PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 75(89) (2004), 9–24

ON A PARAMETRIC METHOD FOR CONFORMAL MAPS WITH QUASICONFORMAL EXTENSIONS

Alexander Vasil'ev

ABSTRACT. The Löwner-Kufarev equation gives a complete description of the class S of all univalent holomorphic functions f in the unit disk normalized by $f(0)+1=f'(0)=1$. We consider the class S^{qc} of all functions from S that admit quasiconformal extension to the whole Riemann sphere fixing ∞. There is a well known Becker's sufficient condition for the Löwner-Kufarev equation that guarantees a function from S to be from S^{qc}. We study subordination chains of quasidisks bounded by analytic curves and corresponding motions on the modelling universal Teichmüller space. This leads to a specific form of the Löwner-Kufarev equation.

1. Introduction

Let U denote the unit disk in the Riemann sphere $\hat{\mathbb{C}}$ and $U^* = \hat{\mathbb{C}} \setminus \hat{U}$, where $\hat{U}$ is the closure of U, $\mathbb{T} := \partial U$. By S we denote the class of all holomorphic univalent functions in U normalized by $f(z) = z + a_2 z^2 + \cdots$, $z \in U$. Any function $f \in S$ can be represented as a limit

$$\lim_{t\to\infty} e^t w(z,t), \tag{1.1}$$

where the function $w(z,t)$ is a solution of the equation

$$\frac{dw}{dt} = -wp(w,t), \tag{1.2}$$

almost everywhere in $t \in [0,\infty)$, with the initial condition $w(z,0) = z$. The function $p(z,t) = 1 + p_1(t)z + \cdots$ is analytic in U, measurable with respect to $t \in [0,\infty)$, and its real part $\operatorname{Re} p(z,t)$ is positive for almost all $t \in [0,\infty)$. The equation (1.2) is known as the *Löwner-Kufarev equation.* First its important particular form appeared almost 80 years ago in a seminal paper by K. Löwner [**21**], who studied a one-parameter semigroup of conformal one-slit maps of U, taken

2000 *Mathematics Subject Classification*: Primary 30C35, 30C62; Secondary 30F60.
Key words and phrases: univalent function, quasiconformal map, Löwner-Kufarev equation, universal Teichmüller space.
This work is partially supported by projects: Fondecyt (Chile) #1030373, #1040333, and UTFSM #12.03.23.

$p(z,t) = (e^{i\mu(t)} + z)(e^{i\mu(t)} - z)^{-1}$ with a continuous function $\mu(t)$. Löwner's result was generalized, then, in several ways. Attempts have been made to derive an equation that allowed to describe a representation of the whole class S. Nowadays, it is rather difficult to follow the correct history line of the development of the parametric method because in the middle of the 20-th century a number of works dedicated to this general equation appeared independently. In particular, P. P. Kufarev [**18**] studied a one-parameter family of domains $\Omega(t)$ and regular functions $f(z,t)$ defined in $\Omega(t)$. He proved differentiability of $f(z,t)$ with respect to t for z from the Carathéodory kernel $\Omega(t_0)$ of $\Omega(t)$, and derived a generalization of the Löwner equation. Ch. Pommerenke [**24**] proposed to consider subordination chains of domains that led him to a general equation. We mention here also papers by V. Gutlyanskiĭ [**14**] and V. Goryainov [**11**] in this direction. One can learn more about this method in monographs [**1, 8, 25**] (see also the references therein).

The importance of the parametric representation (1.2) is shown by a number of results obtained with a help of this method. The most intricate problem for the class S posed by L. Bieberbach in 1916 [**6**] finally has been solved in 1984 by L. de Branges [**7**] making use of the parametric representation. He proved that $|a_n| \leqslant n$ for any $f \in S$ and the equality is attained only for the Koebe function $k(z) = z(1 - ze^{i\theta})^{-2}$, $\theta \in [0, 2\pi)$.

From the geometric point of view one can consider a *subordination chain* of simply connected domains $\Omega(t)$ in the Riemann sphere $\hat{\mathbb{C}}$, $\infty \notin \Omega(t)$, which is defined for $0 \leqslant t < t_0$ for some $t_0 > 0$. This means that $\Omega(t) \subset \Omega(s)$ when $t < s$. By the Riemann Mapping Theorem one constructs a subordination chain of mappings $f(z,t)$, $z \in U$, where the function $f(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, $a(t) > 0$, is a holomorphic univalent map of U onto $\Omega(t)$ for each fixed t. Ch. Pommerenke [**24, 25**] first introduced such chains in order to generalize Löwner's equation. His result says that given a subordination chain of domains $\Omega(t)$ of the conformal radius $a(t)$ which is differentiable with respect to t, there exists a holomorphic function $p(z,t) = p_0(t) + p_1(t)z + p_2(t)z^2 + \cdots$, $z \in U$, such that $\operatorname{Re} p(z,t) > 0$ in $z \in U$ and

$$\frac{\partial f(z,t)}{\partial t} = z\frac{\partial f(z,t)}{\partial z} p(z,t), \tag{1.3}$$

almost everywhere with respect to t. The coefficient $a_1(t) = a_1(0)\exp(\int_0^t p_0(\tau)d\tau)$ is the conformal radius of $\Omega(t)$.

The connection between (1.2) and (1.3) can be thought of as follows. Solving (1.3) by the method of characteristics and assuming s as the parameter along the characteristics we have

$$\frac{dt}{ds} = 1, \quad \frac{dz}{ds} = -zp(z,t), \quad \frac{df}{ds} = 0,$$

with the initial conditions $t(0) = 0$, $z(0) = z_0$, $f(z,0) = f_0(z)$, where z_0 is in U. We see that the equation (1.2) is exactly the characteristic equation for (1.3). Unfortunately, this approach requires the extension of $f_0(w^{-1}(z,t))$ into U because the solution of the function $f(z,t)$ is given as $f_0(w^{-1}(z,t))$, where $z = w(z_0, s)$ is the solution of the initial value problem for the characteristic equation. Thus, assuming certain properties about the family $f(z,t)$ enables us to derive the equation (1.2) as a necessary condition.

Several attempts have been launched to specialize the Löwner-Kufarev equation to obtain conformal maps that admit quasiconformal extensions (see [**2, 3, 4, 15**]). Precisely, if $f(z,t)$ is a solution to the equation (1.3) where $p(z,t)$ satisfies the condition

$$\left|\frac{p(z,t)-1}{p(z,t)+1}\right| \leqslant k < 1, \tag{1.4}$$

then $\Omega(t)$ is a Jordan domain bounded by a k-quasicircle. The corresponding function generated by (1.1) belongs to the class S_k of all functions from S that admit k-quasiconformal extension to the Riemann sphere fixing ∞. Thus, Becker's condition (1.4) is a sufficient condition for a function $f \in S$ to belong to $S_k \subset S^{qc}$, where $S^{qc} = \bigcup_{k\in[0,1)} S_k$.

We ask reciprocally: **what does $p(z,t)$ look like when $\partial\Omega(t)$ is a quasicircle?** Generally, we are not able to give a complete answer for this question, but assuming that the corresponding quasiconformal mapping of $\mathbb{C}$ has the Beltrami coefficient vanishing in some neighbourhood of $\mathbb{T}$, we will deduce some analogues of the Löwner-Kufarev equations.

Another problem we are concerned with is as follows: **given a one-parameter family of complex functions $\mu(z,t)$ vanishing in a neighbourhood of U, measurable with respect to $z \in U^*$, $\|\mu\|_\infty < 1$, and differentiable in $t \in [0,t_0)$, we solve the Beltrami equation $f_{\bar z} = \mu(z,t) f_z$ normalizing f by $f(0) = 0$, $f'(0) = e^t$, $f(\infty) = \infty$. What is the condition on μ that guarantees the family $\Omega(t) = f(U,t)$ to be a subordination chain?** For the obvious case $\mu(z,t) \equiv \mu(z)$ we have $f(z,t) = e^t f(z)$ which forms a subordination chain of starlike maps if $f(z)$ is starlike.

Parametric methods for quasiconformal maps have been known since 1959. Shah Dao-Shing [**26**] suggested an evolutionary equation for quasiconformal automorohisms of U. In another form this method appeared in the paper by F. Gehring and E. Reich [**10**], and then, in [**19**]. Later, Cheng Qi He [**17**] obtained an analogous equation for classes of quasiconformally extendable univalent functions (in terms of inverse functions). Unlike the parametric method for conformal maps, its analogue for quasiconformal maps did not receive so much attention. Moreover, the parametric representations mentioned above have no precise concordance with the Löwner equation.

In 1987 the author attended the Kuban' conference on Geometric Function Theory held in a beautiful village Divnomorsk at the Black Sea Coast. There V. V. Goryainov gave a nice lecture on a semi-group approach to the Löwner-Kufarev equation. Later he published several new results obtained by this approach (see, e.g., [**11, 12, 13**]). Recently D. Shoikhet published a monograph [**27**] where he presented a systematic treatment of this method. Inspired by these works we consider differentiable paths on the universal Teichmüller space as a natural realization of one-parameter families of quasicircles and link them to semigroups of conformal maps with quasiconformal extension. Then, based on variational formulas, we derive a special form of the function p as a necessary condition and shed some light onto proposed questions.

Acknowledgement. I would like to thank Jochen Becker and Victor Goryainov for helpful conversations. In particular, J. Becker made several valuable remarks leading to improvement of the paper.

2. Infinitesimal structure of the universal Teichmüller space

Let us consider the family $\mathcal{F}$ of all quasiconformal automorphisms of U. Every such map f satisfies the Beltrami equation $f_{\bar{z}} = \mu_f(z) f_z$ in U in distributional sense, where μ_f is a measurable essentially bounded function ($L^\infty(U^*)$) in U, $\|\mu_f\| = \operatorname{ess\,sup}_U |\mu_f(z)|_\infty < 1$. Conversely, for each measurable Beltrami coefficient μ essentially bounded as above, there exists a quasiconformal automorphism of U, that satisfies the Beltrami equation, which is unique if provided with some conformal normalization, e.g., three point normalization $f(\pm 1) = \pm 1$, $f(i) = i$. Two normalized maps f_1 and f_2 are said to be equivalent $f_1 \sim f_2$ if being extended onto the unit circle $\mathbb{T}$, the superposition $f_1 \circ f_2^{-1}$ restricted to $\mathbb{T}$ is the identity map. The quotient set $\mathcal{F}/\sim$ is called *the universal Teichmüller space* T. It is a covering space for all Teichmüller spaces of analytically finite Riemann surfaces. By definition we have two realizations of T: as a set of equivalence classes of quasiconformal maps and, due to the relation between $\mathcal{F}/\sim$ and the unit ball $B \subset L^\infty(U)$, as a set of equivalence classes of corresponding Beltrami coefficients.

The normalized maps from $\mathcal{F}$ form a group $\mathcal{F}_0$ with respect to superposition and the maps that act identically on $\mathbb{T}$ form its normal subgroup $\mathcal{I}$. Thus, T is the quotient $T = \mathcal{F}_0/\mathcal{I}$.

If $g \in \mathcal{F}$, $f \in \mathcal{F}_0$, then there exists a Möbius transformation h, such that $h \circ f \circ g^{-1} \in \mathcal{F}_0$. Let us denote by $[f] \in T$ the equivalence class represented by $f \in \mathcal{F}_0$. Then, one defines *the universal modular group* $\mathcal{M}$, $\omega \in \mathcal{M}$, $\omega : T \to T$, by the formula $\omega([f]) = [h \circ f \circ g^{-1}]$. Its subgroup $\mathcal{M}_0$ of right translations on T is defined by $\omega_0([f]) = [f \circ g^{-1}]$, where $f, g \in \mathcal{F}_0$.

An important fact (see [**20**, Chapter III, Theorem 1.1]) is that there are real analytic mappings in any equivalence class $[f] \in T$.

Given a Beltrami coefficient $\mu \in B \subset L^\infty(U^*)$ let us extend it by zero into $z \in U$. We normalize the corresponding quasiconformal map f which is conformal in U by $f(z) = z + a_2 z^2 + \cdots$ about the origin. Then, two Beltrami coefficients μ and ν are equivalent if and only if the corresponding normalized mappings f^μ and f^ν map U onto one and the same domain in $\hat{\mathbb{C}}$. Thus, the universal Teichmüller space can be thought of as the family of all normalized conformal maps of U admitting quasiconformal extension. Moreover, any compact subset of T consists of conformal maps f of U that admit quasiconformal extension to U^* with $\|\mu_f\|_\infty \leqslant k < 1$ for some k.

As we mentioned above, a normalized conformal map $f \in [f] \in T$ defined in U can have a quasiconformal extension to U^* which is real analytic in U^*, but on the unit circle f can behave quite irregularly. For example, the resulting quasicircle $f(\mathbb{T})$ can have the Hausdorff dimension greater than 1.

REMARK 2.1. For a bounded k-quasicircle Γ in the plane let $N(\varepsilon,\Gamma)$ denote the minimal number of disks of radius $\varepsilon>0$ that are needed to cover Γ. Let

$$\beta(k)=\sup_{\Gamma}\limsup_{\varepsilon\to 0}\frac{\log N(\varepsilon,\Gamma)}{\log(1/\varepsilon)}$$

denote the supremum of the Minkowski dimension of curves Γ where Γ ranges over all bounded k-quasicircles. The Hausdorff dimension of Γ is bounded from above by $\beta(k)$ (see [**5**]). In [**5**] it was also established several explicit estimates for $\beta(k)$, e.g., $\beta(k)\leqslant 2-cK^{-3.41}$, where $K=(1+k)/(1-k)$.

Let us denote by $S^{qc}\subset S$ the class of all those univalent conformal maps f defined in U that admit a quasiconformal extension into U^*, normalized by $f(z)=z+a_2z^2+\cdots$. Let $x,y\in T$ and $f,g\in S^{qc}$ be such that $\mu_f\in x$ and $\mu_g\in y$. Then, *the Teichmüller distance* $\tau(x,y)$ on T is defined as

$$\tau(x,y)=\inf_{\substack{\mu_f\in x\\ \mu_g\in y}}\frac{1}{2}\log\frac{1+\|\mu_{g\circ f^{-1}}\|_\infty}{1-\|\mu_{g\circ f^{-1}}\|_\infty}.$$

For a given $x\in T$ we consider the extremal Beltrami coefficient μ^* such that $\|\mu^*\|_\infty=\inf_{\nu\in x}\|\nu\|_\infty$. Let us remark that the extremal μ^* need not be unique. A geodesic on T can be described in terms of the extremal coefficient μ^* as a continuous homomorphism $x_t:[0,1]\mapsto T$ such that $\tau(0,x_t)=t\tau(0,x_1)$. Due to the above remark the geodesic need not be unique as well.

We consider the Banach space $B(U)$ of all functions holomorphic in U equipped with the norm

$$\|\varphi\|_{B(U)}=\sup_{z\in U}|\varphi(z)|(1-|z|^2)^2.$$

For a function f from S the Schwarzian derivative

$$S_f(z)=\frac{\partial}{\partial z}\left(\frac{f''(z)}{f'(z)}\right)-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is defined and Nehari's [**23**] estimate $\|S_f\|_{B(U)}\leqslant 6$ holds. Given $x\in T$, $\mu\in x$ we construct the mapping $f^\mu\in S^{qc}$ and have the homeomorphic embedding $T\mapsto B(U)$ by the Schwarzian derivative.

The universal Teichmüller space T is an analytic infinite dimensional Banach manifold modelled on $B(U)$. The Banach space $B(U)$ is an infinite dimensional vector space that can be thought of as a co-tangent space to T at the initial point. More rigorously, the map $f^\mu\in S^{qc}$ has a Fréchet derivative with respect to μ in a direction ν. Let us construct the variation in S^{qc}

$$f^{\tau\nu}(z)=z+\tau V(z)+o(\tau),\quad z\in U.$$

Taking the Schwarzian derivative in U we get

$$S_{f^{\tau\nu}}=\tau V'''(z)+o(\tau),\quad z\in U,$$

locally uniformly in U. Taking into account the normalization of the class S^{qc} we have (see, e.g., [**16, 20**])

$$V(z) = -\frac{z^2}{\pi} \iint\limits_{U^*} \frac{\nu(w)\, d\sigma_w}{w^2(w-z)},$$

$$V'''(z) = -\frac{6}{\pi} \iint\limits_{U^*} \frac{\nu(w)\, d\sigma_w}{(w-z)^4} = -\frac{6}{\pi} \iint\limits_{U} \frac{\bar{w}^2}{w^2} \frac{\nu(1/\bar{w})\, d\sigma_w}{(1-\bar{w}z)^4}.$$

The integral formula implies $V'''(A(z))A'(z)^2 = V'''(z)$ (subject to the relation for the Beltrami coefficient $\mu(A(z))\overline{A'(z)} = \mu(z)A'(z)$) for any Möbius transform A.

Let us extend $\nu(z)$, $z \in U^*$ into U by putting $\nu(1/\bar{z}) = \overline{\nu(z)}z^2/\bar{z}^2$, $z \in U$. Taking $\Lambda_\mu(z) = S_{f^{\tau\mu}}(z)$ and $\dot{\Lambda}_\mu(z) = V'''(z)$ we have (see, e.g., [**9**, Section 6.5, Theorem 5]) that

$$\Lambda_\nu(z) - \tau\dot{\Lambda}_\nu(z) = \frac{o(\tau)}{(1-|z|^2)^2}, \quad z \in U$$

or $\dot{\Lambda}_\nu$ is the derivative of Λ_ν at the initial point of the universal Teichmüller space with respect to the norm of the Banach space $B(U)$. The reproducing case of the Bergman integral gives

$$\varphi(z) = \frac{3}{\pi} \iint\limits_{U} \frac{\varphi(w)(1-|w|^2)^2\, d\sigma_w}{(1-\bar{w}z)^4}, \quad \varphi \in B(U). \tag{2.1}$$

Changing variables $w \to 1/\bar{w}$ in the latter integral we come to the so-called harmonic (Bers') Beltrami differential

$$\nu(z) = \Lambda^*_\varphi(z) = -\frac{1}{2}\overline{\varphi(z)}(1-|z|^2)^2, \quad z \in U.$$

Let us denote by $A(U)$ the Banach space of analytic functions with finite L^1 norm. Then $A(U) \hookrightarrow B(U)$ is a continuous inclusion ([**22**], Section 1.4.2). On $L^\infty(U^*) \times A(U)$ one can define a coupling

$$\langle \nu, \varphi \rangle := \iint\limits_{U} \nu(z)\varphi(z)\, d\sigma_z,$$

where $d\sigma_z$ means the area element in U. Denote by N the space of *locally trivial Beltrami coefficients*, which is a subspace of $L^\infty(U)$ that forms the kernel of the operator $\langle \cdot, \varphi \rangle$ for all $\varphi \in A(U)$. Then, one can identify the tangent space to T at the initial point with the space $H := L^\infty(U)/N$. It is natural to relate it to a subspace of $L^\infty(U^*)$. The superposition $\dot{\Lambda}_\nu \circ \Lambda^*_\varphi$ acts identically on $A(U)$ due to (2.1). The space N is also the kernel of the operator $\dot{\Lambda}_\nu$. Thus, the operator Λ^* splits the following exact sequence

$$0 \longrightarrow N \hookrightarrow L^\infty(U) \xrightarrow{\dot{\Lambda}_\nu} A(U) \longrightarrow 0.$$

Then, $H = \Lambda^*(A(U)) \cong L^\infty(U)/N$. The coupling $\langle \mu, \varphi \rangle$ defines $A(U)$ as a cotangent space.

Let $A^2(U)$ denote the Banach space of analytic functions φ with the finite norm

$$\|\varphi\|_{A^2(U)} = \iint\limits_U |\varphi(z)|^2(1-|z|^2)^2\, d\sigma_z.$$

Then $A(U) \hookrightarrow A^2(U)$ and Petersson's Hermitian product [**28**] is defined on $A^2(U)$ as

$$(\varphi_1, \varphi_2) = \iint\limits_U \varphi_1(z)\overline{\varphi_2(z)}(|z|^2-1)^2\, d\sigma_z.$$

The Kählerian Weil-Petersson metric $\{\nu_1, \nu_2\} = \langle \nu_1, \dot{\Lambda}_{\nu_2}\rangle$ can be defined on the tangent space to T that gives a Kählerian manifold structure to T.

The universal Teichmüller space is a smooth manifold on which a Lie group of real sense preserving diffeomorfisms $\operatorname{Diff} T$ is defined. The tangent bundle is defined on T and is represented by the harmonic differentials from H translated to all points of T. We will consider tangent vectors from H at the initial point of T represented by the map $f(z) \equiv z$. The Weil-Petersson metric defines a Lie algebra of vector fields on T by the Poisson bracket $[\nu_1, \nu_2] = \{\nu_2, \nu_1\} - \{\nu_1, \nu_2\}$, where $\nu_1, \nu_2 \in H$. One can define the Poisson bracket at all other points of T by left translations from $\operatorname{Diff} T$. To each element $[x]$ from $\operatorname{Diff} T$ an element x from T is associated as an image of the initial point. Therefore, a curve in $\operatorname{Diff} T$ generates a traced curve in T that can be realized by a one-paramter family of quasiconfromal maps from S^{qc}.

For each tangent vector $\nu \in H$ there is a one-parameter semi-flow in $\operatorname{Diff} T$ and a corresponding flow $x^\tau \in T$ with the velocity vector ν. To make an explicit representation we use the variational formula for the subclass S^{qc} of S of functions with quasiconformal extension (see, e.g., [**16**]) to $\hat{\mathbb{C}}$, fixing ∞. If $f^\mu \in S^{qc}$, $\nu \in H$, and

$$\mu_f(z,\tau) = \begin{cases} \tau\nu(z) + o(\tau), & \text{if } z \in U^*, \\ 0, & \text{if } z \in U, \end{cases} .$$

then the map

$$f^\mu(z) = z - \frac{\tau z^2}{\pi} \iint\limits_{U^*} \frac{\nu(w)\, d\sigma_w}{w^2(w-z)} + o(\tau)$$

locally describes the semi-flow x^τ on T.

3. Semigroups of conformal maps

We consider the semigroup $\mathcal{G}$ of conformal univalent maps from U into itself with composition as the semigroup operation. This makes $\mathcal{G}$ a topological semigroup with respect to the topology of local uniform convergence on U. We impose the natural normalization to such conformal maps: $\Phi(z) = \beta z + b_2 z^2 + \cdots$, $z \in U$, $\beta > 0$. The unit of the semigroup is the identity. Let us construct on $\mathcal{G}$ a one-parameter semi-flow Φ^τ, that is a continuous homomorphism from $\mathbb{R}^+$ into $\mathcal{G}$, with the parameter $\tau \geqslant 0$. For any fixed $\tau \geqslant 0$ the element Φ^τ is from $\mathcal{G}$ and is represented by a conformal map $\Phi(z,\tau) = \beta(\tau)z + b_2(\tau)z^2 + \cdots$ from U onto the domain $\Phi(U,\tau) \subset U$. The element Φ^τ satisfies the following properties:

- $\Phi^0 = id$;
- $\Phi^{\tau+s} = \Phi(\Phi(z,\tau), s)$, for $\tau, s \geqslant 0$;
- $\Phi(z,\tau) \to z$ locally uniformly in U as $\tau \to 0$.

In particular, $\beta(0) = 1$. This semi-flow is generated by a vector field $v(z)$ if for each $z \in U$ the function $w = \Phi(z,\tau)$, $\tau \geqslant 0$ is a solution of an autonomous differential equation $dw/d\tau = v(w)$ with the initial condition $w|_{\tau=0} = z$. The semi-flow can be extended to a symmetric interval $(-t,t)$ by putting $\Phi^{-\tau} = \Phi^{-1}(z,\tau)$. Certainly, the latter function is defined on the set $\Phi(U,\tau)$. Admitting this restriction for negative τ we define a one-parameter family Φ^τ for $\tau \in (-t,t)$.

For a semi-flow Φ^τ on $\mathcal{G}$ there is an infinitesimal generator at $\tau = 0$ constructed by the following procedure. Any element Φ^τ is represented by a conformal map $\Phi(z,\tau)$ that satisfies the Schwarz Lemma for the maps $U \to U$, and hence,

$$\operatorname{Re}\frac{\Phi(z,\tau)}{z} \leqslant \left|\frac{\Phi(z,\tau)}{z}\right| \leqslant 1, \quad z \in U,$$

where the equality sign is attained only for $\Phi^0 = id \simeq \Phi(z,0) \equiv z$. Therefore, the following limit exists (see, e.g., [**11, 12, 27**])

$$\lim_{\tau\to 0}\operatorname{Re}\frac{\Phi(z,\tau)-z}{\tau z} = \operatorname{Re}\frac{\left.\frac{\partial\Phi(z,\tau)}{\partial\tau}\right|_{\tau=0}}{z} \leqslant 0,$$

and the representation

$$\left.\frac{\partial\Phi(z,\tau)}{\partial\tau}\right|_{\tau=0} = -zp(z)$$

holds, where $p(z) = p_0 + p_1 z + \cdots$ is an analytic function in U with positive real part, and

$$\left.\frac{\partial\beta(\tau)}{\partial\tau}\right|_{\tau=0} = -p_0. \tag{3.1}$$

In [**13**] it was shown that Φ^τ is even C^∞ with respect to τ. The function $-zp(z)$ is an infinitesimal generator for Φ^τ at $\tau = 0$, and the following variational formula holds

$$\Phi(z,\tau) = z - \tau\, zp(z) + o(\tau), \quad \beta(\tau) = 1 - \tau p_0 + o(\tau). \tag{3.2}$$

The convergence is thought of as local uniform. We rewrite (3.2) as

$$\Phi(z,\tau) = (1-\tau p_0)z + \tau\, z(-p(z)+p_0) + o(\tau) = \beta(\tau)z + \tau\, z(-p(z)+p_0) + o(\tau). \tag{3.3}$$

Now let us proceed with a semigroup $\mathcal{G}^{qc} \subset \mathcal{G}$ of quasiconformal automorphisms of $\hat{\mathbb{C}}$. A quasiconformal map Φ representing an element of $\mathcal{G}^{qc}$ satisfies the Beltrami equation in $\hat{\mathbb{C}}$

$$\Phi_{\bar z} = \mu_\Phi(z)\Phi_z,$$

with the distributional derivatives $\Phi_{\bar z}$ and Φ_z, where $\mu_\Phi(z)$ is a measurable function vanishing in U and essentially bounded in U^* by

$$\|\mu_\Phi\| = \operatorname{ess\,sup}_{U^*}|\mu_\Phi(z)| \leqslant k < 1,$$

for some k. If k is sufficiently small, then the function Φ/β satisfies the variational formula (see, e.g., [**16**])

$$\frac{\Phi(z)}{\beta} = z - \frac{z^2}{\pi} \iint\limits_{U^*} \frac{\mu_\Phi(w)\, d\sigma_w}{w^2(w-z)} + o(k), \tag{3.4}$$

where $d\sigma_w$ stands for the area element in the w-plane.

Now for each τ small and $\Phi^\tau \in \mathcal{G}^{qc}$, the mapping $h(z,\tau) = \Phi(z,\tau)/\beta(\tau)$ is from S^{qc} and represents an equivalence class $[h^\tau] \in T$. Consider the one-parameter curve $x^\tau \in T$ that corresponds to $[h^\tau]$ and a velocity vector $\nu(z) \in H$ (that is not trivial), such that

$$\mu_h(z,\tau) = \mu_\Phi(z,\tau) = \tau\nu(z) + o(\tau).$$

We take into account that $\Phi(z,0) \equiv z$ in U and is extended up to the identity map of $\hat{\mathbb{C}}$.

Comparing (3.3) and (3.4) we come to the conclusion about Φ:

$$\Phi(z,\tau) = \beta(\tau) z - \frac{\tau z^2}{\pi} \iint\limits_{U^*} \frac{\nu(w)\, d\sigma_w}{w^2(w-z)} + o(\tau). \tag{3.5}$$

The relations (3.2, 3.3, 3.5) imply that

$$p(z) = p_0 + \frac{z}{\pi} \iint\limits_{U^*} \frac{\nu(w)\, d\sigma_w}{w^2(w-z)}. \tag{3.6}$$

The constant p_0 and the function ν must be such that $\operatorname{Re} p(z) > 0$ for all $z \in U$.

We summarize these observations in the following theorem.

THEOREM 3.1. *Let Φ^τ be a semi-flow in $\mathcal{G}^{qc}$. Then it is generated by the vector field $v(z) = -zp(z)$,*

$$p(z) = p_0 + \frac{z}{\pi} \iint\limits_{U^*} \frac{\nu(w)\, d\sigma_w}{w^2(w-z)},$$

where $\nu(z) \in H$ is a harmonic differential and the holomorphic function $p(z)$ has positive real part in U.

This theorem implies that at any point $\tau \geqslant 0$ we have

$$\frac{\partial \Phi(z,\tau)}{\partial \tau} = -\Phi(z,\tau) p(\Phi(z,\tau)).$$

4. Evolution families and differential equations

A subset $\Phi^{t,s}$ of $\mathcal{G}$, $0 \leqslant s \leqslant t$ is called an *evolution family* in $\mathcal{G}$ if

- $\Phi^{t,t} = id$;
- $\Phi^{t,s} = \Phi^{t,r} \circ \Phi^{r,s}$, for $0 \leqslant s \leqslant r \leqslant t$;
- $\Phi^{t,s} \to id$ locally uniformly in U as $t, s \to \tau$.

In particular, if Φ^τ is a one-parameter semi-flow, then Φ^{t-s} is an evolution family. We consider a subordination chain of mappings $f(z,t)$, $z \in U$, which is defined in an interval $t \in [0, t_0)$, where the function $f(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ is a holomorphic univalent map $U \to \hat{\mathbb{C}}$ for each fixed t and $f(U,s) \subset f(U,t)$ for $s < t$.

Let us pass to the semigroup $\mathcal{G}^{qc}$. So $\Phi^{t,s}$ now has a quasiconformal extension to U^* and being restricted to U is from $\mathcal{G}$. Moreover, $\Phi^{t,s} \to id$ locally uniformly in $\mathbb{C}$ as $t, s \to \tau$.

For each t fixed in $[0, t_0)$ the map $f(z,t)$ has a quasiconformal extension into U^* (that can be assumed even real analytic). An important presupposition is that $f(z,t)$ generates a *non-trivial path* in T. This means, in particular, that for any $t_1, t_2 \in [0, t_0)$, $t_1 \neq t_2$ the image $f(U, t_1)$ can not be obtained from $f(U, t_2)$ by a Möbius transform, or taking into account our normalization, multiplying by a constant. We construct the superposition $f^{-1}(f(z,s),t)$ for $t \in [0,t_0)$, $s \leqslant t$. Putting $\tau = t - s$ we denote this mapping by $\Phi(z,t,\tau)$.

Now we suppose the following conditions for $f(z,t)$.

(i) The maps $f(z,t)$ form a subordination chain.
(ii) The map $f(z,t)$ is holomorphic in U, $f(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, where $a_1(t) > 0$ and differentiable with respect to t.
(iii) The map $f(z,t)$ is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, $f(\infty, t) = \infty$.
(iv) The chain of maps $f(z,t)$ is not trivial.
(v) The Beltrami coefficient $\mu_f(z,t)$ of this map is differentiable with respect to t locally uniformly in U^*, vanishes in some neighbourhood of U (indepentently of t).

The function $\Phi(z,t,\tau)$ is embedded into an evolution family in $\mathcal{G}$. It is differentiable with regard to τ and t in $[0,t_0)$, and $\Phi(z,t,0) = z$. Fix t and let $D_\tau = \Phi^{-1}(U,t,\tau) \setminus U$. Then there exists $\nu \in H$ such that the Beltrami coefficient is of the form $\mu_\Phi(z,t,\tau) = \tau\nu(z,t) + o(\tau)$ in $U^* \setminus D_\tau$ with some ν, $\mu_\Phi(z,t,\tau) = \mu_f(z,t-\tau)$ in D_τ, and vanishes in $\hat{U}$. We make τ sufficiently small such that $\mu_\Phi(z,t,\tau)$ vanishes in D_τ too. Therefore, $z = \lim_{\tau\to 0} \Phi(z,t,\tau)$ locally uniformly in $\mathbb{C}$ and $\Phi(z,t,\tau)$ is embedded now into an evolution family in $\mathcal{G}^{qc}$. The identity map is embedded into a semi-flow $\Phi^\tau \subset \mathcal{G}^{qc}$ (which is smooth) as the initial point with the same velocity vector

$$\frac{\partial \Phi(z,t,\tau)}{\partial \tau}\bigg|_{\tau=0} = -zp(z,t), \quad z \in U,$$

that leads to (1.3) (the semi-flow Φ^τ is tangent to the evolution family at the origin). Actually, the differentialble trajectory $f(z,t)$ generates a pencil of tangent smooth semi-flows with starting tangent vectors $-zp(z,t)$ (that can be only measurable with respect to t).

The requirement of non-triviality makes it possible to use the variation (3.4). Therefore, the conclusion is that the function $f(z,t)$ satisfies the equation (1.3) where the function $p(z,t)$ is given by

$$p(z,t) = p_0(t) + \frac{z}{\pi}\iint\limits_{U^*} \frac{\nu(w,t)\, d\sigma_w}{w^2(w-z)}, \tag{4.1}$$

and has positive real part. The existence of $p_0(t)$ comes from the existence of the subordination chain. We can assign the normalization to $f(z,t)$ controlling the growth of the conformal radius of the subordination chain by e^t. Changing variables we obtain $p_0 = 1$.

Summarizing the conclusions about the function $p(z,t)$ we come to the following result.

THEOREM 4.1. *Let $f(z,t)$ be a normalized subordination chain of maps that exists for $t \in [0,t_0)$ and satisfies the conditions (i–v). Then, there is a harmonic differential $\nu(z,t)$, $\nu \in H$, such that $\operatorname{Re} p(z,t) > 0$ for $z \in U$, $t \in [0,t_0)$,*

$$p(z,t) = 1 + \frac{z}{\pi} \iint\limits_{U^*} \frac{\nu(w,t)\, d\sigma_w}{w^2(w-z)}, \quad z \in U,$$

and $f(z,t)$ satisfies the differential equation

$$\frac{\partial f(z,t)}{\partial t} = z \frac{\partial f(z,t)}{\partial z} p(z,t), \quad z \in U, \tag{4.2}$$

in $t \in [0,t_0)$.

In the above theorem the function ν belongs to the space of harmonic differentials. We ask now about another but equivalent form of ν. Writing $w = f(z, t-\tau)$, $\Phi(z,t,\tau) = f^{-1}(w,t)$ we have

$$\mu_\Phi = \frac{\Phi_{\bar z}}{\Phi_z} = \frac{f_w^{-1} w_{\bar z} + f_{\bar w}^{-1} \bar w_{\bar z}}{f_w^{-1} w_z + f_{\bar w}^{-1} \bar w_z} = \frac{w_{\bar z} + \mu_{f^{-1}} \bar w_{\bar z}}{w_z + \mu_{f^{-1}} \bar w_z} = \frac{\bar w_{\bar z}}{w_z} \frac{\mu_w \frac{w_z}{\bar w_{\bar z}} - \mu_f \frac{f_z}{\bar f_{\bar z}}}{1 - \mu_f \overline{\mu_w} \frac{f_z \bar w_{\bar z}}{w_z \bar f_{\bar z}}}.$$

We use that $\mu_{f^{-1}} \circ f = -\mu_f f_z / \bar f_{\bar z}$. Finally, μ_f, f_z, $f_{\bar z}$ are differentiable by t in $t \in [0,t_0)$ for $z \in U^*$, and

$$\nu_0(z,t) = \lim_{\tau \to 0} \frac{\mu_\Phi}{\tau} = -\frac{\bar f_{\bar z}}{f_z} \frac{\frac{\partial}{\partial t}\left(\mu_f \frac{f_z}{\bar f_{\bar z}}\right)}{1 - |\mu_f|^2},$$

where the limit exists a.e. with respect to $t \in [0,t_0)$ locally uniformly in $z \in U^*$, or in terms of the inverse function

$$\nu_0(z,t) = \frac{f_w^{-1}}{\bar f_{\bar w}^{-1}} \frac{\frac{\partial \mu_{f^{-1}}}{\partial t}}{1 - |\mu_{f^{-1}}|^2} \circ f(z,t).$$

Sometimes, it is much better to operate just with dilatations avoiding functions, so we can rewrite the last expression as

$$\nu_0(z,t) = -\mu_f(z,t) \left[\frac{\frac{\partial \log \mu_{f^{-1}}}{\partial t}}{1 - |\mu_{f^{-1}}|^2} \circ f(z,t) \right].$$

REMARK 4.1. The function $\nu(z,t)$ in Theorem 4.1 may be replaced by the function $\nu_0(z,t)$ that belongs to the same equivalence class in H.

Let us consider one-parameter families of maps in U normalized by $f(z,t) = e^t z + a_2(t)z^2 + \cdots$. The inverse result to the Löwner-Kufarev equation in partial derivatives states that given a holomorphic function $p(z,t) = 1+p_1 z+\cdots$ in U with positive real part the solutions to the equation (4.2) give a subordination chain (see, e.g., [**25**]). This enable us to give a condition for ν_0 that guarantees a normalized one-parameter non-trivial family of maps $f(z,t)$ to be a subordination chain.

THEOREM 4.2. *Let $f(z,t)$ be a normalized one-parameter non-trivial family of maps for $z \in U$ which satisfies the conditions (ii–v) and is defined in an interval $[0,t_0)$. Let each $f(z,t)$ be a homeomorphism of $\hat{\mathbb{C}}$ which is holomorphic in U, is normalized by $f(z,t) = e^t z + a_2(t)z^2 + \cdots$, and satisfies (4.2). Let the quasiconformal extension to U^* be given by a Beltrami coefficient $\mu_f = \mu(z,t)$ which is differentiable with respect to t almost everywhere in $t \in [0,t_0)$. If*

$$\|\nu_0\|_\infty < \frac{\pi}{4\int_0^1 \mathbf{K}(s)\,ds} \approx 0.434\ldots,$$

where $\nu_0(z,t)$ is as above, and $\mathbf{K}(\cdot)$ is the complete elliptic integral, then $f(z,t)$ is a normalized subordination chain.

PROOF. Let $|z| = \rho$, $w = re^{i\theta}$. We calculate

$$\begin{aligned}
\left|\frac{z}{\pi}\iint\limits_{U^*}\frac{\nu(w,t)\,d\sigma_w}{w^2(w-z)}\right| &\leqslant \frac{\rho\|\nu\|_\infty}{\pi}\iint\limits_{U^*}\frac{d\sigma_w}{|w|^2|w-z|} \leqslant \frac{\rho\|\nu\|_\infty}{\pi}\iint\limits_{U}\frac{d\sigma_w}{|w||1-wz|} \\
&= \frac{\rho\|\nu\|_\infty}{\pi}\int_0^1\int_0^{2\pi}\frac{dr\,d\theta}{|1-re^{i\theta}z|} = \frac{\rho\|\nu\|_\infty}{\pi}\int_0^1\int_0^{2\pi}\frac{dr\,d\theta}{|1-re^{i\theta}\rho|} \\
&= \frac{\rho\|\nu\|_\infty}{\pi}\int_0^1\int_0^{2\pi}\frac{dr\,d\theta}{\sqrt{1+r^2\rho^2-2r\rho\cos\theta}} \\
&= \frac{\|\nu\|_\infty}{\pi}\int_0^\rho\int_0^{2\pi}\frac{ds\,d\theta}{\sqrt{1+s^2-2s\cos\theta}} \\
&\leqslant \frac{\|\nu\|_\infty}{\pi}\int_0^1\int_0^{2\pi}\frac{ds\,d\theta}{\sqrt{1+s^2-2s\cos\theta}} \\
&= \frac{\|\nu\|_\infty}{\pi}\iint\limits_{U^*}\frac{d\sigma_w}{|w|^2|w-1|} = \frac{4\|\nu\|_\infty}{\pi}\int_0^1\mathbf{K}(s)\,ds < 1.
\end{aligned}$$

Then $\mathrm{Re}\, p(z,t) > 0$ that implies the statement of the theorem. □

REMARK 4.2. If $\|\nu(\cdot,t)\|_\infty \leqslant q$, then

$$\frac{1+|\mu(z,t)|}{1-|\mu(z,t)|} \leqslant e^{2tq}\frac{1+|\mu(z,0)|}{1-|\mu(z,0)|}.$$

This obviously follows from the inequality

$$\frac{\partial|\mu_f|}{\partial t} = \frac{\partial|\mu_{f^{-1}}|}{\partial t} \leqslant |\dot{\mu}_{f^{-1}}|.$$

The equation (4.2) is just the Löwner-Kufarev equation in partial derivatives with a special function $p(z,t)$ given in the above theorems.

A dual result for the Löwner-Kufarev equation in partial derivatives is the Löwner-Kufarev ordinary differential equation (1.2). The solutions to (1.2) form a retracting subordination chain $w = g(z,t)$, i.e., $g(U,t) \subset U$, $g(U,t) \subset g(U,s)$ for $t > s$, and $g(z,0) \equiv z$.

Let a one-parameter family of maps $w = g(z,t)$ satisfy the following conditions.

(i) The maps $g(z,t)$ form a retracting subordination chain $g(U,0) \subset U$.

(ii) The map $g(z,t)$ is holomorphic in U, $g(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$, where $a_1(t) > 0$ and differentiable with respect to t.

(iii) The map $g(z,t)$ is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, $g(\infty,t) = \infty$.

(iv) The chain of maps $g(z,t)$ is not trivial.

(v) The Beltrami coefficient $\mu_g(z,t)$ of this map is differentiable with respect to t locally uniformly in U^*.

Note that in this case we need not a stong assumption (v) of the previous case. Set

$$H(z,t,\tau) = g(g(z,t),\tau) = \beta(\tau)w + b_2(\tau)w^2 + \cdots,$$

where $w = g(z,t)$. For each fixed t the mapping $g(z,t)$ forms a smooth semi-flow H^τ in $\mathcal{G}^{qc}$ which is tangent to the path $g(z,t+\tau)$ at $\tau = 0$. Therefore, we use the velocity vector $-wp(w,t)$ (that can be only measurable regarding to t) with $w = g(z,t)$ and obtain

$$\left.\frac{\partial H(z,t,\tau)}{\partial \tau}\right|_{\tau=0} = -g(z,t)p(g(z,t),t).$$

As before, the trajectory $g(z,t)$ generates a pencil of tangent smooth semi-flows with the tangent vectors $-wp(w,t)$, $w = g(z,t)$. Since $g(U,t) \in U$ for any $t > 0$, we can consider the limit

$$\lim_{\tau\to 0} \frac{H(z,t,\tau) - g(z,t)}{\tau g(z,t)}.$$

We have that

$$\left.\frac{\partial H(z,t,\tau)}{\partial \tau}\right|_{\tau=0} = \frac{\partial g(z,t)}{\partial t} = -g(z,t)p(g(z,t),t), \tag{4.3}$$

where $p(z,t) = p_0(t) + p_1(t)z + \cdots$ is an analytic function in U that has positive real part for almost all fixed t. The equation defined by (4.3) is an evolutionary equation for the path $g(z,t)$ and the initial condition is given by $g(z,0) = z$.

We suppose that all $g(z,t)$ admit real analytic quasiconformal extensions. The function $g(w,\tau) = H(z,t,\tau)/\beta(\tau)$ can be extended to a function from S^{qc} and it represents an equivalence class $[g^\tau] \in T$. There is a one-parameter path $y^\tau \in T$ that corresponds to a tangent velocity vector $\nu(w,t)$ such that

$$\mu_g(w,\tau) = \tau\nu(w,t) + o(\tau), \quad w = g(z,t).$$

We calculate explicitly the velocity vector making use of the Beltrami coefficient for a superposition:

$$\nu(w,t) = \lim_{\tau\to 0}\frac{\mu_{g(w,\tau)}\circ g(z,t)}{\tau} = \lim_{\tau\to 0}\frac{1}{\tau}\frac{\mu_{H(z,t,\tau)}-\mu_{g(z,t)}}{1-\bar{\mu}_{g(z,t)}\mu_{H(z,t,\tau)}}\frac{g_z(z,t)}{\bar{g}_{\bar{z}}(z,t)},$$

or

$$\nu(w,t) = \frac{\partial\mu_{g(z,t)}/\partial t}{1-|\mu_{g(z,t)}|^2}\frac{g_z}{\bar{g}_{\bar{z}}}\circ g^{-1}(w,t), \quad z\in U^*. \tag{4.4}$$

It is natural to implement an intrinsic parametrization using the Teichmüller distance $\tau_T(0,[g^t]) = t$, and assume the conformal radius to be $b_1(t) = e^{-t}$ that implies $p_0 = 1$. Then we use the variational formula (3.5) to state the following theorem.

THEOREM 4.3. *Let $g(z,t)$ be a retracting non-trivial subordination chain of maps defined for $t\in[0,t_0)$ and $z\in U$. Each $g(z,t)$ is a homeomorphism of $\hat{\mathbb{C}}$ which is holomorphic in U, $g(z,t) = e^{-t}z + b_2(t)z^2+\cdots$, with a e^{2t}-quasiconformal extension to U^* given by a Beltrami coefficient $\mu(z,t)$ which is differentiable regarding to t a.e. in $[0,t_0)$. The initial condition is $g(z,0)\equiv z$. Then, there is a function $p(z,t)$ such that that $\operatorname{Re} p(z,t) > 0$ for $z\in U$, and*

$$p(w,t) = 1 + \frac{w}{\pi}\iint\limits_{g(U^*,t)}\frac{\nu(u,t)\,d\sigma_u}{u^2(u-w)}, \quad w\in g(U,t),$$

where $\nu(u,t)$ is given by the formula (4.4), $\|\nu\|_\infty < 1$, and $w = g(z,t)$ is a solution to the differential equation

$$\frac{dw}{dt} = -wp(w,t), \quad w\in g(U,t)$$

with the initial condition $g(z,0) = z$.

REMARK 4.3. Taking into account the superposition we have

$$p(g(z,t),t) = 1 + \frac{g(z,t)}{\pi}\iint\limits_{U^*}\frac{\dot{\mu}_g g_\zeta^2(\zeta,t)\,d\sigma_\zeta}{g^2(\zeta,t)(g(\zeta,t)-g(z,t))},$$

where $\zeta\in U^*$, $z\in U$.

REMARK 4.4. The function $wp(w,t)$ has a continuation into $g(U^*,t)$ given by

$$\frac{dw}{dt} = F(w,t),$$

where the function $F(w,t)$ is a solution to the equation

$$\frac{\partial F}{\partial\bar{w}} = \frac{g_z^2\dot{\mu}_g}{|g_z|^2-|g_{\bar{z}}|^2}\circ g^{-1}(w,t).$$

The solution exists by the Pompeiu integral and can be written as

$$F(w,t) = h(w,t) - \frac{1}{\pi} \iint\limits_{g(U^*,t)} \frac{g_z^2 \dot{\mu}_g}{|g_z|^2 - |g_{\bar{z}}|^2} \circ g^{-1}(u,t) \frac{d\sigma_u}{u-w}$$
$$= h(w,t) - \frac{1}{\pi} \iint\limits_{g(U^*,t)} \frac{\nu(u,t) d\sigma_u}{u-w},$$

where $w \in g(U^*,t)$, $h(w,t)$ is a holomorphic functions with respect to w, that can be written as

$$h(w,t) = \frac{1}{2\pi i} \int\limits_{\partial g(U^*,t)} \frac{up(u,t)}{u-w}\, du.$$

Reciprocally, given the function $F(u,t)$, $u \in g(U^*,t)$, we can write the function $p(w,t)$ as

$$p(w,t) = 1 + \frac{w}{\pi} \iint\limits_{g(U^*,t)} \frac{F_{\bar{u}}(u,t)\, d\sigma_u}{u^2(u-w)},$$

where $w \in g(U,t)$.

References

[1] I. A. Aleksandrov, *Parametric Continuations in the Theory of Univalent Functions*, Nauka, Moscow, 1976. (in Russian)

[2] J. Becker, *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23–43.

[3] J. Becker, *Löwnersche Differentialgleichung und Schlichtheitskriterien*, Math. Ann. **202** (1973), 321–335.

[4] J. Becker, *Conformal mappings with quasiconformal extensions*, Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979), Academic Press, London and New York, 1980, 37–77.

[5] J. Becker, Ch. Pommerenke, *On the Hausdorff dimension of quasicircles*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12**:2 (1987), 329–333.

[6] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. (1916), 940–955.

[7] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985) no. 1–2, 137–152.

[8] P. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.

[9] F. Gardiner, N. Lakić, *Quasiconformal Teichmüller Theory*, Mathematicsl Surveys and Monographs, vol. 76, Amer. Math. Soc., 2000.

[10] F. W. Gehring, E. Reich, *Area distortion under quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I, No. 388, 1966, 15 pp.

[11] V. V. Goryainov, *Fractional iterates of functions that are analytic in the unit disk with given fixed points*, Mat. Sb. **182**:9 (1991), 1281–1299; Engl. Transl. in Math. USSR-Sb. **74**:1 (1993), 29–46.

[12] V. V. Goryainov, *One-parameter semigroups of analytic functions*, Geometric function theory and applications of complex analysis to mechanics: studies in complex analysis and its applications to partial differential equations, 2 (Halle, 1988), Pitman Res. Notes Math. Ser., 257, Longman Sci. Tech., Harlow, 1991, 160–164.

[13] V. V. Goryainov, *One-parameter semigroups of analytic functions and a compositional analogue of infinite divisibility*, Proceedings of the Institute of Applied Mathematics and Mechanics, Vol. 5, Tr. Inst. Prikl. Mat. Mekh., 5, Nats. Akad. Nauk Ukrainy Inst. Prikl. Mat. Mekh., Donetsk, 2000, 44–57. (in Russian)
[14] V. Ya. Gutlyanskiĭ, *Parametric representation of univalent functions*, Dokl. Akad. Nauk SSSR **194** (1970), 750–753; Engl. Transl. in Soviet Math. Dokl. **11** (1970), 1273–1276.
[15] V. Ya. Gutlyanskiĭ, *On some classes of univalent functions*, Theory of functions and mappings, "Naukova Dumka", Kiev, 1979, 85–97. (in Russian)
[16] V. Ya. Gutljanskiĭ, *The method of variations for univalent analytic functions with a quasiconformal extension*, Sibirsk. Mat. Zh. **21**:2 (1980), 61–78; Engl. Transl. in Siberian Math. J. **21**:2 (1980), 190–204.
[17] Cheng Qi He, *A parametric representation of quasiconformal extensions*, Kexue Tongbao, **25**:9 (1980), 721–724.
[18] P. P. Kufarev, *On one-parameter families of analytic functions*, Rec. Math. [Mat. Sbornik] N.S. **13(55)** (1943), 87–118.
[19] J. Ławrynowicz, J. Krzyż, *Quasiconformal Mappings in the Plane: Parametric Methods*, Lecture Notes in Math. 978, Springer-Verlag, Berlin-New York, 1983.
[20] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Graduate Texts in Mathematics 109, Springer-Verlag, New York, 1987.
[21] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89** (1923), 103–121.
[22] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Wiley–Interscience, New York, 1988.
[23] Z. Nehari, *Schwarzian derivatives and schlicht functions*, Bull. Amer. Math. Soc. **55**:6 (1949), 545–551.
[24] Ch. Pommerenke, *Über die Subordination analytischer Funktionen*, J. Reine Angew. Math. **218** (1965), 159–173.
[25] Ch. Pommerenke, *Univalent Functions, with a Chapter on Quadratic Differentials by G. Jensen*, Vandenhoeck & Ruprecht, Göttingen, 1975.
[26] Shah Dao-Shing, *Parametric representation of quasiconformal mappings*, Science Record **3** (1959), 400–407.
[27] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Kluwer, Dordrecht, 2001.
[28] S. Wolpert, *Thurston's Riemannian metric for Teichmüller space*, J. Differential Geom. **23**:2 (1986), 143–174.

Departamento de Matemática
Universidad Técnica Federico Santa María Casilla
110-V, Valparaíso Chile

alexander.vasiliev@usm.cl