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ABOUT AN OLD PROBLEM OF KARAMATA

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ABSTRACT. We give a unified solution of problems posed by Karamata–Blanuša–Mitrinović more than 50 years ago on the determination of algebraic functions with prescribed asymptotic and monotonicity properties.

1. Introduction

One of the oldest unsolved problems published in the American Mathematical Monthly is Problem 5626 [6], proposed in 1968 by D. S. Mitrinović:

PROBLEM A. *Determine those algebraic functions* $A_n(x)$, $n = 2, 3, \dots$ *which have the following properties:*

$$1.\ \frac{\log x}{x-1} \leqslant A_n(x) \quad (x > 0); \qquad 2.\ A_n(x) \sim x^{-1/n} \quad (x \to 0^+);$$

$$3.\ xA_n(x) \sim x^{1/n} \quad (x \to +\infty); \qquad 4.\ A_n(x) - \frac{\log x}{x-1} \sim a_n(x-1)^{2n-2} \quad (x \to 1),$$

where a_n *is independent of* x.

PROBLEM B. *Find also algebraic functions* $A_n(x)$, $n = 2, 3, \dots$ *such that, for* $x > 0$,

$$\frac{\log x}{x-1} \leqslant A_m(x) \quad \text{and} \quad A_m(x) \geqslant A_n(x) \quad \text{for} \quad 2 \leqslant m \leqslant n.$$

Mitrinović has also put this problem in his well-known book "Analytic Inequalities" [7] with a comment that "so far no solution has been published to the above problems".

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In fact, this problem is much older and was originally posed in 1949 [3] by the distinguished Serbian mathematician Jovan Karamata (1902–1967) in connection with some approximation problems of Ramanujan [4]. In [3] he indicated that

$$A_2(x) = \frac{1}{\sqrt{x}}; \qquad A_3(x) = \frac{1 + x^{1/3}}{x + x^{1/3}}.$$

D. Blanuša in [2] gives the form of $A_4(x)$:

$$A_4(x) = \frac{7 + 16t + 7t^2}{7t - t^2 + 18t^3 - t^4 + 7t^5} \quad \text{for } t = \sqrt[4]{x}.$$

He also noticed that, for each $x > 0$,

$$A_2(x) \geqslant A_3(x) \geqslant A_4(x).$$

Therefore Problem 5626 stays unsolved for more than 50 years and there is a logical question: what is the reason for this?

Our investigation, which includes a lot of digging among old papers, shows the following: while it is quite obvious that Mitrinović just rewrote problems posed by Karamata and Blanuša into the form of 5626, his demand to determine (all of) those algebraic functions $A_n(x)$ satisfying the conditions of Problems A and B, leads to difficulties. They seem to be so big that it makes 5626 practically unsolvable.

To illustrate this point of view, we give just an example.

Let $A_k(x)$ be a solution of Problem A and let $B_m(x)$ be some algebraic function defined and positive for $x \geqslant 0$. Then

(i) $$A_n(x) := A_k(x) + b_n B_m(x), \qquad n = 2, 3, \ldots,$$

is a solution of Problem B for any positive decreasing sequence (b_n).

(ii) $$A_k(x) \cdot \frac{B_m(x) + x(x-1)^{m-1}}{B_m(x)},$$

is also a solution of Problem A for any $m \geqslant 2k - 1$ and $B_m(x)/x^m \to \infty$ $(x \to \infty)$.

The presence of an almost arbitrary algebraic function $B_m(x)$ in (i) and (ii), shows that it is hard to give a definite general form of the solutions of Problems A and B.

Nevertheless, some symmetry among solutions should also persist since $A_k(x)$ and $A_k^*(x) := x^{-1} A_k(x^{-1})$ are both solutions of cited problems.

Anyway, the situation becomes entirely different if we take an insight into the original problems posed by Karamata and Blanuša in [2] and [3]. Instead of asking for "all possible solutions", they looked for "the simplest algebraic function $A_n(x)$" satisfying the conditions of Problems A and B. Although the word "simplest" is a little bit ambigous, we can conclude from the given forms of $A_2(x)$, $A_3(x)$ and $A_4(x)$ what the authors meant by it. The algebraic function $A_n(x)$ is "simple" if it is a *rational function* of $\sqrt[n]{x}$ with integer coefficients and with the further symmetry property that $A_n(x) = A_n^*(x)$.

If there is a couple of solutions of this form, we can always call "better" the one which gives better approximation to $\frac{\log x}{x-1}$ in the neiborhood of the point $x = 1$ i.e., for which a_n is smaller.

In this way, we can consider solutions of 5626 "in the sense of Karamata".

Our task in this article is, beside the former discussion, to give an explicit form of a "simple solution" $A_n(x)$ for each $n \geqslant 2$. It happens that our solutions coincide with Karamata–Blanuša versions for $n = 2$ and are "better" for $n = 3$ and $n = 4$.

2. Results

PROPOSITION. *The algebraic functions $A_n(x)$, $x > 0$, $n = 2, 3, \dots$ which represent a common solution of Problems A and B in the sense of Karamata, are defined by the following expression*

$$A_n(x) := \frac{1}{x-1}\biggl((x^{1/2n}+x^{-1/2n})^2\Bigl(\frac{x^{1/n}-1}{x^{1/n}+1}\Bigr)^{2n-1}+2n\sum_{k=1}^{n-1}\frac{1}{2k-1}\Bigl(\frac{x^{1/n}-1}{x^{1/n}+1}\Bigr)^{2k-1}\biggr).$$

Proof. We shall prove first that $A_n(x)$ satisfy the conditions 1, 2, 3 and 4 of Problem A.

Since for fixed n,

$$\frac{x}{x-1} \sim 1, \quad \frac{x^{1/n}-1}{x^{1/n}+1} \sim 1 \quad (x \to +\infty); \quad \frac{x^{1/n}-1}{x^{1/n}+1} \sim -1 \quad (x \to +0),$$

it is evident that $A_n(x)$ satisfy the conditions 2 and 3.

To prove the validity of the assertion 1, put $x = e^t$, $t \in R$ and rewrite the inequality $\frac{\log x}{x-1} \leqslant A_n(x)$ in the form

$$\text{(A.1)} \qquad \frac{t}{2\sinh(t/2)} \leqslant \frac{n}{\sinh(t/2)}\biggl(\frac{2}{n}\frac{X_n^{2n-1}}{1-X_n^2} + \sum_{k=1}^{n-1}\frac{X_n^{2k-1}}{2k-1}\biggr),$$

where, for the sake of simplicity, we put $X_n = X_n(t) := \tanh(t/2n)$.

Now, for $0 \leqslant w \leqslant X_n < 1$; $n = 2, 3, \dots$, integrating the identity

$$\text{(A.2)} \qquad \frac{1}{1-w^2} = \frac{w^{2n-2}}{1-w^2} + \sum_{k=1}^{n-1} w^{2k-2}$$

we get

$$\text{(A.3)} \quad \frac{t}{2n} = \frac{1}{2}\log\frac{1+X_n}{1-X_n} = \int_0^{X_n}\frac{dw}{1-w^2} = \sum_{k=1}^{n-1}\frac{1}{2k-1}X_n^{2k-1} + \int_0^{X_n}\frac{w^{2n-2}}{1-w^2}dw$$

Since

$$\int_0^{X_n} \frac{w^{2n-2}}{1-w^2} dw \leqslant \frac{1}{1-X_n^2} \int_0^{X_n} w^{2n-2} dw \leqslant \frac{2X_n^{2n-1}}{n(1-X_n^2)},$$

putting this in $(A.3)$ and dividing by $\frac{\sinh(t/2)}{n}$, $t > 0$, we obtain $(A.1)$ for $t > 0$.

Since both functions on the left and right-hand side of $(A.1)$ are even, we see that this inequality is also valid for the negative values of t.

Therefore $(A.1)$ is proved.

In the sequel we need this simple lemma.

LEMMA 1.

$$\tanh^m y \sim y^m; \qquad \frac{1}{2} \log \frac{1+y}{1-y} - \sum_{k=1}^{n} \frac{y^{2k-1}}{2k-1} \sim \frac{y^{2n+1}}{2n+1} \quad (y \to 0).$$

Now, we shall prove the assertion 4. Applying Lemma 1 and A.3, we have

$$\begin{aligned} A_n(e^t) - \frac{t}{e^t-1} &= \frac{2n}{e^t-1}\Big(\frac{2X_n^{2n-1}}{n(1-X_n^2)} - \Big(\frac{1}{2}\log\frac{1+X_n}{1-X_n} - \sum_{k=1}^{n-1}\frac{1}{2k-1}X_n^{2k-1}\Big)\Big) \\ &\sim \frac{2n}{t}\Big(\frac{2}{n}X_n^{2n-1} - \frac{1}{2n-1}X_n^{2n-1}\Big) \sim \frac{1}{t}\Big(4 - \frac{2n}{2n-1}\Big)\Big(\frac{t}{2n}\Big)^{2n-1} \\ &\sim \frac{2(3n-2)}{(2n-1)(2n)^{2n-1}} t^{2n-2} \qquad (t \to 0). \end{aligned}$$

Hence we obtain a very precise approximation

$$A_n(x) - \frac{\log x}{x-1} \sim \frac{3n-2}{(2n-1)4^{n-1}n^{2n-1}}(x-1)^{2n-2} \qquad (x \to 1),$$

and the proof that $A_n(x)$ are solutions of Problem A is finished.

We shall also prove that the algebraic functions $A_n(x)$ are monotone decreasing in n i.e., that they satisfy the conditions of Problem B.

For this purpose, note that we can restrict ourselves to the case $t > 0$; hence (X_n) is monotone decreasing in n.

Therefore, for fixed $t > 0$, using A.3 and partial integration, we have

$$\begin{aligned} A_n(e^t) &= \frac{1}{e^t-1}\Big(4\frac{X_n^{2n-1}}{1-X_n^2} + 2n\sum_{k=1}^{n-1}\frac{1}{2k-1}X_n^{2k-1}\Big) \\ &= \frac{1}{e^t-1}\Big(t + 4\frac{X_n^{2n-1}}{1-X_n^2} - 2n\int_0^{X_n}\frac{w^{2n-2}}{1-w^2}dw\Big) \\ &= \frac{1}{e^t-1}\Big(t + \int_0^{X_n} 4w^{n/2} d\Big(\frac{w^{3n/2-1}}{1-w^2}\Big)\Big). \end{aligned}$$

For $0 \leqslant w \leqslant X_n < 1$ an easy computation shows that $W_n(w) := 4w^{n/2}\frac{d}{dw}\left(\frac{w^{3n/2-1}}{1-w^2}\right)$ is positive and decreasing in n.

Hence

$$A_n(e^t) = \frac{1}{e^t-1}\left(t + \int_0^{X_n} W_n(w)\,dw\right) \geqslant \frac{1}{e^t-1}\left(t + \int_0^{X_n} W_{n+1}(w)\,dw\right)$$
$$\geqslant \frac{1}{e^t-1}\left(t + \int_0^{X_{n+1}} W_{n+1}(w)\,dw\right) = A_{n+1}(e^t),$$

and the proof of Proposition is completed.

Since this year is the hundredth anniversary of Karamata's birthday, we should say a few words about this extraordinary man. He become famous around 1930 by his brilliant (and short) proof of Hardy–Littlewood's Tauberian Theorem for Power Series [5].

His proof is cited in many books on Real or Complex Analysis as an example of an ingenious thinking (see [8, pp. 234–236]). This spirit of beauty and clearness is present in each of Karamata's 130 papers.

Karamata is also the founder of well-known Theory of Regular Variation. This theory has large applications in Probability, Real and Complex Analysis, Number Theory, Theory of Distributions (see [1], [9]) etc.

In honor of his anniversary, the Serbian Academy of Sciences is preparing the Collected Papers of Jovan Karamata.

Remark 1. This article is originally written for "The American Mathematical Monthly" but a very shortened version is published in [11].

Remark 2. Another solution of Karamata's problem is in [10]. There we treat separately Problems A and B. Among other results, we construct two monotone sequences of algebraic functions which are well-approximating the target function from both sides. This paper completes the investigations started in [10].

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