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ON BOUNDED DUAL-VALUED DERIVATIONS ON CERTAIN BANACH ALGEBRAS

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Abstract. We consider the class $\mathfrak{D}(\mathcal{U})$ of bounded derivations $\mathcal{U} \overset{d}{\to} \mathcal{U}^*$ defined on a Banach algebra $\mathcal{U}$ with values in its dual space $\mathcal{U}^*$ so that $\langle x, d(x)\rangle = 0$ for all $x \in \mathcal{U}$. The existence of such derivations is shown, but lacking the simplest structure of an inner one. We characterize the elements of $\mathfrak{D}(\mathcal{U})$ if $\operatorname{span}(\mathcal{U}^2)$ is dense in $\mathcal{U}$ or if $\mathcal{U}$ is a unitary dual Banach algebra.

1. Introduction

Throughout this article let $\mathcal{U}$ be a complex Banach algebra endowed with a norm $\|\circ\|$. Let $\mathcal{U}^\sharp$ be the algebra $\mathcal{U}$ plus an adjoined unit element e with the usual Banach algebra structure. As usual, by $\mathcal{U}^*$ and $\left(\mathcal{U}^\sharp\right)^*$ we will denote the dual spaces of $\mathcal{U}$ and $\mathcal{U}^\sharp$ respectively. Let $j : \mathcal{U} \hookrightarrow \mathcal{U}^\sharp$ and $p : \mathcal{U}^\sharp \to \mathcal{U}$ be the natural injection and the corresponding projection of $\mathcal{U}$ into $\mathcal{U}^\sharp$ and of $\mathcal{U}^\sharp$ onto $\mathcal{U}$ respectively. Then $\mathcal{U}^\sharp = \mathbb{C}\cdot e \bigoplus j(\mathcal{U})$, i.e., any element $\eta \in \mathcal{U}^\sharp$ can be written in a unique way as $\eta = ae + j(x)$, with $x \in \mathcal{U}$ and $a \in \mathbb{C}$, and its $\mathcal{U}^\sharp$-norm is given as $\|\eta\|_{\mathcal{U}^\sharp} = |a| + \|x\|$. Indeed, since j is an isometric homomorphism then $j(\mathcal{U})$ becomes a closed ideal of $\mathcal{U}^\sharp$. Further, $p \circ j = \mathrm{Id}_{\mathcal{U}}$ while $j \circ p$ is the linear projection of $\mathcal{U}^\sharp$ onto $j(\mathcal{U})$. Thus, let $e^* \in \left(\mathcal{U}^\sharp\right)^*$ be defined as $\langle \eta, e^*\rangle \triangleq a$ if $\eta = ae + j(x)$ in $\mathcal{U}^\sharp$. Then
$$\left(\mathcal{U}^\sharp\right)^* = \mathbb{C}\cdot e^* \bigoplus \operatorname{rank}(p^*),$$
where $p^* : \mathcal{U}^* \to \left(\mathcal{U}^\sharp\right)^*$ is the adjoint operator of p. It is well known that $\mathcal{U}^*$ admits a Banach $\mathcal{U}$-bimodule structure if for $x, y \in \mathcal{U}$ and $x^* \in \mathcal{U}^*$ we write
$$\langle y, xx^*\rangle \triangleq \langle yx, x^*\rangle \text{ and } \langle y, x^*x\rangle \triangleq \langle xy, x^*\rangle .$$

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The $\mathcal{U}^\sharp$-bimodule structure on $\left(\mathcal{U}^\sharp\right)^*$ is given as

$$(ae + j(x))(be^* + p^*(x^*)) \triangleq (ab + \langle x, x^* \rangle)e^* + p^*(ax^* + xx^*),$$
$$(be^* + p^*(x^*))(ae + j(x)) \triangleq (ab + \langle x, x^* \rangle)e^* + p^*(ax^* + x^*x),$$

where $a, b \in \mathbb{C}$, $x \in \mathcal{U}$, $x^* \in \mathcal{U}^*$.

Given a Banach $\mathcal{U}$-bimodule $\mathfrak{X}$ let $\mathcal{Z}^1(\mathcal{U}, \mathfrak{X})$ be the Banach space of bounded derivations $d : \mathcal{U} \to \mathfrak{X}$, i.e., those $d \in \mathcal{B}(\mathcal{U}, \mathfrak{X})$ that satisfy the Leibnitz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{U}$. A bounded derivation d is said to be inner if there is an element $\phi \in \mathfrak{X}$ so that $d(x) = x\phi - \phi x$ if $x \in \mathcal{U}$. In that case we write $d = \mathrm{ad}_\phi$ and the class of inner derivations from $\mathcal{U}$ into $\mathfrak{X}$ is denoted as $\mathcal{N}^1(\mathcal{U}, \mathfrak{X})$. The quotient $\mathcal{H}^1(\mathcal{U}, \mathfrak{X}) \triangleq \mathcal{Z}^1(\mathcal{U}, \mathfrak{X})/\mathcal{N}^1(\mathcal{U}, \mathfrak{X})$ defines the first Hochschild cohomology group of $\mathcal{U}$ with coefficients in $\mathfrak{X}$. Kamowitz lay the functional analytic overtones required to adapt the theory of Banach algebras to the Hochschild algebraic setting (cf. [**6**]; see also [**4**]). The theory of *amenable Banach algebras* was greatly influenced by Johnson's memoire in 1972 (cf. [**5**]). A Banach algebra $\mathcal{U}$ is called *amenable* if $\mathcal{H}^1(\mathcal{U}, \mathfrak{X}^*) = \{0\}$ for any Banach $\mathcal{U}$-bimodule $\mathfrak{X}$. A Banach algebra $\mathcal{U}$ it is called *weakly amenable* if $\mathcal{H}^1(\mathcal{U}, \mathcal{U}^*) = \{0\}$. This last notion generalizes that introduced by Bade, Curtis and Dales in [**2**].

In [**3**] it was proved that a non-unital abelian Banach algebra $\mathcal{U}$ is weakly amenable if and only if $\mathcal{U}^\sharp$ is weakly amenable but the general case still remains open. Our goal in this article is to seek relationships between derivations on a non-abelian non-unital Banach algebra $\mathcal{U}$ with values in $\mathcal{U}^*$ and derivations in $\mathcal{U}^\sharp$ with values in $(\mathcal{U}^\sharp)^*$. Our investigation naturally bring us to introduce the notion of $\mathfrak{D}$-*derivations on* $\mathcal{U}$ in Definition 2.1. Althought any element of $\mathcal{N}^1(\mathcal{U}, \mathcal{U}^*)$ is a $\mathfrak{D}$-derivation on $\mathcal{U}$ sometimes there exist non-inner $\mathfrak{D}$-derivations as we will see in the examples 2.2 and 2.3. In Proposition 2.1 we consider certain Banach projective tensor products all of whose derivations are $\mathfrak{D}$-derivations. In Theorem 2.1 we characterize $\mathfrak{D}$-derivations on $\mathcal{U}$ on Banach algebras $\mathcal{U}$ so that $\mathcal{U}^2$ is dense in $\mathcal{U}$, where $\mathcal{U}^2 = \mathrm{span}\{xy : x, y \in \mathcal{U}\}$. The relationship between $\mathfrak{D}$-derivations on $\mathcal{U}$ and their extensions to the unitization $\mathcal{U}^\sharp$ are studied in Proposition 2.2. In this context, inner $\mathfrak{D}$-derivations on $\mathcal{U}$ are characterized in Corollary 2.1. Finally, in Proposition 2.3 we characterize $\mathfrak{D}$-derivations on dual Banach algebras.

2. On $\mathfrak{D}$-derivations

DEFINITION 2.1. A derivation $d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*)$ is called a $\mathfrak{D}$-derivation on $\mathcal{U}$ if $\langle x, d(x) \rangle = 0$ for all $x \in \mathcal{U}$. Let $\mathfrak{D}(\mathcal{U})$ be the set of $\mathfrak{D}$-derivations on $\mathcal{U}$.

EXAMPLE 2.1. All inner $\mathcal{U}^*$-valued derivations on $\mathcal{U}$ are $\mathfrak{D}$-derivations on $\mathcal{U}$.

EXAMPLE 2.2. Let $\mathcal{U} \triangleq C_0^{(1)}[a, b]$ be the commutative Banach algebra of functions $x : [a, b] \to \mathbb{C}$ with continuous derivative $\dot{x}$ so that $x(a) = x(b)$ endowed with the norm $\|x\| \triangleq \|x\|_\infty + \|\dot{x}\|_\infty$. Then we define $d : \mathcal{U} \to \mathcal{U}^*$ as

$$\langle y, d(x) \rangle = \int_a^b y\, dx \quad \text{if } x, y \in \mathcal{U}.$$

The above Riemann–Stieltjes integral is well defined, d becomes clearly a $\mathbb{C}$-linear functional and

$$|\langle y, d(x)\rangle| \leqslant \int_a^b |y|\, d|x| \leqslant \|y\|_\infty |\dot{x}\|_\infty \leqslant \|y\|\,\|x\|,$$

i.e., $d \in \mathcal{B}(\mathcal{U},\mathcal{U}^*)$ and $\|d(x)\| \leqslant \|x\|$ for all $x \in \mathcal{U}$. Indeed,

$$\begin{aligned}\langle y, d(x_1x_2)\rangle &= \int_a^b y\frac{d}{dt}\,(x_1x_2)\,dt \\ &= \int_a^b y\Big(\frac{dx_1}{dt}x_2 + x_1\frac{dx_2}{dt}\Big)dt \\ &= \int_a^b yx_2dx_1 + \int_a^b yx_1dx_2 \\ &= \langle yx_2, d(x_1)\rangle + \langle yx_1, d(x_2)\rangle \\ &= \langle y, d(x_1)x_2 + x_1d(x_2)\rangle,\end{aligned}$$

i.e., $d \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$. Certainly, it is a nonzero $\mathfrak{D}$-derivation since for $x \in \mathcal{U}$ we see that

$$\langle x, d(x)\rangle = \int_a^b x\frac{dx}{dt}\,dt = \frac{x^2}{2}\Big|_a^b = 0.$$

Further, d is not inner because $\mathcal{U}$ is abelian and so $\mathcal{N}^1(\mathcal{U},\mathcal{U}^*) = \{0\}$.

REMARK 2.1. Given a *dual Banach pair* $(\mathfrak{X}, \mathfrak{Y}, \langle\circ,\circ\rangle)$ by the universal property characteristic of general tensor products there is a unique operation on $\mathfrak{X}\otimes\mathfrak{Y}$ so that

$$(x_1\otimes y_1)\,(x_2\otimes y_2) = \langle x_2, y_1\rangle\,(x_1\otimes y_2) \ \text{ if } x_1, x_2 \in \mathfrak{X},\ y_1, y_2 \in \mathfrak{Y}.$$

Then $\mathfrak{X}\otimes\mathfrak{Y}$ becomes an algebra. Further, if for $u \in \mathfrak{X}\otimes\mathfrak{Y}$ we write

$$\|u\|_\pi = \inf\Big\{\sum_{j=1}^n \|x_j\|\,\|y_j\| : u = \sum_{j=1}^n x_j\otimes y_j\Big\}$$

then $(\mathfrak{X}\otimes\mathfrak{Y}, \|\circ\|_\pi)$ becomes a normed algebra. The completion of this algebra is the well known projective Banach tensor algebra $\mathfrak{X}\hat{\otimes}\mathfrak{Y}$ (cf. [**8**, B.2.2, p. 250]). Then, $\mathfrak{X}\hat{\otimes}\mathfrak{Y}$ is amenable if and only if $\dim(\mathfrak{X}) = \dim(\mathfrak{Y}) < \infty$ (cf. [**8**, Th. 4.3.5, p. 98]). So, if $\mathfrak{X}$ is an infinite dimensional Banach space the determination of structure theorems of bounded derivations on $\mathfrak{X}\hat{\otimes}\mathfrak{X}^*$ has its own interest. Moreover, several Banach operator algebras can be represented as certain tensor products of the above type. For instance, if the Banach space $\mathfrak{X}$ has the approximation property, then $\mathcal{N}_{\mathfrak{X}^*}(\mathfrak{X}) \approx \mathfrak{X}\hat{\otimes}\mathfrak{X}^*$, where $\approx$ denotes an isometric isomorphism and $\mathcal{N}_{\mathfrak{X}^*}(\mathfrak{X})$ is the Banach space of $\mathfrak{X}^*$-nuclear operators on $\mathfrak{X}$ (cf. [**8**, Th. C.1.5, p. 256]). In this setting the authors recently researched on structure theorems and properties of derivations on some non-amenable nuclear Banach algebras (see [**1**]).

PROPOSITION 2.1. *Let $\mathfrak{X}$ is an infinite dimensional Banach space endowed with and shrinking basis $\{x_n\}_{n=1}^\infty$ and an associated sequence of coefficient functionals $\{x_n^*\}_{n=1}^\infty$ and let $\mathcal{U} \triangleq \mathfrak{X}\hat{\otimes}\mathfrak{X}^*$. Then $\mathfrak{D}(\mathcal{U}) \supseteq \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$.*

PROOF. Let $d \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$. The system of basic tensor products $z_{n,m} \triangleq x_n \otimes x_m^*$ can be arranged into a basis $\{z_{n,m}\}_{n,m=1}^{\infty}$ of $\mathfrak{X} \hat{\otimes} \mathfrak{X}^*$ (the reader can see [**9**], or else [**10**, Th. 18.1, p. 172]). Given $p, q, r, s, t \in \mathbb{N}$ we have

$$\begin{aligned} \langle z_{p,q}, d(z_{r,t})\rangle &= \langle z_{p,q}, d(z_{r,s}) \cdot z_{s,t} + z_{r,s} \cdot d(z_{s,t})\rangle \\ &= \langle z_{s,t} \cdot z_{p,q}, d(z_{r,s})\rangle + \langle z_{p,q} \cdot z_{r,s}, d(z_{s,t})\rangle \\ &= \delta_{p,t}\langle z_{s,q}, d(z_{r,s})\rangle + \delta_{q,r}\langle z_{p,s}, d(z_{s,t})\rangle, \end{aligned} \tag{2.1}$$

where δ denotes the usual Kronecker's symbol. By (2.1), $\langle z_{p,q}, d(z_{r,t})\rangle = 0$ if $p \neq t$ and $q \neq r$. Using (2.1) we also get

$$\langle z_{p,q}, d(z_{q,p})\rangle = \langle z_{s,q}, d(z_{q,s})\rangle + \langle z_{p,s}, d(z_{s,p})\rangle \tag{2.2}$$

if $p, q, s \in \mathbb{N}$. By (2.2) we see that

$$\langle z_{p,p}, d(z_{p,p})\rangle = 0 \quad \text{if } p \in \mathbb{N}. \tag{2.3}$$

On the other hand, by (2.1) we obtain

$$\langle z_{p,q}, d(z_{p,q})\rangle = 0 \quad \text{if } p, q \in \mathbb{N},\ p \neq q. \tag{2.4}$$

Now, let F be a finite subset of $\mathbb{N} \times \mathbb{N}$, $\left\{\lambda_{(n,m)}\right\}_{(n,m)\in F} \subseteq \mathbb{C}$ and let

$$u = \sum_{(n,m)\in F} \lambda_{(n,m)} z_{n,m}.$$

By (2.3) and (2.4) we see that

$$\langle u, d(u)\rangle = \sum_{(n,m),(p,q)\in F} \lambda_{(n,m)}\lambda_{(p,q)}[\langle z_{n,m}, d(z_{p,q})\rangle + \langle z_{p,q}, d(z_{n,m})\rangle]. \tag{2.5}$$

As we already observed, those summands in (2.5) so that $n \neq q$ and $m \neq p$ are zero. By symmetry, it suffices to consider $n = q$ and then

$$\begin{aligned} \langle z_{n,m}, d(z_{p,n})\rangle + \langle z_{p,n}, d(z_{n,m})\rangle &= \langle z_{n,n}, z_{n,m} \cdot d(z_{p,n}) + d(z_{n,m}) \cdot z_{p,n}\rangle \\ &= \langle z_{n,n}, d(z_{n,n})\rangle \\ &= 0. \end{aligned}$$

Consequently $\langle u, d(u)\rangle = 0$. Finally, the result holds since $u \to \langle u, d(u)\rangle$ is continuous on $\mathcal{U}$ and $\mathfrak{X} \otimes \mathfrak{X}^*$ is dense in $\mathcal{U}$. □

EXAMPLE 2.3. Let $\mathcal{U} \triangleq l^p \widehat{\otimes} l^q$, with $1 < p, q < \infty$, $1/p + 1/q = 1$. If $x \in l^p$, $x^* \in l^q$ let

$$\underline{d}_{x,x^*} : l^p \times l^q \to \mathbb{C}, \underline{d}_{x,x^*}(y, y^*) \triangleq \langle x, y^*\rangle - \langle y, x^*\rangle.$$

Then $\underline{d}_{x,x^*} \in \mathcal{B}^2(l^p, l^q, \mathbb{C})$, i.e., $\underline{d}_{x,x^*}$ is a bounded bilinear form between $l^p \times l^q$ and $\mathbb{C}$. By the universal characteristic property of the projective tensor product of Banach spaces there is a unique $\overline{d}_{x,x^*} \in \mathcal{U}^*$ so that $\|\overline{d}_{x,x^*}\| = \|\underline{d}_{x,x^*}\|$ and $\langle y \otimes y^*, \overline{d}_{x,x^*}\rangle = \underline{d}_{x,x^*}(y, y^*)$ if $y \in l^p$, $y^* \in l^q$. The following map is then induced

$$\overline{d} : l^p \times l^q \to \mathcal{U}^*, \overline{d}(x, x^*) \triangleq \overline{d}_{x,x^*}.$$

It is readily seen that $\overline{d} \in \mathcal{B}^2(l^p, l^q, \mathcal{U}^*)$ and so there is a unique $d \in \mathcal{B}(\mathcal{U}, \mathcal{U}^*)$ so that $\|d\| = \|\overline{d}\|$ and $d(x \otimes x^*) = \overline{d}(x, x^*)$ if $x \in l^p$, $x^* \in l^q$. Consequently, the following identity

$$\langle y \otimes y^*, d(x \otimes x^*)\rangle = \langle x, y^*\rangle - \langle y, x^*\rangle$$

holds if $x, y \in l^p$ and $x^*, y^* \in l^q$. It is straightforward to see that $d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*)$ and hence it is a $\mathfrak{D}$-derivation. Let us see that $d \notin \mathcal{N}^1(\mathcal{U}, \mathcal{U}^*)$. For, let us assume that d is inner, say $d = \mathrm{ad}_T$ for some $T \in \mathcal{U}^*$. Let us consider the usual basis $\{x_n\}_{n=1}^{\infty}$ of l^p, $x_n = \{\delta_{n,m}\}_{m=1}^{\infty}$ if $n \in \mathbb{N}$. So, $\{x_n\}_{n=1}^{\infty}$ is obviously a shrinking basis and its associated sequence of coefficient functionals are $x_n^* = \{\delta_{n,m}\}_{m=1}^{\infty}$ if $n \in \mathbb{N}$. With the notation of Proposition 2.1, since $\langle z_{n,m}, d(z_{p,q})\rangle = \delta_{m,p} - \delta_{n,q}$ for all $n, m, p, q \in \mathbb{N}$ we deduce that $T(z_{n,m}) = 1$ if $n, m \in \mathbb{N}$ and $n \neq m$. However, let us write

$$u \triangleq \frac{1}{\zeta(q)^{1/q}} \sum_{n=1}^{\infty} \frac{1}{n} \cdot z_{1,1+n},$$

where ζ denotes the Riemann zeta function. Then $u \in \mathcal{U}$ is well defined,

$$\begin{aligned} \|u\|_\pi &= \lim_{N\to\infty} \left\| \frac{1}{\zeta(q)^{1/q}} \sum_{n=1}^{N} \frac{1}{n} \cdot z_{1,1+n} \right\|_\pi \\ &= \frac{1}{\zeta(q)^{1/q}} \lim_{N\to\infty} \left\| x_1 \otimes \sum_{n=1}^{N} \frac{1}{n} x_{n+1}^* \right\|_\pi \\ &= \frac{1}{\zeta(q)^{1/q}} \lim_{N\to\infty} \left\| \sum_{n=1}^{N} \frac{1}{n} x_{n+1}^* \right\|_{l^q} \\ &= 1, \end{aligned}$$

and as

$$T(u) = \frac{1}{\zeta(q)^{1/q}} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

then T can not be bounded.

THEOREM 2.1. *Let $\mathcal{U}$ be a Banach algebra so that $\mathcal{U}^2$ is dense in $\mathcal{U}$. Let us denote $k_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{U}^{**}$ to the usual isometric embedding of $\mathcal{U}$ into its second dual space $\mathcal{U}^{**}$ by means of evaluations. Given $d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*)$ the following assertions are equivalent:*

(i) $d \in \mathfrak{D}(\mathcal{U})$.
(ii) $\langle x, d(y)\rangle + \langle y, d(x)\rangle = 0$ *for all* $x, y \in \mathcal{U}$.
(iii) $d^* \circ k_{\mathcal{U}} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*)$.
(iv) $d + d^* \circ k_{\mathcal{U}} = 0$.

PROOF. (i) $\Rightarrow$ (ii) Given $x, y \in \mathcal{U}$ we have

$$0 = \langle x + y, d(x + y)\rangle = \langle x, d(y)\rangle + \langle y, d(x)\rangle.$$

(ii) $\Rightarrow$ (iii) If $x, y, z \in \mathcal{U}$ we have

$$\begin{aligned}
\langle z, (d^* \circ k_{\mathcal{U}})(xy)\rangle &= \langle (d(z), k_{\mathcal{U}}(xy))\rangle \\
&= \langle xy, d(z)\rangle \\
&= \langle x, yd(z)\rangle \\
&= \langle x, d(yz) - d(y)z\rangle \\
&= \langle x, d(yz)\rangle - \langle zx, d(y)\rangle \\
&= \langle d(yz), k_{\mathcal{U}}(x)\rangle + \langle y, d(zx)\rangle \\
&= \langle yz, d^*(k_{\mathcal{U}}(x))\rangle + \langle d(zx), k_{\mathcal{U}}(y)\rangle \\
&= \langle z, d^*(k_{\mathcal{U}}(x))y\rangle + \langle z, xd^*(k_{\mathcal{U}}(y))\rangle \\
&= \langle z, (d^* \circ k_{\mathcal{U}})(x)y + x(d^* \circ k_{\mathcal{U}})(y)\rangle.
\end{aligned}$$

(iii) $\Rightarrow$ (iv) For $x, y, z \in \mathcal{U}$ we have

$$\begin{aligned}
\langle xy, d(z)\rangle &= \langle d(z), k_{\mathcal{U}}(xy)\rangle \\
&= \langle z, (d^* \circ k_{\mathcal{U}})(xy)\rangle \\
&= \langle z, (d^* \circ k_{\mathcal{U}})(x)y + x(d^* \circ k_{\mathcal{U}})(y)\rangle \\
&= \langle yz, (d^* \circ k_{\mathcal{U}})(x)\rangle + \langle zx, (d^* \circ k_{\mathcal{U}})(y)\rangle \\
&= \langle x, d(y)z + yd(z)\rangle + \langle y, d(z)x + zd(x)\rangle \\
&= \langle zx, d(y)\rangle + 2\langle xy, d(z)\rangle + \langle yz, d(x)\rangle.
\end{aligned}$$

Therefore,

$$\langle z, d(xy)\rangle = -\langle xy, d(z)\rangle = -\langle d(z), k_{\mathcal{U}}(xy)\rangle = -\langle z, (d^* \circ k_{\mathcal{U}})(xy)\rangle,$$

i.e., $(d + d^* \circ k_{\mathcal{U}})(xy) = 0$ if $x, y \in \mathcal{U}$. Since $\mathcal{U}^2$ is dense in $\mathcal{U}$ the claim follows.

(iv) $\Rightarrow$ (i) If $x \in \mathcal{U}$ then

$$0 = \langle x, d(x) + (d^* \circ k_{\mathcal{U}})(x)\rangle = 2\langle x, d(x)\rangle. \qquad \square$$

Proposition 2.2. *Let $\mathcal{U}$ be a Banach algebra and let $d \in \mathfrak{D}(\mathcal{U})$. There is a unique $d^\sharp \in \mathfrak{D}\left(\mathcal{U}^\sharp\right)$ so that the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\ d\ } & \mathcal{U}^* \\
{\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle p^*} \\
\mathcal{U}^\sharp & \xrightarrow{\ d^\sharp\ } & t(\mathcal{U}^\sharp)^*.
\end{array}$$

Proof. Consider $d^\sharp \triangleq p^* \circ d \circ p$. Thus, $d^\sharp \in \mathcal{B}\big(\mathcal{U}^\sharp, (\mathcal{U}^\sharp)^*\big)$ and $d^\sharp \circ j = p^* \circ d$. If $\eta, \mu \in \mathcal{U}^\sharp$ we get

$$\langle \eta, d^\sharp(\eta)\rangle = \langle p(\eta), d(p(\eta))\rangle = 0 \tag{2.6}$$

and if $\eta = ae + j(x)$, $\mu = be + j(y)$ for uniquely determined $a, b \in \mathbb{C}$ and $x, y \in \mathcal{U}$ then

$$\begin{aligned} d^\sharp(\eta)\mu + \eta d^\sharp(\mu) &= p^*(d(x))(be + j(y)) + (ae + j(x))p^*(d(y)) \\ &= (\langle y, d(x)\rangle + \langle x, d(y)\rangle)e^* + p^*(ad(y) + bd(x) + d(xy) \\ &= ad^\sharp(y) + bd^\sharp(x) + p^*(d(xy)) \\ &= p^*(d(ay + bx + xy)) \\ &= d^\sharp(\eta\mu). \end{aligned} \tag{2.7}$$

Thus, by (2.6) and (2.7) we conclude that $d^\sharp \in \mathfrak{D}(\mathcal{U}^\sharp)$. As we already observed, $j \circ p$ projects $\mathcal{U}^\sharp$ onto $j(\mathcal{U})$. Since $j(\mathcal{U})$ is complemented in $\mathcal{U}^\sharp$ by $\mathbb{C}{\cdot}e$ then $d^\sharp$ is uniquely determined. □

Corollary 2.1. *A $\mathfrak{D}$-derivation on $\mathcal{U}$ is inner if and only if its associated derivation $d^\sharp : \mathcal{U}^\sharp \to \left(\mathcal{U}^\sharp\right)^*$ by Proposition 2.2 is inner.*

Proof. Let $x^* \in \mathcal{U}^*$, $a \in \mathbb{C}$. Hence, it is easy to see that $(\mathrm{ad}_{x^*})^\sharp = \mathrm{ad}_{p^*(x^*)}$ and if $d^\sharp = \mathrm{ad}_{ae^*+p^*(x^*)}$, then $d = \mathrm{ad}_{x^*}$. □

Remark 2.2. Let us consider a dual Banach algebra $\mathcal{U}$, i.e., $\mathcal{U} \approx (\mathcal{U}_*)^*$, where $\mathcal{U}_*$ is a closed submodule of $\mathcal{U}^*$. Although $\mathcal{U}_*$ need not be unique, we will assume that $\mathcal{U}$ is realized as the dual space of a fixed closed submodule $\mathcal{U}_*$ of $\mathcal{U}^*$. It is known that a dual Banach algebra has a unit if and only if it has a bounded approximate identity (see [**7**, Prop. 1.2]).

Proposition 2.3. *Let $\mathcal{U}$ be a dual Banach algebra with unit and let $d \in \mathcal{Z}^1(\mathcal{U},\mathcal{U}^*)$ so that $d(\mathcal{U}) \subseteq k_{\mathcal{U}_*}(\mathcal{U}_*)$. Then $d \in \mathfrak{D}(\mathcal{U})$ if and only if $d^* + d \circ k^*_{\mathcal{U}_*} = 0$.*

Proof. ($\Rightarrow$) Given $x \in \mathcal{U}$ let $x_* \in \mathcal{U}_*$ be the unique element so that $d(x) = k_{\mathcal{U}_*}(x_*)$. If $x^{**} \in \mathcal{U}^{**}$ by Theorem 2.1(iv) we have

$$\begin{aligned} \langle x, (d \circ k^*_{\mathcal{U}_*})(x^{**})\rangle &= \langle d(k^*_{\mathcal{U}_*}(x^{**})), k_{\mathcal{U}}(x)\rangle \\ &= \langle k^*_{\mathcal{U}_*}(x^{**}), (d^* \circ k_{\mathcal{U}})(x)\rangle \\ &= -\langle k^*_{\mathcal{U}_*}(x^{**}), d(x)\rangle \\ &= -\langle x_*, k^*_{\mathcal{U}_*}(x^{**})\rangle \\ &= -\langle k_{\mathcal{U}_*}(x_*), x^{**}\rangle \\ &= -\langle d(x), x^{**}\rangle \\ &= -\langle x, d^*(x^{**})\rangle. \end{aligned}$$

($\Leftarrow$) If $x, y \in \mathcal{U}$ we obtain

$$\langle y, (d^* \circ k_{\mathcal{U}})(x)\rangle = -\langle y, (d \circ k^*_{\mathcal{U}_*} \circ k_{\mathcal{U}})(x)\rangle = -\langle y, d(x)\rangle,$$

i.e., $d + d^* \circ k_{\mathcal{U}} = 0$ and our claim follows. □

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References

[1] A. L. Barrenechea, C. C. Peña, *On the structure of derivations on certain non-amenable nuclear Banach algebras*, New York J. Math. 15 (2009), 199–209; Zbl pre05561324.
[2] W. Bade, P. C. Curtis, H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. (3) 55 (1987), 359–377; Zbl 0634.46042.
[3] H. G. Dales, F. Ghahramani, N. Grønbæk, *Derivations into iterated duals of Banach algebras*, Stud. Math. **128**, 1 (1998), 19–54; Zlb 0903.46045.
[4] A. Guichardet, *Sur l'homologie et la cohomologie des algèbres de Banach*, C. R. Acad. Sci. Paris, Ser. A **262** (1966), 38–42; Zlb 0131.13101.
[5] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127**, 1972; MR 51#11130.
[6] H. Kamowitz, *Cohomology groups of commutative Banach algebras*, Trans. Amer. Math. Soc. **102**, 1962, 352–372; MR 30#458.
[7] V. Runde, *Amenability for dual Banach algebras*, Stud. Math. **148** (2001), 47–66.
[8] V. Runde, *Lectures on amenability*, Lect. Notes Math. 1774 (2002), Springer-Verlag, Berlin, xiv+296 pp. ISBN 3-540-42852-6, MR 1874893 (2003h:46001), Zlb 0999.46022.
[9] R. Schatten, *A theory of cross spaces*, Ann. Math. Stud. 26, Princeton University Press, 1950, vii+153 pp; MR 0036935 (12,186e), Zlb 0041.43502.
[10] I. Singer, *Bases in Banach spaces. I*, Grundlehren Math. Wiss. 154, Springer-Verlag, New York and Berlin, 1970, viii+668 pp; MR 0298399 (45#7451), Zlb 0198.16601.

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