}_{jm} + \underset{2}{\psi}_{jm}).
\end{aligned}
$$

Let us start from the curvature tensor of the first kind. After using conditions (4.8) we obtain

$$
(4.10)\quad \overline{\underset{1}{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + \omega(\delta^i_m g_{\underline{jn}} - \delta^i_n g_{\underline{jm}}) - (\delta^i_m \Gamma^p_{\underset{\vee}{jn}} - \delta^i_n \Gamma^p_{\underset{\vee}{jm}})\psi_p + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j.
$$

Contracting by indices $i$ and $n$ in the previous equation we get

$$
\overline{\underset{1}{R}}_{jm} = \underset{1}{R}_{jm} + \omega(1-N) - (1-N)\Gamma^p_{\underset{\vee}{jm}}\psi_p,
$$

wherefrom we have

$$
(4.11)\quad \omega g_{\underline{jm}} = \frac{1}{N-1}(\underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm}) + \Gamma^p_{\underset{\vee}{jm}}\psi_p.
$$

Put (4.11) into (4.10) and obtain

$$
(4.12)\quad \overline{\underset{1}{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{1}{R}_{jn} - \overline{\underset{1}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm}) + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j.
$$

In the similar way, starting from the second equation in (4.9) and using (4.8) we obtain the following equation for the curvature tensor of the second kind

$$
(4.13)\quad \overline{\underset{2}{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{2}{R}_{jn} - \overline{\underset{2}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{2}{R}_{jm} - \overline{\underset{2}{R}}_{jm}) + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j.
$$

Let us sum up (4.12) and (4.13). We get

$$
\begin{aligned}
\overline{\underset{1}{R}}{}^i_{jmn} + \overline{\underset{2}{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + \underset{2}{R}{}^i{}_{jmn} &+ \frac{\delta^i_m}{N-1}[\underset{1}{R}_{jn} + \underset{2}{R}_{jn} - (\overline{\underset{1}{R}}_{jn} + \overline{\underset{2}{R}}_{jn})]\\
&- \frac{\delta^i_n}{N-1}[\underset{1}{R}_{jm} + \underset{2}{R}_{jm} - (\overline{\underset{1}{R}}_{jm} + \overline{\underset{2}{R}}_{jm})].
\end{aligned}
$$

If we introduce the notation

$$
\underset{1}{Q}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + \underset{2}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{1}{R}_{jn} + \underset{2}{R}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{1}{R}_{jm} + \underset{2}{R}_{jm}),
$$

we obtain $\overline{\underset{1}{Q}}{}^i_{jmn} = \underset{1}{Q}{}^i_{jmn}$, which means that $\underset{1}{Q}{}^i_{jmn}$ is an invariant object of equitorsion geodesic mapping. Obviously, this object is a tensor.

Analogously to the previous consideration we can determine the relationships between the curvature tenors of the third, the fourth and the fifth kind under equitorsion geodesic mapping satisfying condition (4.8). Thus we have

$$\begin{aligned}
\overline{\underset{3}{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{3}{R}_{jn} - \overline{\underset{3}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{3}{R}_{jm} - \overline{\underset{3}{R}}_{jm}) \\
&\qquad + 2\delta^i_m \Gamma^p_{\underset{\vee}{jn}}\psi_p + 2\psi_n \Gamma^i_{\underset{\vee}{mj}} + 2\psi_m \Gamma^i_{\underset{\vee}{nj}}, \\
\overline{\underset{4}{R}}{}^i_{jmn} &= \underset{4}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{4}{R}_{jn} - \overline{\underset{4}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{4}{R}_{jm} - \overline{\underset{4}{R}}_{jm}) \\
&\qquad + 2\delta^i_m \Gamma^p_{\underset{\vee}{jn}}\psi_p + 2\psi_n \Gamma^i_{\underset{\vee}{mj}} + 2\psi_m \Gamma^i_{\underset{\vee}{nj}}, \\
\overline{\underset{5}{R}}{}^i_{jmn} &= \underset{5}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{5}{R}_{jn} - \overline{\underset{5}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{5}{R}_{jm} - \overline{\underset{5}{R}}_{jm}).
\end{aligned} \tag{4.14}$$

After subtraction of the second equations from the first in (4.14) we obtain

$$\begin{aligned}
\overline{\underset{3}{R}}{}^i_{jmn} - \overline{\underset{4}{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} - \underset{4}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}[\underset{3}{R}_{jn} - \underset{4}{R}_{jn} - (\overline{\underset{3}{R}}_{jn} - \overline{\underset{4}{R}}_{jn})] \\
&\qquad - \frac{\delta^i_n}{N-1}[\underset{3}{R}_{jm} - \underset{4}{R}_{jm} - (\overline{\underset{3}{R}}_{jm} - \overline{\underset{4}{R}}_{jm})].
\end{aligned}$$

After introducing the denotation

$$\underset{2}{Q}{}^i_{jmn} = \underset{3}{R}{}^i{}_{jmn} - \underset{4}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{3}{R}_{jn} - \underset{4}{R}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{3}{R}_{jm} - \underset{4}{R}_{jm}),$$

we obtain $\overline{\underset{2}{Q}}{}^i_{jmn} = \underset{2}{Q}{}^i_{jmn}$, which presents the second invariant tensor of the mapping. From the last equation in (4.14) we get

$$\underset{3}{Q}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}\underset{5}{R}_{jn} - \frac{\delta^i_n}{N-1}\underset{5}{R}_{jm},$$

which is the third invariant tensor, i.e., $\overline{\underset{3}{Q}}{}^i_{jmn} = \underset{3}{Q}{}^i_{jmn}$.

Further, let us use (4.2) and the fact that the torsions of the corresponding spaces are equal under equitorsion mapping in (4.12). Then we have

$$\begin{aligned}
\overline{\underset{1}{R}}{}^i_{jmn} &= \underset{1}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{1}{R}_{jn} - \overline{\underset{1}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm}) \\
&\qquad + 2\Gamma^i_{\underset{\vee}{mn}} \frac{1}{N+1}(\overline{\Gamma}^p_{\underline{jp}} - \Gamma^p_{\underline{jp}}) \\
&= \underset{1}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}(\underset{1}{R}_{jn} - \overline{\underset{1}{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{1}{R}_{jm} - \overline{\underset{1}{R}}_{jm}) \\
&\qquad - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{mn}}\Gamma^p_{\underline{jp}}.
\end{aligned}$$

Let us denote

$$\underset{4}{Q}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}\underset{1}{R}_{jn} - \frac{\delta^i_n}{N-1}\underset{1}{R}_{jm} - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{mn}}\Gamma^p_{\underline{jp}}.$$

The object $\underset{4}{Q}{}^i_{jmn}$ is invariant of the mapping, but is not a tensor. In the similar way we can write the second, the third and the fourth curvature tensor

$$\begin{aligned}
\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} &+ \frac{\delta^i_m}{N-1}(\underset{2}{R}_{jn} - \underset{2}{\overline{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{2}{R}_{jm} - \underset{2}{\overline{R}}_{jm}) \\
&+ \frac{2}{N+1}\overline{\Gamma}{}^i_{\underset{\vee}{nm}}\overline{\Gamma}{}^p_{\underline{jp}} - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{nm}}\Gamma^p_{\underline{jp}}. \\
\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i{}_{jmn} &+ \frac{\delta^i_m}{N-1}(\underset{3}{R}_{jn} - \underset{3}{\overline{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{3}{R}_{jm} - \underset{3}{\overline{R}}_{jm}) \\
&+ \frac{2}{N+1}\delta^i_m(\overline{\Gamma}{}^p_{\underset{\vee}{jn}}\overline{\Gamma}{}^q_{\underline{pq}} - \Gamma^p_{\underset{\vee}{jn}}\Gamma^q_{\underline{pq}}) \\
&+ \frac{2}{N+1}(\overline{\Gamma}{}^i_{\underset{\vee}{mj}}\overline{\Gamma}{}^p_{\underline{np}} - \Gamma^i_{\underset{\vee}{mj}}\Gamma^p_{\underline{np}}) + \frac{2}{N+1}(\overline{\Gamma}{}^i_{\underset{\vee}{nj}}\overline{\Gamma}{}^p_{\underline{mp}} - \Gamma^i_{\underset{\vee}{nj}}\Gamma^p_{\underline{mp}}), \\
\underset{4}{\overline{R}}{}^i_{jmn} = \underset{4}{R}{}^i{}_{jmn} &+ \frac{\delta^i_m}{N-1}(\underset{4}{R}_{jn} - \underset{4}{\overline{R}}_{jn}) - \frac{\delta^i_n}{N-1}(\underset{4}{R}_{jm} - \underset{4}{\overline{R}}_{jm}) \\
&+ \frac{2}{N+1}\delta^i_m(\overline{\Gamma}{}^p_{\underset{\vee}{jn}}\overline{\Gamma}{}^q_{\underline{pq}} - \Gamma^p_{\underset{\vee}{jn}}\Gamma^q_{\underline{pq}}) \\
&+ \frac{2}{N+1}(\overline{\Gamma}{}^i_{\underset{\vee}{mj}}\overline{\Gamma}{}^p_{\underline{np}} - \Gamma^i_{\underset{\vee}{mj}}\Gamma^p_{\underline{np}}) + \frac{2}{N+1}(\overline{\Gamma}{}^i_{\underset{\vee}{nj}}\overline{\Gamma}{}^p_{\underline{mp}} - \Gamma^i_{\underset{\vee}{nj}}\Gamma^p_{\underline{mp}})
\end{aligned}$$

wherefrom we get three more invariant parameters of the mapping which are not tensors:

$$\begin{aligned}
\underset{5}{Q}{}^i_{jmn} &= \underset{2}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}\underset{2}{R}_{jn} - \frac{\delta^i_n}{N-1}\underset{2}{R}_{jm} - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{nm}}\Gamma^p_{\underline{jp}}, \\
\underset{6}{Q}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}\underset{3}{R}_{jn} - \frac{\delta^i_n}{N-1}\underset{3}{R}_{jm} - \frac{2}{N+1}\delta^i_m\Gamma^p_{\underset{\vee}{jn}}\Gamma^q_{\underline{pq}} \\
&\qquad - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{mj}}\Gamma^p_{\underline{np}} - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{nj}}\Gamma^p_{\underline{mp}}, \\
\underset{7}{Q}{}^i_{jmn} &= \underset{4}{R}{}^i{}_{jmn} + \frac{\delta^i_m}{N-1}\underset{4}{R}_{jn} - \frac{\delta^i_n}{N-1}\underset{4}{R}_{jm} - \frac{2}{N+1}\delta^i_m\Gamma^p_{\underset{\vee}{jn}}\Gamma^q_{\underline{pq}} \\
&\qquad - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{mj}}\Gamma^p_{\underline{np}} - \frac{2}{N+1}\Gamma^i_{\underset{\vee}{nj}}\Gamma^p_{\underline{mp}}.
\end{aligned}$$

Thus, we proved the following theorem.

THEOREM 4.2. *Let $f : \mathbb{GR_N} \to \mathbb{G\overline{R}_N}$ be an equitorsion geodesic mapping of generalized equidistant space $\mathbb{GR_N}$ in which the equidistant congruence is generated by the vector $\psi_i$ such that $\varphi = e^{-\psi}$. Then $\underset{1}{Q}{}^i_{jmn}$, $\underset{2}{Q}{}^i_{jmn}$, $\underset{3}{Q}{}^i_{jmn}$ are invariant tensors and $\underset{4}{Q}{}^i_{jmn}$, $\underset{5}{Q}{}^i_{jmn}$, $\underset{6}{Q}{}^i_{jmn}$, $\underset{7}{Q}{}^i_{jmn}$ are invariant parameters (not tensors) of this mapping.*

## 5. Conclusion

The notion of generalized equidistant spaces first appeared in our paper [**1**]. We here continued the idea of generalized equidistant spaces, studying conformal and geodesic mappings of such spaces. We proved the existence of mentioned nontrivial mappings and constructed examples of conformal and geodesic mapping of a generalized equidistant space $\mathbb{GR}_3$.

Also, we found three invariant tensors and four invariant objects which are not tensors under geodesic mapping. By linear combinations of the obtained objects one can form new interesting invariant objects, but the question is how many of them are linearly independent and what they are.

The equidistant spaces are defined as the spaces satisfying the condition $\varphi^i_{;j} = \rho\delta^i_j$, for some vector field $\varphi$ and a function $\rho$.

Due to the fact that in the generalized Riemannian spaces there are four kinds of covariant derivatives with respect to the connection $\Gamma^i_{jk}$, we can also define generalized equidistant spaces of the first and second kinds respectively by the conditions

$$\varphi^i_{\underset{1}{|}j} = \rho\delta^i_j, \quad \varphi^i_{\underset{2}{|}j} = \rho\delta^i_j.$$

(Note that the covariant derivatives of the third and fourth kinds are reduced to the second and the first kind, respectively, in the case of $\varphi^i$.)

All this opens new questions and gives quite interesting ideas for the further investigation.

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