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GEODESIC MAPPINGS ON COMPACT RIEMANNIAN MANIFOLDS WITH CONDITIONS ON SECTIONAL CURVATURE

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Abstract. We found new criteria for sectional curvatures on compact Riemannian manifolds for which geodesic mappings are affine, and, moreover, homothetic.

1. Introduction

To the theory of geodesic mappings and their transformations have been devoted many papers, these results are formulated in a large number of research papers and monographs [**2**, **4**–**12**, **16**–**19**, **21**–**26**, **30**, **33**], etc.

In 1953, Takeno and Ikeda [**31**] considered geodesic mappings of spherically symmetric spaces V_4, in 1954 Sinyukov [**26**, p. 88] studied the case of symmetric and recurrent spaces and, in 1976 Mikeš ([**13**,**16**], [**21**, p. 206], [**26**, pp. 151–155]) proved that generalized recurrent (pseudo-) Riemannian spaces V_n with nonconstant curvature do not admit nontrivial geodesic mappings. In this topic Prvanović [**23**] and Sobchuk [**20**, **29**] also have been interested. These results were obtained "locally" and they are contained in [**14**, **16**, **21**, **26**].

Global results for geodesic mappings of compact Riemannian manifolds were obtained by Vrançeanu [**33**], Sinyukova [**27**, **28**], Mikeš [**15**, **16**], etc.

The above results are related to questions of projective rigidity of (pseudo-) Riemannian manifolds and also of manifolds with affine connections.

In [**10**] and [**11**] we proved that these mappings preserve the smoothness class of metrics of geodesically equivalent (pseudo-) Riemannian manifolds. In [**10**] it was sufficient to suppose the metrics to be of differentiability class C^2, and in [**11**] to be of class C^1.

We present new results on geodesic mappings of compact Riemannian manifolds with certain conditions on the sectional curvature of the Ricci directions.

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2. Geodesic mapping theory

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be n-dimensional (pseudo-) Riemannian manifolds with metrics g and $\bar{g}$, respectively.

DEFINITION 2.1. A diffeomorphism $f : V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

We restricted ourselves to the study of a coordinate neighborhood (U, x) of the points $x \in V_n$ and $f(x) \in \bar{V}_n$. The points x and $f(x)$ have the same coordinates $x = (x^1, \dots, x^n)$. We assume that V_n, $\bar{V}_n \in C^1$ $(g, \bar{g} \in C^1)$ if their components $g_{ij}(x), \bar{g}_{ij}(x) \in C^1$ on (U, x), respectively.

It is known [**12**], see [**6**, pp. 131–133], [**21**, p. 167], that V_n admits a geodesic mapping onto $\bar{V}_n$ if and only if the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} \tag{2.1}$$

hold, where ∇ is the Levi-Civita connection on V_n and

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{n+1} \ln \sqrt{|\det \bar{g}/\det g|}, \quad \partial_i = \partial/\partial x^i.$$

Sinyukov [**26**, p. 121], see [**21**, p. 167], proved that the Levi-Civita equations (2.1) are equivalent to

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{2.2}$$

where

$$\text{(a)}\ \ a_{ij} = e^{2\Psi}\, \bar{g}^{\alpha\beta} g_{i\alpha} g_{j\beta}; \quad \text{(b)}\ \ \lambda_i = -e^{2\Psi}\, \bar{g}^{\alpha\beta} \psi_\alpha g_{i\beta}, \tag{2.3}$$

and, moreover, $\lambda_i = \partial_i \Lambda$, $\Lambda = \frac{1}{2}\, a_{\alpha\beta} g^{\alpha\beta}$. Here $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ and $(g^{ij}) = (g_{ij})^{-1}$.

On the other hand:

$$\bar{g}_{ij} = e^{2\Psi} \hat{g}_{ij}, \quad \Psi = \ln \sqrt{|\det \hat{g}/\det g|}, \quad (\hat{g}_{ij}) = (a_{\alpha\beta} g^{i\alpha} g^{j\beta})^{-1}.$$

Furthermore, we assume that $V_n = (M, g) \in C^2$ and $\bar{V}_n = (M, \bar{g}) \in C^2$. In this case, the integrability conditions of the equations (2.2), due to the Ricci identity

$$\nabla_l \nabla_k a_{ij} - \nabla_k \nabla_l a_{ij} = a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl}, \tag{2.4}$$

have the following form

$$a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl} = g_{ik} \nabla_l \lambda_j + g_{jk} \nabla_l \lambda_i - g_{il} \nabla_k \lambda_j - g_{jl} \nabla_k \lambda_i, \tag{2.5}$$

where R^h_{ijk} are components of the Riemannian tensor R on V_n, and after contraction with g^{ik} we get [**26**, p. 133]

$$n \nabla_l \lambda_j = \mu\, g_{jl} - a_{j\alpha} R^\alpha_l - a_{\alpha\beta} R_j{}^{\alpha\beta}{}_l, \tag{2.6}$$

where $\mu = \nabla_\alpha \lambda^\alpha$, $R^\alpha_i = g^{\alpha\beta} R_{\beta i}$ and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Ricci tensor Ric on V_n.

3. Integral formula

We introduce the vector field ξ on $V_n \in C^2$ in the following way

$$\xi^i = a^\alpha_\beta \nabla_\alpha a^{i\beta} - a^i_\beta \nabla_\alpha a^{\alpha\beta}, \tag{3.1}$$

where $a^l_i = g^{l\alpha} a_{\alpha i}$, $a^{ij} = a_{\alpha\beta} g^{i\alpha} g^{j\beta}$. Using formula (3.1), the Ricci identity (2.4) and Sinyukov's equations (2.2) we obtained that the divergence of the vector ξ has the following representation

$$\operatorname{div} \xi = \Phi(a) - (n-1)(n+2)\, \lambda_\alpha \lambda_\beta g^{\alpha\beta},$$

where $\Phi(a) = R_{ij} a^{ik} a^j_k - R_{ijkl} a^{ik} a^{jl}$.

Suppose that the Riemannian manifold (M, g) is compact and without boundary, then on the basis of the Gauß theorem $\int_M \operatorname{div} \xi \, d\nu = 0$ we obtain the integral formula

$$\int_M \Phi(a)\, d\nu = (n-1)(n+2) \int_M \lambda_\alpha \lambda_\beta g^{\alpha\beta} d\nu. \tag{3.2}$$

For applying the Gauss theorem it is necessary to require the orientability of M, if M is a non-orientable manifold, then we'll look at the oriented double cover.

Let $g(e_i, e_j) = \delta_{ij}$ and $a(e_i, e_j) = \alpha_i \delta_{ij}$ with the Kronecker symbol δ_{ij}, i.e., $\{e_1, \dots, e_n\}$ is the orthonormal basis of eigenvectors to the eigenvalues $\alpha_1, \dots, \alpha_n$ of the tensor $a = (a_{ij})$ of $T_x M$ at any point $x \in M$. As we can see from direct calculation, $\Phi(a)$ has the following form (see [**3**, p. 592]):

$$\Phi(a) = \sum_{i<j} K(e_i, e_j)(\alpha_i - \alpha_j)^2, \tag{3.3}$$

where $K(e_i, e_j)$ are sectional curvatures in the two-directions $e_i \wedge e_j$.

It is easy to see:

$$\begin{aligned}
\Phi(a) = R_{ij} a^{ik} a^j_k - R_{ijkl} a^{ik} a^{jl} &= \sum_{i,j} (\alpha_i)^2 R_{ijij} - \sum_{i,j} \alpha_i \alpha_j R_{ijij} \\
&= \sum_{i<j} ((\alpha_i)^2 + (\alpha_j)^2) \cdot R_{ijij} - 2 \sum_{i<j} \alpha_i \alpha_j R_{ijij} \\
&= \sum_{i<j} (\alpha_i - \alpha_j)^2 \cdot R_{ijij} = \sum_{i<j} (\alpha_i - \alpha_j)^2 \cdot K(e_i, e_j),
\end{aligned}$$

where

$$K(e_i, e_j) = \frac{R(e_i, e_j, e_i, e_j)}{g(e_i, e_i) \cdot g(e_j, e_j) - (g(e_i, e_j))^2} = R_{ijij}.$$

4. Principal orthonormal basis

Eisenhart [**6**, pp. 113–114] introduced a *principal direction* in a Riemannian manifold (M, g), as an eigenvector of the Ricci tensor. He showed that at any point $x \in M$ there exists the orthonormal basis $\{e_1, \dots, e_n\}$ in which

$$g_{ij} = \delta_{ij} \quad \text{and} \quad R_{ij} = \rho_i \delta_{ij},$$

i.e., $e_1, \dots, e_n$ are the vectors of the principal directions and $\rho_1, \dots, \rho_n$ are their eigenvalues. This basis is called the *principal orthonormal basis*.

This means that the existence of this basis is a property only of the Riemannian manifold (M, g), independent of the solution a_{ij} of equation (2.2). Generally the set of principal orthonormal bases is a proper subset of the set of orthonormal bases. Because the vector field λ_i is gradient-like, formula (2.6) implies [**26**, p. 138]

$$a_{i\alpha}R^{\alpha}_{j} = a_{j\alpha}R^{\alpha}_{i}.$$

So the tensors a_{ij} and R_{ij} commute and have common eigenvectors. From this fact it follows that there exist a principal orthonormal basis in which $g_{ij} = \delta_{ij}$ and $a_{ij} = \alpha_i\delta_{ij}$ hold. This basis is called a *joint principal orthonormal basis*. Note that we do not restrict the signature of the Ricci tensor and the tensor a_{ij}. In the following we restrict ourselves to the study of formulas (3.2) and (3.3) on joint principal orthonormal bases.

5. Main Theorems

For the following we recall that a compact Riemannian manifold V_n admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n$.

If we assume that at each point $x \in M$ all sectional curvatures $K(e_i, e_j)$ are non-positive in the two-directions $e_i \wedge e_j$ of the joint principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main directions of the Ricci tensor, then from integral formula (3.2) it follows

$$\text{(5.1)} \qquad \text{(a)} \int_M \Phi(a)\, d\nu = 0 \ \text{ and } \ \text{(b)} \int_M \lambda_\alpha\lambda_\beta g^{\alpha\beta}\, d\nu = 0.$$

From integral (5.1b) follows $\lambda_\alpha\lambda_\beta g^{\alpha\beta} = 0$ and this fact implies that λ_i is vanishing on M, i.e., $\lambda_1 = \dots = \lambda_n = 0$. In this case, the geodesic mapping is *affine* (see [**21**, p. 150]). We proved the following theorem:

Theorem 5.1. *Assume a compact Riemannian manifold (M, g) without boundary of dimension $n \geqslant 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive for any two-direction $e_i \wedge e_j$ from all the principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main direction of the Ricci tensor, then any geodesic mapping of (M, g) is affine.*

Moreover, we suppose at each point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and that there is a certain point $x_0 \in M$ where the sectional curvature $K(e_i, e_j)$ in any two-direction $e_i \wedge e_j$ of the joint principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main directions of the Ricci tensor is negative. Then from integral (3.2) follows equation (5.1). On the basis of Theorem 5.1 it follows $\lambda_1 = \dots = \lambda_n = 0$ and the geodesic mapping is affine.

Further, from integral (5.1a) follows $\Phi(a) = 0$ on M. Then from formula (3.3) at the point $x_0 \in M$ we obtain $\alpha_1 = \dots = \alpha_n = \alpha$. Hence $a_{ij} = \alpha\delta_{ij}$, i.e., $a_{ij}(x_0) = \alpha g_{ij}(x_0)$.

In this case, the affine mapping is homothetic, i.e., $\bar{g} = \alpha' g$, where $\alpha' = \text{const}$. This fact follows from the uniqueness of solutions of the fundamental equations of affine mappings $V_n \to \bar{V}_n : \nabla_k \bar{g}_{ij} = 0$ with initial values $\bar{g}_{ij}(x_0) = \alpha' g_{ij}(x_0)$. This is equivalent to $a_{ij}(x_0) = \alpha g_{ij}(x_0)$, this fact follows from equation (2.3).

We proved the following theorem:

THEOREM 5.2. *Assume a compact Riemannian manifold (M, g) without boundary of dimension $n \geqslant 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and if there is a certain point $x_0 \in M$, where the sectional curvature $K(e_i, e_j)$ is negative in any two-direction $e_i \wedge e_j$ of all the principal orthonormal basis $\{e_1, \dots, e_n\}$ of vectors of the main directions of the Ricci tensor, then any geodesic mapping of (M, g) is homothetic.*

These Theorems generalize the results of Mikeš [**15**] (see [**16**]), which were obtained by means of modifications of integral inequalities obtained by Švec [**1**, p. 10].

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