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INFINITE COMBINATORICS IN FUNCTION SPACES: CATEGORY METHODS

N. H. Bingham and A. J. Ostaszewski

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Abstract. The infinite combinatorics here give statements in which, from some sequence, an infinite subsequence will satisfy some condition – for example, belong to some specified set. Our results give such statements generically – that is, for 'nearly all' points, or as we shall say, for quasi all points – all off a null set in the measure case, or all off a meagre set in the category case. The prototypical result here goes back to Kestelman in 1947 and to Borwein and Ditor in the measure case, and can be extended to the category case also. Our main result is what we call the Category Embedding Theorem, which contains the Kestelman–Borwein–Ditor Theorem as a special case. Our main contribution is to obtain functionwise rather than pointwise versions of such results. We thus subsume results in a number of recent and related areas, concerning e.g., additive, subadditive, convex and regularly varying functions.

1. Introduction and motivation

The theory of regular variation was initiated by Jovan Karamata, to whom this paper is dedicated, in 1930 and developed by himself and his pupils till 1963, as well as by others. This subject is given monograph treatment in [**5**]. The main result of the subject is the Uniform Convergence theorem (UCT) of slow variation which is as follows (see e.g., [**5**, Ch. 1]).

UCT: If $l : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $l(\lambda x)/l(x) \to 1$ (as $x \to \infty$) for all $\lambda > 0$ and l is measurable or (has the property of) Baire – then the convergence is uniform on compact λ-sets in $\mathbb{R}_+$. Two points need emphasis here.

(i) Some regularity on l is required. For counterexamples showing this, see [**5**] e.g., Th.1.2.2.

(ii) So l measurable/Baire are sufficient for UCT. Neither includes the other.

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In memoriam Jovan Karamata (1902–1967).

The principal foundational question in the theory of regular variation (explicitly raised in [**5**, p. 11]) is the search for a minimal common generalization of measurability and the Baire property to serve as a *necessary and sufficient condition* on l in UCT. This question has now been fully answered; see [**6**] and [**7**]. The answer involves *infinite combinatorics*, hence the title of this paper. This is the subject matter of Section 2 and 3. One consequence of our approach is that it reveals the Baire case, rather than the more traditional measurable case, to be the more important.

The arguments λx in the definition above refer to the multiplication group of positive reals. The question arises of the extent to which the theory can be developed in more general settings—Euclidean space $\mathbb{R}^d$, Hilbert space, suitable topological groups etc. The question is raised explicitly in [**5**, Appendix 1], where the (then rather sparse) literature was reviewed.

In the two decades since [**5**] a great deal of work has been done on such questions. This has been largely motivated by *extreme value theory* within probability theory. For extremal value theory in one dimension see [**5**, Section 8.11]; the motivation here is the greatest flood height in a set of readings, or the greatest wind-speed, etc., since it is the maximum that is most damaging or dangerous, or in a financial context the highest (or lowest) stock price. Extensions to higher dimensions are natural: in climatic contexts because one may have data from a number of recording stations, and in the financial context because one may hold portfolios of stocks to diversify one's risks. Infinite-dimensional extensions are equally natural, one classic example being the difference in profiles between the sea dykes protecting the Netherlands and the sea level. There are a number of recent monographs on such higher-dimensional theory, including e.g., de Haan and Fereira [**30**], Resnick [**43**], [**44**], and in the financial case Balkema and Embrechts [**3**].

In Section 4 we turn to *normed groups* to provide a suitable setting in which a unified theory of regular variation can be developed. This permits the same kind of infinite combinatorics to play the key role. We work in Section 4 in the setting of function spaces over normed groups. Similarly in Section 5 we extend to these settings topological results on deformation, in the spirit of [**40**], a matter which we approach within the framework of homotopy theory in a forthcoming paper.

Equally relevant to the foundations of regular variation is the question of when $k(xy) = k(x)k(y)$ ($\forall x, y \in \mathbb{R}_+$) implies that $k(x) \equiv x^\rho$ for some ρ (called the index of regular variation, see e.g., [**5**, Section 1.4]). This can be reformulated as when an additive function – i.e., one satisfying

$$f(x + y) = f(x) + f(y), \quad \forall x, y \in \mathbb{R}$$

– satisfies $f(x) \equiv cx$ for some $c \in \mathbb{R}$. For these, one has a dichotomy – such functions are either very good or very bad. Additivity and continuity clearly give $f(x) = cx$, so this question reduce to one of *automatic continuity*, for additive functions (see [**18**] for the algebraic background, and [**31**] for a topological view). Regularity conditions discriminating between the two extremes of behaviour may be given in either measure or category forms (see e.g., [**9**]); here again it turns out that the underlying explanation hinges on the same kind of infinite combinatorics as in the UCT question. We give elsewhere a unified treatment, including as special cases classical results of Steinhaus and Ostrowski. Additivity may be weakened to *subadditivity,* with $f(x+y) \leqslant f(x) + f(y)$; the subadditive case is treated along similar lines in [**8**]. It may also be weakened to (mid-point) *convexity*

$$f(\tfrac{1}{2}(x+y)) \leqslant \tfrac{1}{2}(f(x)+f(y)),$$

a matter we study in a forthcoming publication; yet again infinite combinatorics underpin regularity considerations.

The recurring theme in these examples is group structure (or additive structure in the commutative case), as all of the above defining functional equations or inequalities may be restated in the language of normed groups, and of normed vector spaces in particular.

Thus the motivation and theme of this paper rests on extending the recurring feature of infinite combinatorics (e.g., the Kestelman–Borwein–Ditor Theorem below) to function spaces in general.

The advantage of applying Baire category methods − and thereby making the Baire case the primary one, rather than the classical measurable case − is that it shows the natural setting here to be Baire spaces.

It is category questions that are crucial, not questions of compactness or local compactness. This assists the generalization from finite dimensions to infinite dimensional settings: Hilbert space, for example, is not locally compact (the unit ball is not compact in infinitely many dimensions), but is Baire so our methods do apply to it. We recall (see e.g., [**24**, 3.3]) that Baire spaces, i.e., spaces in which the Baire Theorem applies, include Polish spaces and locally compact spaces.

2. Preliminaries

We shall be concerned here with both measure and category (cf. [**41**]), and need concepts of smallness for each. On the measure side, we deal with the class $\mathcal{L}$ of (Lebesgue) measurable sets, and interpret small sets as (Lebesgue) null sets; on the category side we deal with the class $\mathcal{B}a$ of sets with the Baire property (briefly, Baire sets), and interpret small sets as meagre sets (those of the first category). We use *quasi everywhere* (q.e.), or for *quasi all points*, to mean *for all points off a meagre set.* For Γ in $\mathcal{L}$ or $\mathcal{B}a$, we say that $P \in \Gamma$ holds *for generically all* t if $\{t : t \notin P\}$ is null/meagre according as Γ is $\mathcal{L}$ or $\mathcal{B}a$. Our starting-point is the following result, due to Kestelman [**33**] and to Borwein and Ditor [**11**]. This exemplifies the infinite combinatorics of the title, but concerns scalars, rather than functions.

Theorem 2.1. (Kestelman–Borwein–Ditor Theorem). *Let* $\{z_n\} \to 0$ *be a null sequence of reals. If* T *is measurable and non-null/Baire and non-meagre, then for generically all* $t \in T$ *there is an infinite set* $\mathbb{M}_t$ *such that* $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T$.

This result (briefly, the KBD theorem) is a corollary of a topological result, the Category Embedding Theorem (CET), given in two forms in Section 3 below. The starting point there is that $h_n(t) := t + z_n$ is a sequence of autohomeomorpisms (or, self-homeomorphisms) of the line which converges uniformly to the identity.

Our object here is to give a unified treatment of such infinite combinatorics on function spaces in general, thus providing a common perspective on all these results. In Section 3 below we give the CET, in what we call its *consecutive* form (the motivation being results of van der Waerden type, and also the need to handle bilateral shifts, $t-z_m$ and $t+z_m$). In Section 4 we work in normed groups, an area that we study in extenso elsewhere, applying the bitopological approach of CET to this more general group setting for shifts. What motivates such a broader context is the re-interpretation of a sequence of autohomeomorphisms $h_n(t)$ uniformly converging to the identity as giving rise to null function sequences $z_n(t) := h_n(t) - t$ (converging in supremum norm to zero), which need not be constant as in the KBD Theorem. In Section 5 we give generic forms of some results appearing in Kuczma [**34**, Ch. IX], which we term reflection theorems, and we close with a treatment in this vein of a genericity result, due to Császár [**17**], which makes explicit the ideas implicit in arguments presented in [**34**, IX.7]. Section 6 illustrates how the combinatorics may be applied to deduce automatic continuity of (mid-point) convex functions.

We will need the *density topology* (introduced in [**29**], [**26**], [**38**] and studied also in [**27**] – see also [**16**], and for textbook treatments [**32**], [**36**]). Recall that for T measurable, t is a (metric) *density point* of T if $\lim_{\delta\to 0} |T \cap I_\delta(t)|/\delta = 1$, where $I_\delta(t) = (t - \frac{\delta}{2}, t + \frac{\delta}{2})$. By the Lebesgue Density Theorem almost all points of T are density points ([**28**, Section 61], [**41**, Th. 3.20], or [**25**]). A set U is d-open (open in the density topology) if (U is measurable and) each of its points is a density point of U. We mention five properties:

(i) The density topology (d, say) is finer than (contains) the Euclidean topology [**32**, 17.47(ii)].

(ii) A set is Baire in the density topology iff it is (Lebesgue) measurable [**32**, 17.47(iv)].

(iii) A Baire set is meagre in the density topology iff it is null [**32**, 17.47(iii)]. So (since a countable union of null sets is null), the Baire theorem holds for the line under d:

(iv) $(\mathbb{R}, d)$ is a Baire space.

(v) A function is d-continuous iff it is approximately continuous in Denjoy's sense [**19**], [**36**, pp. 1,149].

The reader unfamiliar with the density topology may find it helpful to think, in the style of Littlewood's Three Principles: general situations are 'nearly' the easy situations – i.e., are easy situations modulo small sets. See [**35**, Ch. 4], [**46**, Section 3.6, p. 72].

3. Consecutive Category Embedding Theorem

We formulate two variant forms of a notion that we call weak category convergence (to distinguish it from the related notion of category convergence introduced by Wilczyński and his collaborators – see e.g., [**42**]). The first two definitions refer to homeomorphisms which form a sequence of 'approximations' to the identity in the sense of (approx) below, while the third introduces a relaxation. We follow set-theoretic usage and write $\omega := \{0, 1, 2, \dots\}$.

Definition 3.1 (weak category convergence). A sequence of autohomeomorphisms h_n of a topological space X satisfies the *weak category convergence* condition if: for any non-empty open set U, there is an non-empty open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\text{(wcc)} \qquad \bigcap_{n \geqslant k} V \setminus h_n^{-1}(V) \text{ is meagre.}$$

Equivalently, for each $k \in \omega$, there is a meagre set $M_k = M_k(V)$ in X such that, for $t \notin M_k$,

$$\text{(approx)} \qquad t \in V \Longrightarrow (\exists n \geqslant k)\ h_n(t) \in V.$$

We say that the homeomorphisms h_n satisfy the *weak category convergence conjunctively* if, for each $k \in \omega$,

$$\text{(wccc)} \qquad \bigcap_{n \geqslant k} V \setminus [h_{2n}^{-1}(V) \cap h_{2n+1}^{-1}(V)] \text{ is meagre.}$$

Equivalently, for each $k \in \omega$, there is a meagre set M_k in X such that, for $t \notin M_k$,

$$t \in V \Longrightarrow (\exists n \geqslant k)\ h_{2n}(t) \in V \text{ and } h_{2n+1}(t) \in V.$$

Finally, we formulate a local version of (wcc) appropriate to the case $X = \mathbb{R}$ (but generalizable to X a group in the context of Section 3 below), which allows some rescaling of h_n.

Say that the sequence of homeomorphisms h_n satisfies the *re-scaled weak category convergence condition* at u if for every open set U with $u \in U$ there is an open set V with $u \in V \subset U$ and $\eta = \eta_u > 0$ such that, for each $k \in \omega$,

$$\text{(rwcc)} \qquad \bigcap_{n \geqslant k} \eta V \setminus h_n^{-1}(V) \text{ is meagre.}$$

Equivalently, for each $k \in \omega$, there is a meagre set M_k in X such that, for $t \notin M_k$,

$$t \in \eta V \Longrightarrow (\exists n \geqslant k)\ h_n(t) \in V,$$

or, writing ηs for t and ηN_k for M_k, for each $k \in \omega$, there is a meagre set N_k in X such that, for $s \notin N_k$,

$$\text{(approx-eta)} \qquad s \in V \Longrightarrow (\exists n \geqslant k)\ h_n(\eta s) \in V.$$

Remarks. 1. In the case of the line with Euclidean topology the functions $h_n(t) = t \pm z_n$, with sign selected according to parity, are autohomeomorphisms. The condition (wccc) is used to deduce the bilateral embedding result

$$\{t - z_m, t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Multiple consecutive forms, p-fold ones, may also be considered by working modulo p rather than 2 in (wccc). These forms are important in connection with van der Waerden's Theorem and its relatives; we treat such extensions elsewhere.

2. Note that $t \in \limsup h_n^{-1}(T) := \bigcap_{k \in \omega} \bigcup_{n \geqslant k} h_n^{-1}(T)$ iff for some infinite $\mathbb{M}_t \subset \omega$

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

The theorem below implies that for Baire T the sets $\limsup h_n^{-1}(T)$ and T are equal modulo a meagre set.

3. Taking $h_{2n+1} = h_{2n}$ reduces (wccc) to (wcc).

4. Consider the affine homeomorphisms $A_n(t) = \alpha_n t + z_n$ with $\alpha_n \geqslant 2\eta > 0$ and $z_n \to 0$. For any symmetric interval I_δ about the origin of radius δ, we have

$$\alpha_n I_\delta + z_n \supseteq 2\eta I_\delta + z_n = I_{2\eta\delta} + z_n.$$

For n large enough we have $z_n \in I_{\eta\delta}$, so $\alpha_n I_\delta + z_n \supseteq I_{\eta\delta}$, i.e., $A_n[I_\delta] \supseteq I_{\eta\delta}$, so that $\eta I_\delta \setminus A_n[I_\delta]$ is meagre. Thus A_n^{-1} satisfies (rwcc) at the origin.

Note that if M is meagre then $T := I_\delta \setminus M$ is Baire non-meagre, and we have

$$A_n[T] = A_n[I_\delta \setminus M] \supseteq \eta I_\delta \setminus A_n[M],$$

so $\eta T \setminus A_n[T]$ is meagre.

5. When X is a group one may interpret the condition (rwcc) as referring to group multiplication by η – specifically on the left when adopting the formal context (approx-eta).

Theorem 3.1. (Category Embedding Theorem—Consecutive form). *Let X be a Baire space. Suppose the homeomorphisms $h_n : X \to X$ satisfy the weak category convergence condition conjunctively. Then, for any Baire set T, for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that $\{h_m(t), h_{m+1}(t) : m \in \mathbb{M}_t\} \subseteq T$.*

Proof. We may assume that $T = U \setminus M$ with U open, non-empty and M meagre. Consider homeomorphisms $h_n : X \to X$ satisfying the weak category convergence condition conjunctively. By assumption, there is $V \subseteq U$ satisfying (wccc). Since the functions h_n are homeomorphisms, the set $M' := M \cup \bigcup_n h_n^{-1}(M)$ is meagre. Put

$$W = \mathbf{h}(V) := \bigcap_{k\in\omega} \bigcup_{n\geqslant k} V \cap \left[h_{2n}^{-1}(V) \cap h_{2n+1}^{-1}(V)\right] \subseteq V \subseteq U.$$

Then W is co-meagre in V. Indeed

$$V \setminus W = \bigcup_{k\in\omega} \bigcap_{k\geqslant n} V \setminus \left[h_{2n}^{-1}(V) \cap V \setminus h_{2n}^{-1}(V)\right],$$

which by assumption is meagre.

Let $t \in V \cap W \setminus M'$ so that $t \in T$. Now there exists an infinite set $\mathbb{M}_t$ such that, for $m \in \mathbb{M}_t$, there are points $v_{2m}, v_{2m+1} \in V$ with $t = h_{2m}^{-1}(v_{2m}) = h_{2m+1}^{-1}(v_{2m+1})$. Since $h_{2m}^{-1}(v_{2m}) = t \notin h_{2m}^{-1}(M)$, we have $v_{2m} \notin M$, and hence $v_{2m} \in T$; likewise $v_{2m+1} \in T$. Thus $\{h_{2m}(t), h_{2m+1}(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a co-meagre set, as asserted.

To deduce that quasi all $t \in T$ satisfy the conclusion of the theorem, put $S := T \setminus \mathbf{h}(T)$; then S is Baire and $S \cap \mathbf{h}(T) = \emptyset$. If S is non-meagre, then by the preceding argument there are $s \in S$ and an infinite $\mathbb{M}_s$ such that $\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S$, i.e., $s \in \mathbf{h}(S) \subseteq \mathbf{h}(T)$, a contradiction. □

Following Remark 3 above, we obtain as a special case a result derived by us elsewhere.

Corollary 3.1. (Category Embedding Theorem—single form). *Let X be a Baire space. Suppose the homeomorphisms $h_n : X \to X$ satisfy the weak category convergence condition* (wcc). *Then, for any Baire set T, for quasi all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$.*

We close with a further strengthening obtained by reworking the proof so as replace (wccc) with (rwcc).

Corollary 3.2. (Locally rescaled CET). *Let $\mathbb{R}$ be given a Baire topology and let T be Baire. Suppose that h_n are homeomorphisms satisfying* (rwcc) *at* 0*. Then, for quasi all $u \in T$ and quasi all $t \in T$ near u (i.e., in some open set U with $u \in U$), there is an infinite $\mathbb{M}_{t,u}$ such that $u + h_m(t - u) \in T$, for all $m \in \mathbb{M}_{t,u}$.*

Proof. Let $T = U \setminus M \cup N$ with U open and M, N meagre. As our conclusions concern quasi all members of T, we may take $N = \emptyset$, which means that 'for quasi all $u \in T$' is synonymous with 'for all $u \in U \setminus M$'. Fix $u \in T$. Then $0 \in U - u$. Let the autohomeomorphisms h_n satisfy (rwcc) at 0. Thus we may select V with $u \in V \subset U$ and $\eta = \eta_u > 0$ such that $0 \in V \subseteq U - u$ and $\bigcap_{n \geqslant k} \eta V \setminus h_n^{-1}(V)$ is meagre. Further, select open $W \subset V$ (e.g., $W = \eta^{-1}V$) with $0 \in \eta W \subseteq V \subseteq U - u$. Put

$$S = \eta W \cap \bigcap_{k \in \omega} \bigcup_{n \geqslant k} h_n^{-1}(T_u);$$

then

$$M' = \eta W \setminus S = \bigcup_{k \in \omega} \bigcap_{n \geqslant k} \eta W \setminus h_n^{-1}(T_u) \subset \bigcup_{k \in \omega} \bigcap_{n \geqslant k} \eta V \setminus h_n^{-1}(T_u)$$

is meagre. But $\eta W \setminus (M - u) \subseteq (U - u) \setminus (M \setminus u)$, so for $t \in (u + \eta W) \cap T$ with $t \notin (M' + u) \cup M$ we have $x := t - u \in (T_u \cap S)$, and so there is an infinite set $\mathbb{M}_{t,u}$ such that

(equiv) $$t - u = x \in h_m^{-1}(T_u), \text{ for } m \in \mathbb{M}_{t,u}.$$

Thus $u + h_m(t - u) \in T$, for $m \in \mathbb{M}_{t,u}$. □

4. Shift-embeddings

We now specialize Corillary 3.1 to a metric group setting in order to consider sequences of autohomeomorphisms generated as shifts $h_n(x) = xz_n$. We say that the group X is *normed* if it has a group-norm as defined below (cf. [**20**]).

Definition 4.1. We say that $\| \cdot \| : X \to \mathbb{R}_+$ is a *group-norm* if the following properties hold:

(i) Subadditivity (Triangle inequality): $\|xy\| \leqslant \|x\| + \|y\|$;
(ii) Positivity: $\|x\| > 0$ for $x \neq e$;
(iii) Inversion (Symmetry): $\|x^{-1}\| = \|x\|$.

By the Birkhoff–Kakutani Theorem, a metrizable topological group T has a right-invariant invariant metric d^T generating the topology. (For a proof see that offered in [**47**] for Th. 1.24 (pp. 18–19), which derives a metrization of a topological vector space in the form $d(x, y) = p(x - y)$ and makes no use of the scalar field.

That proof may be rewritten verbatim with xy^{-1} substituting for the additive notation $x - y$.) Such a metric induces the group-norm $\|t\| := d^T(t, e_T)$. Thus $d^T(x,y) = d^T(e_T, yx^{-1}) = \|yx^{-1}\|$. The conjugate metric, defined by $\tilde{d}^T(x,y) = \|xy^{-1}\| = d^T(e_T, xy^{-1}) = d^T(x^{-1}, y^{-1})$, is continuous relative to d^T iff T is a topological group. That is, the continuity of inversion characterizes topological groups as a subcategory of normed groups. We turn to an exhaustive account of the subject in a forthcoming publication.

Let $\mathcal{A} = \mathcal{A}(T)$ denote the set of bounded autohomeomorphisms h from T to T (i.e., having $\sup_T d(h(t), t) < \infty$) with composition $\circ$ as group operation. Thus $e_{\mathcal{A}}(t) \equiv t$. Then $\mathcal{A}$ has the right-invariant metric $d_{\mathcal{A}}(h, h') = \sup_T d(h(t), h'(t))$, which generates the norm

$$\|h\|_{\mathcal{A}} := d_{\mathcal{A}}(h, e_{\mathcal{A}}) = \sup_T d(h(t), t).$$

For the purposes of studying topological flows one is interested in topological subgroups of $\mathcal{A}$ either under $d_{\mathcal{A}}$, or under its symmetrization $d^S_{\mathcal{A}} = d_{\mathcal{A}} + \tilde{d}_{\mathcal{A}}$, where $\tilde{d}_{\mathcal{A}}(f,g)$ is the conjugate metric $\tilde{d}_{\mathcal{A}}(f^{-1}, g^{-1})$. We note for completeness the following.

LEMMA 4.1. *Under $d_{\mathcal{A}}$ on $\mathcal{A}$ and d_T on T, the evaluation map $(h,t) \to h(t)$ is continuous.*

PROOF. Fix h_0 and t_0. The result follows from continuity of h_0 at t_0 via

$$\begin{aligned} d_T(h_0(t_0), h(t)) &\leqslant d_T(h_0(t_0), h_0(t)) + d_T(h_0(t), h(t)) \\ &\leqslant d_T(h_0(t_0), h_0(t)) + d_{\mathcal{A}}(h, h_0). \end{aligned}$$ □

REMARK. Here, for background, we mention without proof some pertinent results, which we establish elsewhere. If T is a topological normed group then the left shift $t \to at$, regarded as homeomorphism, is both bounded and uniformly continuous in norm, in fact it is bi-uniformly continuous, since its inverse $t \to a^{-1}t$ is also uniformly continuous in norm. As a subgroup the shifts metrized by $d_{\mathcal{A}}$ form a normed topological group, isometric to T. In general the subgroup $\mathcal{H}_u$ of bi-uniformly continuous bounded homeomorphisms is a topological group under the symmetrized metric $d^S_{\mathcal{A}}$ (and is complete if d_T is complete).

Let $\mathcal{C} = \mathcal{C}_b(T)$ denote the set of continuous functions from T to T with norm-bounded range and with group operation pointwise multiplication:

$$(x \cdot y)(t) = x(t)y(t).$$

Here the identity element is the constant function $e_{\mathcal{C}}(t) \equiv e_T$. To retain a unified setting we give $\mathcal{C}$ the supremum norm; thus $\mathcal{C}$ is now a metric space.

REMARKS. 1. The symmetrized metric $d(h,h') = d_{\mathcal{A}}(h,h') + \tilde{d}_{\mathcal{A}}(h,h')$ is *admissible* in that it endows $\mathcal{H}_u$ with the structure of a topological group. We note that, if a group $\mathcal{G}$ is metrizable, non-meagre and analytic (for which see [**45**]) in the metric, and left and right shifts are continuous, then $\mathcal{G}$ is a topological group (see e.g., [**49**, p. 352]). Our choice of $d_{\mathcal{A}}$ retains metrizability and right-invariance (normability) and is sufficient to ensure that the natural $\mathcal{A}(T)$-flow on T, i.e., the evaluation action $(h,t) \to h(t)$, is continuous (compare the structural assumptions of Ellis' Theorem in [**23**], or [**49**, p. 351]).

2. Rather than use the supremum metric, one may consider the compact-open topology (the topology of uniform convergence on compacts, introduced by Fox and studied by Arens in [**1**], [**2**]). However, in order to ensure the kind of properties we need, the metric space T would need to be restricted to a special case, which we prefer to deal with on its own merits. (On this point see the remarks in [**50**]; for an alternative topology see [**4**, Ch. IV].) From this perspective we recall some salient features of the compact-open topology. For composition to be continuous local compactness is essential ([**22**, Ch. XII.2], [**37**], [**4**, Section 8.2], or [**51**, Ch. 1]). When T is compact the topology is admissible, but the issue of admissibility in the non-compact situation is not currently fully understood (even in the locally compact case for which counter-examples with non-continuous inversion exist, and so additional properties such as local connectedness are usually invoked – see [**21**] for the strongest results). Our focus of interest is on separable function spaces; we recall that, by a theorem of Arens, if T is separable metric and the compact-open topology on $\mathcal{C}\ (T,\mathbb{R})$ is metrizable, then T is necessarily locally compact and σ-compact, and conversely (see e.g., [**24**, pp. 165, 266]). We consider the locally compact, σ-compact case, typified by $\mathbb{R}$, at the end of Section 5.

Definition 4.2. Say that $z_n \in \mathcal{C}$ is a *null sequence in* $\mathcal{C}$ or simply that z_n is *uniformly null*, if $z_n \to e_T$, in sup norm, i.e., $\|z_n\| := \sup d_T(z_n(t), e_T) \to 0$.

Thus z_n is a null sequence in $\mathcal{C}$ iff z_n^{-1} is a null sequence in $\mathcal{C}$ (where $z_n^{-1}(t) := z_n(t)^{-1}$). Put $\theta_n(t) = z_n(t)t$; then

$$\|\theta_n\|_{\mathcal{A}} := \sup d^T(\theta_n(t), t) = \sup d^T(z_n(t)t, t) = \sup d^T(z_n(t), e_T) = \|z_n\|_{\mathcal{C}}.$$

One thus has the following result.

Lemma 4.2. *For z_n in $\mathcal{C}$, the sequence θ_n converges to the identity in $\mathcal{A}$ iff z_n is a uniformly null sequence* (*in* $\mathcal{C}$).

The next two theorems verify circumstances when the condition (wccc) holds thus permitting Corollary 3.1 to apply.

Theorem 4.1. (Norm topology verification theorem). *If ψ_n in $\mathcal{A}$ converges to the identity, then ψ_n satisfies the weak category convergence condition* (wcc). *Indeed the sequence satisfies* (wccc).

Proof. It is more convenient to prove the equivalent statement that ψ_n^{-1} satisfies the category convergence condition.

Put $z_n = \psi_n(z_0)$, so that $z_n \to z_0$. Let k be given.

Suppose that $y \in B_\varepsilon(z_0)$, i.e., $r = d(y, z_0) < \varepsilon$. For some $N > k$, we have $\varepsilon_n = d(\psi_n, id) < \frac{1}{3}(\varepsilon - r)$, for all $n \geqslant N$. Now

$$d(y, z_n) \leqslant d(y, z_0) + d(z_0, z_n) = d(y, z_0) + d(z_0, \psi_n(z_0)) \leqslant r + \varepsilon_n.$$

For $y = \psi_n(x)$ and $n \geqslant N$,

$$\begin{aligned} d(z_0, x) &\leqslant d(z_0, z_n) + d(z_n, y) + d(y, x) \\ &= d(z_0, z_n) + d(z_n, y) + d(x, \psi_n(x)) \\ &\leqslant \varepsilon_n + (r + \varepsilon_n) + \varepsilon_n < \varepsilon. \end{aligned}$$

So $x \in B_\varepsilon(z_0)$, giving $y \in \psi_n(B_\varepsilon(z_0))$. Thus

$$y \notin \bigcap_{n \geqslant N} B_\varepsilon(z_0) \setminus \psi_n(B_\varepsilon(z_0)) \supseteq \bigcap_{n \geqslant k} B_\varepsilon(z_0) \setminus \psi_n(B_\varepsilon(z_0)).$$

It now follows that

$$\bigcap_{n \geqslant k} B_\varepsilon(z_0) \setminus \psi_n(B_\varepsilon(z_0)) = \emptyset,$$

giving (wcc) as required; similarly for (wccc). □

Remark. Taking the viewpoint afforded by Remark 2 of Section 3, the referee has kindly pointed out that the argument above may be re-formulated to say that $B_{\varepsilon s}(z_0) \subseteq \psi_n(B_\varepsilon(z_0))$ for each $0 < s < 1$ and hence implies

$$B_\varepsilon(z_0) \subseteq \liminf \psi_n(B_\varepsilon(z_0)).$$

Theorem 4.2. (Density topology verification theorem). *For T a normed locally compact group with left-invariant Haar measure m, V an m-measurable non-null set and z_n a null sequence in $\mathcal{C}(T)$, let $h_n(t) := tz_n(t)^{-1}$ be an autohomeomorphism. Then for each $k \in \omega$,*

$$H_k := \bigcap_{n \geqslant k} V \setminus \left[h_{2n}^{-1}(V) \cap h_{2n+1}^{-1}(V)\right] \text{ is } m\text{-null, so meagre in the } d\text{-topology.}$$

That is, the sequence $h_n(t) = tz_n(t)^{-1}$ satisfies the weak category convergence condition (wccc).

Proof. Suppose not; then $m(H_k) > 0$ for some k; write H for H_k. We write Vz for $V \cdot z$, etc., so that $t \in h_n^{-1}(V)$ iff $h_n(t) \in V$ iff $t \in Vz_n(t)$.

For $n \geqslant k$ we have, since $H \subseteq V \setminus \left[h_{2n}^{-1}(V) \cap h_{2n+1}^{-1}(V)\right]$, that $\emptyset = H \cap h_{2n}^{-1}(V) \cap h_{2n+1}^{-1}(V)$ and so $\emptyset = H \cap h_{2n}^{-1}(H) \cap h_{2n+1}^{-1}(H)$. Thus, for $h \in H$, either $h \notin h_{2n}^{-1}(H)$ or $h \notin h_{2n+1}^{-1}(H)$; so there are infinitely many odd, or infinitely many even, $n \geqslant k$ for which $h \notin Hz_n(h)$ – that is, $h \notin Hz_n(h)$ for $h \in H$ for infinitely many $n \geqslant k$.

Let u be a metric density point of H. Thus, for some bounded (Borel) neighbourhood $U_\nu u$ we have $m[H \cap U_\nu u] > \frac{3}{4} m[U_\nu u]$. Fix U_ν and put $\delta = m[U_\nu u]$.

Let $E = H \cap U_\nu u$. For any $z_n(t)$, we have $m[(Ez_n(t)) \cap U_\nu u z_n(t)] = m[E] > \frac{3}{4}\delta$. By Theorem A of [**28**, p. 266], for all large enough n, we have

$$m(U_\nu u \triangle U_\nu u z_n(t)) < \delta/4.$$

Hence, for all n large enough we have $|(Ez_n(t)) \setminus U_\nu u| \leqslant \delta/4$.

Put $F = (EB_{\|z_n\|}(e)) \cap U_\nu u$; then $m[F] > \delta/2$ for all large enough n. But

$$\delta \geqslant m[E \cup F] = m[E] + m[F] - m[E \cap F] \geqslant \frac{3}{4}\delta + \frac{1}{2}\delta - m[E \cap F].$$

So for all large enough n and $h \in H$, we have $m[H \cap (Hz_n(h))] \geqslant m[E \cap F] \geqslant \frac{1}{4}\delta$, so $H \cap (Hz_n(h))$ is non-empty. But this contradicts $h \notin Hz_n(h)$, for $h \in H$ and infinitely many n. □

REMARK. The only fact about h_n used in the proof above is that, for some sequence of radii $r(n)$ tending to zero, $h_n(t) \in B_{r(n)}(t)$. One may thus verify the (rwcc) condition in the following context.

COROLLARY 4.1. *For $A_n(t) := \alpha_n t + z_n$, with $\alpha_n \to \alpha > 0$ and z_n uniformly null, and for V bounded and of finite positive measure,*

$$\bigcap_{n \geqslant k} \alpha V \setminus A_n(V) \text{ is } m\text{-null, so meagre in the } d\text{-topology.}$$

PROOF. Put $\alpha_n = \alpha + \varepsilon_n$, so that $\varepsilon_n \to 0$, and let

$$W_n := (\varepsilon_n + z_n)(V) := \{\varepsilon_n v + z_n(v) : v \in V\}$$

so that $(\alpha_n + z_n)(V) \subseteq \alpha V + W_n$. Now $m[W_n] \to 0$ and $\operatorname{diam}(W_n) \to 0$, so since αV is of finite positive measure, Theorem 4.2 yields that $\bigcap_{n \geqslant k} \alpha V \setminus A_n(V)$ is null, as required. □

As an immediate corollary of Corollary 3.1 and Theorem 4.1 we obtain the following special case of Corollary 3.1.

THEOREM 4.3. *If $\mathcal{X}$ is a Baire subset of functions $x(.)$ in $\mathcal{C}[0,1]$ and $f_n \to f$ in $\mathcal{C}[0,1]$ in sup-norm, then for quasi all $x \in \mathcal{X}$ there is an infinite set $\mathbb{M}_x$ such that $\{x + f_m - f : m \in \mathbb{M}_x\} \subseteq \mathcal{X}$.*

PROOF. Let $z_n = f_n - f$; then $z_n \to 0$. Since $\mathcal{C}[0,1]$, a complete metric space, is a Baire space, and $x \to x + z_n$ is a sequence of homeomorphisms, Theorem 4.1 applies. □

We may now deduce two strengthened forms of the KBD theorem. Putting $h_n(t) = tz_n(t)$ we obtain the following corollary.

THEOREM 4.4. (Functionwise Embedding Theorem). *Let T be a normed locally compact group, z_n a null sequence in $\mathcal{C}_b(T)$ such that $t \to tz_n(t)$ is, for each n, an autohomeomorphism. If S is Haar measurable, resp., Baire, then for generically all $t \in S$ there is an infinite set $\mathbb{M}_t$ such that $\{tz_m(t) : m \in \mathbb{M}_t\} \subseteq S$.*

Next let z_n and w_n be null sequences in $\mathcal{C}_b(T)$. Put $h_{2n}(t) = tz_n(t)$ and $h_{2n+1}(t) = tw_n(t)$; then the merged sequence $z_0(t), w_0(t), z_1(t), w_1(t), \dots$ is a null sequence in $\mathcal{C}_b(T)$. Thus one has

THEOREM 4.5. (Functionwise Consecutive Embedding Theorem). *Let T be a normed locally compact group, z_n and w_n null sequences in $\mathcal{C}_b(T)$ such that $t \to tz_n(t)$ and $t \to wt_n(t)$ are, for each n, autohomeomorphisms. If S is Haar measurable, resp., Baire, then for generically all $t \in S$ there is an infinite set $\mathbb{M}_t$ such that $\{tz_m(t), tw_m(t) : m \in \mathbb{M}_t\} \subseteq T$.*

This includes the result on bilateral shifts mentioned earlier.

5. Generic Reflection Theorem

In this section, working again in the context of $T = \mathbb{R}$, we begin by formulating simple conditions ensuring that various null sequences $z_n \to 0$ in $\mathcal{C}_b(\mathbb{R})$ lead to autohomeomorphisms $h_n(t) := t + z_n(t)$ of $\mathbb{R}$ in the usual or in the density topology. This will enable us to apply the functionwise embedding theorems.

Definition 5.1. Say that $h : \mathbb{R} \to \mathbb{R}$ is *bi-Lipschitz* (a notion implicit in [**12**]) if, for some α, β,

$$0 < \alpha \leqslant \frac{h(u) - h(v)}{u - v} \leqslant \beta, \text{ for } u \neq v.$$

In particular, h is continuous and strictly increasing, and so is invertible with continuous and strictly increasing inverse, also bi-Lipschitz, and differentiable, except possibly for at most countably many points. The bi-Lipschitz functions preserve density points – in particular images and preimages of null/meagre sets are null/meagre (see [**12**], [**13**], or [**15**] and [**14**]) – and so are homeomorphisms in the d-topology on $\mathbb{R}$.

Definition 5.2. Call a null sequence z_n in $\mathcal{C}_b$ *bi-Lipschitz* if the mappings $u \to u + z_n(u)$ are bi-Lipschitz uniformly in n, i.e., for some α, β and all n we have

$$0 < \alpha \leqslant 1 + \frac{z_n(u) - z_n(v)}{u - v} \leqslant \beta, \text{ for } u \neq v. \tag{5.1}$$

In particular z_n', where it exists, is bounded away from -1.

Definition 5.3. For z_n a sequence in $\mathcal{C}_b$, the *f-conjugate sequence* $\bar{z}_n$ is defined as follows:

$$\bar{z}_n(t), \text{ or } z_n^f(t), := f(t + z_n(t)) - f(t).$$

Lemma 5.1. *For $f : \mathbb{R} \to \mathbb{R}$ Lipschitz, the f-conjugate sequence is null in $\mathcal{C}_b$. If $z_n(t)$ satisfies* (5.1) *and the derivative $f'(t)$ is continuous near $z = u$ and satisfies $1 + (\alpha - 1)f'(u) > 0$, and is bounded above in a neighbourhood of $t = u$, then the f-conjugate sequence $\{\bar{z}_n(t)\}$ is locally bi-Lipschitz near $t = u$. In particular for z_n differentiable this is so if $1 + f'(u)z_n'(u) > 0$, for all n.*

Proof. For f with Lipschitz constant β_f we have $\|\bar{z}_n\| \leqslant \beta_f \|z_n\|$, as

$$|\bar{z}_n(t)| = |f(t + z_n(t)) - f(t)| \leqslant \beta_f |z_n(t)|.$$

For f differentiable, we may write $f(u) - f(v) = f'(w(u,v))(u - v)$ and

$$f(u + z_n(u)) - f(v + z_n(v)) = f'(w_n(u,v))[z_n(u) - z_n(v) + (u - v)].$$

Thus we have

$$\frac{\bar{z}_n(u) - \bar{z}_n(v)}{u - v} = f'(w_n(u,v))\frac{z_n(u) - z_n(v)}{u - v} + [f'(w_n(u,v)) - f'(w(u,v))].$$

Hence

$$\begin{aligned} 1 + \frac{\bar{z}_n(u) - \bar{z}_n(v)}{u - v} &= 1 + f'(w_n)\frac{z_n(u) - z_n(v)}{u - v} + [f'(w_n(u,v)) - f'(w(u,v))] \\ &\geqslant 1 + (\alpha - 1)f'(w_n) + [f'(w_n(u,v)) - f'(w(u,v))], \end{aligned}$$

and the latter term is positive for v in a small enough neighbourhood of $t = u$. To obtain the differentiable case we note that in the preceding line

$$1 + f'(w_n)\frac{z_n(u) - z_n(v)}{u - v} > 0$$

for v in a small enough neighbourhood of $t = u$. □

As an immediate corollary of the above Lemma, the CET (Th. 3.1) and the two shift theorems (Th. 4.1 and 4.2), we have:

THEOREM 5.1. (Generic Reflection Theorem]. *Let T be measurable/Baire, $f(.)$ be continuously differentiable and non-stationary at generically all points, $z_n \to 0$ in supremum norm a null sequence that is bi-Lipschitz with*

$$1 + f'(t)z_n'(t) > 0, \text{ for all } n, \tag{5.2}$$

for generically all $t \in T$. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that

$$t + f(t + z_n(t)) - f(t) \in T, \text{ for all } n \in \mathbb{M}_t. \tag{5.3}$$

In particular, if in addition f is linear and $f(t) = \alpha t$ with $\alpha \neq 0$, then for generically all $u \in T$ there is an infinite set $\mathbb{M}_u$ such that

$$\alpha u_n + (1 - \alpha)u \in T \text{ for all } n \in \mathbb{M}_u, \text{ where } u_n = u + z_n(u). \tag{5.4}$$

The term 'reflection' above is motivated by Lemma 6.1. For our closing results we need the following.

DEFINITION 5.4. 1. Say that f is *smooth* for z_n if (5.2) holds.

2. More generally, say that the sequence f_n of functions from $\mathbb{R}$ to $\mathbb{R}$ is *smooth* for z_n if:

(i) $\bar{z}_n(t) := f_n(t + z_n(t)) - f_n(t)$ is a null sequence, and

(ii) $h_n(t) := t + \bar{z}_n(t)$ is an autohomeomorphism.

EXAMPLE 5.1. The linear case $f(t) = \alpha t$ is of particular interest. Here

$$h_n(t) := t + f(t + z_n(t)) - f(t) = t + \alpha z_n(t).$$

For $\alpha > 0$, the derivative condition for h_n to be increasing reads

$$1 + \alpha z_n'(t) \geqslant 0, \text{ or } z_n'(t) \geqslant -1/\alpha.$$

So, if the null function sequence is constant (as in KBD theorem), with $z_n(t) \equiv z_n$, the condition is satisfied, as it reduces simply to $0 \geqslant -1/\alpha$.

EXAMPLE 5.2. Let λ_n be a sequence of non-zero reals and z_n a null sequence in $\mathcal{C}_b$. Put $f_n(t) = \lambda_n f(t)$, where $f(.)$ is continuously differentiable. Thus

$$|\bar{z}_n(t)| = |f_n(t + z_n(t)) - f_n(t)| = \lambda_n |z_n(t)| \|f'(v_n(t))|,$$

for some $v_n(t)$. Thus $|\bar{z}_n(t)| \to 0$ on compacts if λ_n is bounded. Now

$$\begin{aligned}\frac{d}{dt}(t+\lambda_n f(t+z_n(t))-\lambda_n f(t)) &= 1+\lambda_n(f'(t+z_n(t))[1+z_n'(t)]-f'(t)) \\ &= 1+\lambda_n f'(t+z_n(t))z_n'(t)+\lambda_n[f'(t+z_n(t))-f'(t)].\end{aligned}$$

Thus, for λ_n bounded, a condition such as $1 + \lambda_n f'(t) z_n'(t) > 0$ ($\forall n \in \mathbb{N}$) ensures that each $t + \bar{z}_n(t)$ is a Euclidean homeomorphism. This will be so when $z_n(t) \equiv z_n$ (constant).

For $f(t) = t$ we have $\bar{z}_n(t) = \lambda_n z_n(t)$. Thus if (5.1) holds for z_n, then, for u, v distinct and $\lambda_n > 0$,

$$1 - \lambda_n < 1 + \lambda_n(\alpha - 1) \leqslant 1 + \lambda_n \frac{z_n(u) - z_n(v)}{u - v} \leqslant 1 + \lambda_n(\beta - 1).$$

So, for $0 < \lambda_n < 1$, we conclude that $\bar{z}_n$ is bi-Lipschitz. If $z_n(t) = z_n$ (constant) then the only condition that needs to be in place is that $\lambda_n \|z_n\| \to 0$. This can be easily arranged by replacing z_n by a subsequence $\hat{z}_n = z_{k(n)}$ such that $\lambda_n \|z_{k(n)}\| \to 0$.

THEOREM 5.2. (Smooth Image Theorem). *Let f and g both be smooth for $z_n \in \mathcal{C}_b$ which is differentiable and bi-Lipschitz. Then, for generically all $t \in T$, there is an infinite set $\mathbb{M}_t$ such that*

$$t + z_n^f \in T, \text{ and } t + z_n^g \in T \text{ for all } n \in \mathbb{M}_u.$$

In particular, for f smooth and $g(t) = t$ the identity map we obtain the simultaneous embedding:

$$t + z_n^f \in T, \text{ and } t + z_n \in T \text{ for all } n \in \mathbb{M}_t.$$

Furthermore, if f and g are smooth and linear and $f(t) = \alpha t$ with $\alpha \neq 0$, $g(t) = \beta t$ with $\beta \neq 0$, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that

$$t + \alpha z_n \in T, \text{ and } t + \beta z_n \in T \text{ for all } n \in \mathbb{M}_t.$$

For instance, taking $\alpha = 1$, $\beta = -1$ we obtain generic bilateral embedding:

$$t + z_n \in T, \text{ and } t - z_n \in T \text{ for all } n \in \mathbb{M}_t.$$

For $\alpha_n = 2^n$ and $z_n(t) = z_n$ constant, the following result (though not its proof) appears implicitly in the proof of Császár's Non-separation Theorem (of a mid-point convex function and its lower hull by a measurable function); we consider its applications in a forthcoming publication.

THEOREM 5.3. (Császár's Genericity Theorem, [**34**, pp. 223–226], [**17**]). *Let T be measurable or Baire.*

(i) *Let $\{\alpha_n\}$ be bounded from below by unity and let $\{z_n\} \to 0$ be uniformly null. For generically all $t \in T$, there are points $t_n \in T$ such that, along some subsequence of n,*

$$t = \alpha_n t_n + (1 - \alpha_n) u_n(t), \text{ where } u_n(t) = t + z_n(t).$$

(ii) *Let $\{\alpha_n\}$ be positive and bounded away from zero and let $\{z_n\} \to 0$ be a null sequence of reals. For generically all $u \in T$ and generically all t near u, there are points $t_n \in T$ such that, along some subsequence of n,*

$$t = \alpha_n t_n + (1 - \alpha_n) u_n, \text{ where } u_n = u + z_n.$$

PROOF. The conclusions concern subsequences; so we may divide the argument according as α_n tends to infinity or is convergent. Suppose first that $a_n \to \infty$, and so also that, for all n, $\alpha_n > 1$. For $\gamma_n := 1/\alpha_n$ and $\lambda_n = 1 - \gamma_n$, we have

$0 < \lambda_n < 1$. Taking $f_n(t) = \lambda_n t = (1-\gamma_n)t$, we conclude from Example 2 above that for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that

$$t_n = t + (1-\gamma_n)z_n(t) \in T, \text{ for } n \in \mathbb{M}_t.$$

So

$$\text{(csa)} \qquad t_n = \gamma_n t + (1-\gamma_n)[t + z_n(t)] \in T,$$

and equivalently $t = \alpha_n t_n + (1-\alpha_n)u_n(t)$.

Now suppose that $\alpha_n \to \alpha > 0$. Thus $(1-\alpha_n)z_n \to 0$. Take $h_n^{-1}(t) = A_n(t) = \alpha_n t + (1-\alpha_n)z_n(t)$. Since (rwcc) holds at 0 in the Euclidean case (by Remark 4 of Section 2), and also in the density case by Cor. 3.2, we conclude that there is an infinite set $\mathbb{M}_{t,u}$ such that $t - u = x \in h_m^{-1}(T_u)$, for $m \in \mathbb{M}_{t,u}$, as in equation (equiv) in the proof of Corollary 3.1 (end of Section 3). Thus again we have

$$t - u = h_n^{-1}(t_n - u) = \alpha_n(t_n - u) + (1-\alpha_n)z_n,$$

or again $t = \alpha_n t_n + (1-\alpha_n)(u + z_n)$. □

REMARKS. 1. Theorem 5.1 applies also to sequences z_n which converge to zero on compacts. This is because all our results are local, by *capping,* as follows. Suppose $z_n(t)$ only converges to zero on compacts and that $t + z_n(t)$ is a Euclidean homeomorphism (i.e., is strictly increasing and continuous). For any interval (a,b) in $\mathbb{R}$, the capped sequence

$$\hat{z}_n(t) = \begin{cases} z_n(a), & \text{for } t \leqslant a,\\ z_n(t), & \text{for } a < t < b,\\ z_n(b), & \text{for } t \geqslant b \end{cases}$$

has $\hat{z}_n \to 0$ in supremum norm, and the substitution of $\hat{z}_n$ for z_n preserves the homeomorphism property (i.e., $t + \hat{z}_n(t)$ is strictly increasing and continuous) as well as equality with $t + z_n(t)$ on (a,b).

For instance, consider $f(t) = t^2$ and a given null sequence of constants $w_n \to 0$. Here its f-conjugate sequence is $z_n(t) := w_n(2t + w_n)$ and

$$h_n(t) := t + z_n(t) = t(1 + 2w_n) + w_n^2$$

is increasing for n large enough; however $z_n \to 0$ uniformly only on compacts. Nevertheless, by the capping procedure, here too, for T Baire /measurable, for generically all t in T there is an infinite set $\mathbb{M}_t$ such that $\{t + z_n(t) : m \in \mathbb{M}_t\} \subset T$.

2. Other examples of smooth generation of null sequences are

$$\bar{z}_n(t) := f(\varphi(t) + z_n(t)) - f(\varphi(t)),$$

where φ is a homeomorphism. Thus if $\psi = \varphi^{-1}$, then $t + \bar{z}_n(t)$ becomes, under the substitution $u = \varphi(t)$,

$$\psi(u) + f(u + z_n(\psi(u))) - f(u).$$

The special case $\psi = f$ then leads to the embedding of the sequence $f(u+z_n(\psi(u)))$.

6. Applications

The theorems of this section illustrate one area of use of the infinite combinatorics asserted by the KBD theorem – in relation to automatic continuity of (mid-point) convex functions $f : \mathbb{R} \to \mathbb{R}$. Call T in $\mathbb{R}$ *subuniversal* if for any null sequence $\{z_n\} \to 0$ in $\mathbb{R}$ there is an infinite $\mathbb{M} \subseteq \omega$, and $t \in \mathbb{R}$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T. \tag{6.1}$$

The term originates with Kestelman, who calls T universal for null sequences when (6.1) holds with $\mathbb{M}$ co-finite. Thus a *Baire non-meagre/measurable non-null set T is subuniversal.* Although subuniversality is the key combinatorial concept, it needs a geometric rephrasing in the Lemma which follows to suit the needs of the arguments below, which are geometric in nature.

LEMMA 6.1. (Averaging-Reflection Lemma). *A set T is subuniversal iff it is 'averaging', that is, for any null sequence $\{z_n\} \to 0$, any given point $u \in T$, and with $u_n := u + z_n$ (thus an arbitrary convergent sequence, but with limit in T), there are $w \in \mathbb{R}$ (an averaging translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have $u_n = \frac{1}{2}w + \frac{1}{2}v_n$.*

Equivalently, there are $w \in \mathbb{R}$ (a reflecting translator) and $\{v_n\} \subseteq T$ such that, for infinitely many $n \in \omega$, we have $v_n = \frac{1}{2}w + \frac{1}{2}u_n$.

PROOF. In the averaging case, it is enough to show that $\frac{1}{2}T$ is subuniversal iff T is averaging. If $\frac{1}{2}T$ is subuniversal then, given $u_n \to u$, there are $w \in \mathbb{R}$ and some infinite $\mathbb{M}$ so that $\{-\frac{1}{2}w + u_n : n \in \mathbb{M}\} \subseteq \frac{1}{2}T$; hence, putting $v_n := 2u_n - w$, we have $\{v_n : n \in \mathbb{M}\} \subseteq T$. Conversely, if T is averaging and $\{z_n\} \to 0$, then for some x and some $\mathbb{M}$, $\{2x + 2z_n : n \in \mathbb{M}\} \subseteq T$, so $\{x + z_n : n \in \mathbb{M}\} \subseteq \frac{1}{2}T$ and hence $\frac{1}{2}T$ is subuniversal. Similar reasoning yields the reflecting case. □

We recall some properties of convex functions, for which we need to define the lower hull $m_f(x)$ of f by

$$m_f(x) = \lim_{\delta \to 0+} \inf\{f(t) : |t - x| < \delta\}.$$

THEOREM 6.1. (Portmanteau Theorem for Convex Functions). *For convex f:*

(i) *If f is locally bounded above at some point, then f is locally bounded above at all points* [**34**, p. 138].

(ii) *If f is locally bounded below at some point, then f is locally bounded below at all points* [**34**, p. 139].

(iii) *If f is locally bounded above at some point, then it is everywhere locally bounded* [**34**, p. 140].

(iv) *If $f(x) \neq m_f(x)$ for some x, then f is not locally bounded at x* [**34**, p. 144].

The common feature here is that the sequence witnessing bad behaviour at one point yields by translation a sequence witnessing bad behaviour at any desired point.

THEOREM 6.2. *If f is convex and bounded below on a subuniversal set T, then f is locally bounded below.*

PROOF. Suppose not. Let K be a lower bound on T. We use the reflecting property of T. If f is not locally bounded from below, then at any point u in $\bar{T}$ there is a sequence $\{u_n\} \to u$ with $\{f(u_n)\} \to -\infty$. For some $w \in \mathbb{R}$, we have $v_n = \frac{1}{2}w + \frac{1}{2}u_n \in T$, for infinitely many n. Then

$$K \leqslant f(v_n) \leqslant \frac{1}{2}f(w) + \frac{1}{2}f(u_n), \text{ or } 2K - f(w) \leqslant f(u_n),$$

i.e., $f(u_n)$ is bounded from below, a contradiction. □

THEOREM 6.3. [**39**, Th. 3]. *If f is convex and bounded above on a subuniversal set T, then f is continuous.*

PROOF. We use the averaging property of T. Suppose that f is not continuous, but is bounded above on T by K. Then f is not locally bounded above at some point of $u \in \bar{T}$. Then there is a null sequence $z_n \to 0$ with $f(u_n) \to \infty$, where $u_n = u + z_n$. Select $\{v_n\}$ in T and w in $\mathbb{R}$ so that, for infinitely many n, we have $u_n = \frac{1}{2}w + \frac{1}{2}v_n$. But for such n, we have $f(u_n) \leqslant \frac{1}{2}f(w) + \frac{1}{2}f(v_n) \leqslant \frac{1}{2}f(w) + \frac{1}{2}K$, contradicting the unboundedness of $f(u_n)$. □

Theorem 6.3, taken together with the KBD theorem, implies the classical result below, an early automaticity theorem.

THEOREM 6.4. (Császár–Ostrowski Theorem, [**17**], [**34**, p. 210]). *A convex function $f : \mathbb{R} \to \mathbb{R}$ bounded above on a set of positive measure/non-meagre set is continuous.*

Likewise Theorem 6.3 implies the following earlier classical result due to Sierpiński [**48**] (cf. [**5**, p. 5]).

COROLLARY 6.1. (Sierpiński's Theorem, [**48**], [**34**, p. 218]). *A measurable/Baire convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

This is immediate since such a function f is bounded above on a set of positive measure/non-meagre set, so is continuous.

THEOREM 6.5. (Császár's First Theorem, [**34**, p. 223]). *Suppose f is convex and bounded below by K on a Baire non-meagre/measurable non-null set T. Then m_f is bounded below by K on the closure of T and hence f is continuous.*

PROOF. Suppose otherwise. Let $\gamma_n \to 0$ rational (e.g., $\gamma_n = 2^{-n}$) and f convex. Suppose $m_f(u) < K$, for some $u \in \bar{T}$; then there is a sequence $u_n \to u$, say with $f(u_n) \to L < K$. By Theorem 5.3 with $\alpha_n = 1/\gamma_n$ (cf. equation (csa)), there is w and $m(n)$ such that $v_n := \gamma_n w + (1 - \gamma_n)u_{m(n)} \in T$. Hence

$$K \leqslant f(v_n) \leqslant \gamma_n f(w) + (1 - \gamma_n) f(u_{m(n)}).$$

Passing to the limit we obtain the contradiction $K \leqslant L$. □

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Department of Mathematics
Imperial College London
London SW7 2AZ
UK
n.bingham@ic.ac.uk
nick.bingham@btinternet.com

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Mathematics Department
London School of Economics
Houghton Street
London WC2A 2AE
UK
a.j.ostaszewski@lse.ac.uk