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A STRUCTURAL FORMULA OF THE WEINSTEIN FUNCTIONS USED IN HIS PROOF OF THE MILIN, ROBERTSON AND BIEBERBACH CONJECTURES

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ABSTRACT. We find a structural formula of the Weinstein functions indicated in the title. This yields a reduced form of our general integral identity for the Milin functional from which the Milin, Robertson and Bieberbach conjectures follow immediately. In particular, the well-known Fitzgerald–Pommerenke special integral identity for this problem is obtained.

1. Introduction

Let S denote the class of all functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \tag{1}$$

analytic and univalent in the unit disk $|z| < 1$. Bieberbach [1] conjectured that the inequalities

$$|a_n| \le n, \quad n = 2, 3, \dots, \tag{2}$$

hold where the equality sign holds only for the Koebe function

$$f_0(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \in S \tag{3}$$

and its rotations $e^{-i\alpha} f_0(ze^{i\alpha}) \in S$ where α is real.

Let S^2 denote the subclass of odd functions

$$f_2(z) = \sqrt{f(z^2)} = \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}, \quad f(z) \in S, \quad b_1 = 1, \tag{4}$$

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of the class S. It follows from (1) and (4) that

$$a_n = \sum_{k=1}^{n} b_{2k-1} b_{2(n+1-k)-1}, \quad n = 1, 2, \ldots, \quad b_1 = 1. \tag{5}$$

It is clear from (5) and the Cauchy inequality that

$$|a_n| \le \sum_{k=1}^{n} |b_{2k-1}|^2, \quad n = 2, 3, \ldots, \quad b_1 = 1. \tag{6}$$

Because of (6), Robertson [2] conjectured that over the class S^2 the inequalities

$$\sum_{k=1}^{n} |b_{2k-1}|^2 \le n, \quad n = 2, 3, \ldots, \quad b_1 = 1, \tag{7}$$

hold where the equality sign holds only for such functions (4) which correspond to the Koebe function (3) and its rotations. Thus the Robertson conjecture (7) over the class S^2 implies the Bieberbach conjecture (2) over the class S. The logarithmic coefficients c_n of the function $f(z) \in S$ are generated by the Taylor expansion

$$\ln \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n z^n, \quad |z| < 1, \quad \ln 1 = 0. \tag{8}$$

For the coefficients in (4) and (8), Lebedev and Milin [3] found the inequalities

$$\sum_{k=1}^{n+1} |b_{2k-1}|^2 \le (n+1) \exp\left[\frac{1}{4(n+1)} \sum_{k=1}^{n} (k|c_k|^2 - 4/k)(n-k+1)\right] \tag{9}$$

for $n = 1, 2, \ldots$. Because of (9), Milin [4] conjectured that over the class S the inequalities

$$\sum_{k=1}^{n} (k|c_k|^2 - 4/k)(n-k+1) \le 0, \quad n = 1, 2, \ldots, \tag{10}$$

hold where the equality sign holds only for the Koebe function (3) and its rotations. Thus it follows from (9) and (6) that the Milin conjecture (10) implies the Robertson conjecture (7) and the Bieberbach conjecture (2).

Louis de Branges [5]–[6] proved the Milin conjecture (10) for all $n = 1, 2, \ldots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all $n = 2, 3, \ldots$. De Branges uses the Loewner differential equation [7]

$$\frac{\partial f(z,t)}{\partial t} = z\varphi(z,t) \frac{\partial f(z,t)}{\partial z} \tag{11}$$

for the family of analytic and univalent functions

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad |z| < 1, \quad 0 \le t < +\infty, \tag{12}$$

where $f(z,0) = f(z)$, and the function $\varphi(z,t)$ is analytic in $|z| < 1$ and measurable in $0 \le t < +\infty$ satisfying

$$\operatorname{Re}\varphi(z,t) > 0, \quad |z| < 1, \quad \varphi(0,t) = 1, \quad 0 \le t < +\infty. \tag{13}$$

If the logarithmic coefficients $c_n(t)$ for the functions (12) are generated by the Taylor expansion

$$\ln \frac{f(z,t)}{e^t z} = \sum_{n=1}^{\infty} c_n(t) z^n, \quad |z| < 1, \quad 0 \le t < +\infty, \quad \ln 1 = 0, \tag{14}$$

then with the help of the Loewner equation (11), de Branges proved the general inequality

$$\sum_{k=1}^{n} (k|c_k(t)|^2 - 4/k)\sigma_{nk}(t) \le 0, \quad 0 \le t < +\infty, \tag{15}$$

for any positive integer $n \ge 1$, where $\sigma_{nk}(t)$ are the de Branges weight functions

$$\sigma_{nk}(t) = k \sum_{\nu=0}^{n-k} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-\nu}}{(k+\nu)\nu!(n-k-\nu)!} e^{-\nu t - kt}, \tag{16}$$

written by us [8, Theorem 1] in the full notations $\sigma_{nk}(t)$ with the two subscripts n and k, $k = 1, 2, \dots, n$, for $0 \le t < +\infty$, $n \ge 1$, where $(a)_\nu$ for an arbitrary number a denotes the Pochhammer symbol

$$(a)_\nu = a(a+1)\cdots(a+\nu-1), \quad \nu = 1, 2, \dots; \quad (a)_0 = 1. \tag{17}$$

The functions (16) are the unique solution of the de Branges system of differential equations

$$\begin{gathered} \sigma_{nk}(t) - \sigma_{n,k+1}(t) = -\sigma'_{nk}(t)/k - \sigma'_{n,k+1}(t)/(k+1), \\ 0 \le t < +\infty, \quad k = 1, 2, \dots, n, \quad n \ge 1, \quad \sigma_{n,n+1}(t) = 0, \end{gathered} \tag{18}$$

with initial conditions

$$\sigma_{nk}(0) = n - k + 1, \quad k = 1, 2, \dots, n, \quad n \ge 1. \tag{19}$$

The system (18) is written by us in the corresponding full notations. De Branges proved that the sign of the derivative with respect to t of the left-hand side of (15) is determined by the signs of the derivatives of the functions (16) which have the representation

$$-\frac{\sigma'_{nk}(t)}{k} = e^{-kt} \binom{n+k+1}{n-k} {}_3F_2\left(\begin{matrix} -n+k,\ n+k+2,\ k+1/2;\ e^{-t} \\ k+3/2,\ 2k+1 \end{matrix} \right) \tag{20}$$

for $0 \le t < +\infty$ and $k = 1, 2, \dots, n$, $n \ge 1$, where

(21)

$${}_3F_2\left(\begin{matrix} -n+k,\ n+k+2,\ k+1/2;\ e^{-t} \\ k+3/2,\ 2k+1 \end{matrix} \right) = \sum_{\nu=0}^{n-k} \frac{(-n+k)_\nu (n+k+2)_\nu (k+1/2)_\nu}{(k+3/2)_\nu (2k+1)_\nu \nu!} e^{-\nu t},$$

having in mind (17), are special cases of the Askey and Gasper polynomials [9, p. 717, Formula (3.1)], which are nonnegative for the examined t, i.e.,

$$\sigma'_{nk}(t) \le 0, \quad 0 \le t < +\infty, \quad k = 1, 2, \dots, n, \quad n \ge 1, \tag{22}$$

where the equality holds only for $t = 0$ if $n - k$ is odd. Therefore, the left-hand side of (15) is an increasing function with respect to t, and hence, the inequality (15) holds since $\sigma_{nk}(+\infty) = 0$ for $k = 1, 2, \dots, n$, $n \ge 1$, according to (16). In particular, for $t = 0$, it follows from (15) and (19) that the inequalities (10) hold where

$$c_k = c_k(0), \quad k = 1, 2, \dots, \tag{23}$$

are the coefficients in (8) for the function $f(z) = f(z, 0)$. Fitzgerald and Pommerenke [10]–[11] simplified the proof of de Branges in the special case when the function $\varphi(z, t)$ in (11) has the simplest form

$$\varphi(z, t) = \frac{1 + \kappa(t) z}{1 - \kappa(t) z}, \quad |z| < 1, \quad 0 \le t < +\infty, \tag{24}$$

where $\kappa(t)$ is a continuous function on $0 \le t < +\infty$ with $|\kappa(t)| = 1$.

Lenard Weinstein [12] proved the Milin conjecture (10) for all $n = 1, 2, \dots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all $n = 2, 3, \dots$, in another way. Weinstein found the identity

$$\sum_{n=1}^{\infty} z^{n+1} \sum_{k=1}^{n} (4/k - k|c_k|^2)(n - k + 1) = \int_0^{\infty} \frac{e^t w}{1 - w^2} \left(\sum_{k=1}^{\infty} A_k(t) w^k \right) dt, \tag{25}$$

where c_k are the coefficients in (8) (or in (23)), $A_k(t)$ are the functions

$$A_k(t) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \varphi(re^{i\theta}, t) \left| 2 \left(1 + \sum_{\nu=1}^{k} \nu c_\nu(t) r^\nu e^{i\nu\theta} \right) - k c_k(t) r^k e^{ik\theta} \right|^2 d\theta \ge 0 \tag{26}$$

for $0 \le t < +\infty$, $k = 1, 2, \dots$, $0 < r < 1$, where φ is the general function in (11) and (13) and $c_\nu(t)$ are the coefficients in (14), and $w = w(z, t)$ is the Pick function [13] determined by the equation

$$\frac{z}{(1 - z)^2} = \frac{e^t w}{(1 - w)^2}, \quad |z| < 1, \quad 0 \le t < +\infty. \tag{27}$$

With the help of the theory of Legendre polynomials, Weinstein showed that if w is determined by (27), in the Taylor expansion

$$\frac{e^t w^{k+1}}{1 - w^2} = \sum_{n=0}^{\infty} \Lambda_k^n(t) z^{n+1}, \quad k = 1, 2, \dots, \tag{28}$$

for $|z| < 1$ and $0 \le t < +\infty$, all coefficients $\Lambda_k^n \ge 0$, which, according to (25)–(26), proves the Milin conjecture (10) for all $n = 1, 2, \dots$, and hence, the Robertson conjecture (7) and the Bieberbach conjecture (2) for all $n = 2, 3, \dots$.

But Weinstein did not find a simple explicit form of the coefficients $\Lambda_k^n(t)$ in (28). In [8, Formula (41)] we found the surprising formula

$$\Lambda_k^n(t) = -\frac{\sigma'_{nk}(t)}{k}, \quad 0 \le t < +\infty, \quad n \ge k, \quad k = 1, 2, \dots, \tag{29}$$

($\Lambda_k^n(t) = 0$ for $0 \le n \le k-1$), which throws a bridge over the de Branges proof and the Weinstein proof of the Milin, Robertson and Bieberbach conjectures, i.e., in their final stages these two proofs are one and the same. There is another proof of this fact in our paper [14]. From (25) with the help of (28)–(29) we obtained the following integral identity for the Milin functional (10) [8, Theorem 2] namely

$$\sum_{k=1}^{n}(k|c_k|^2 - 4/k)(n-k+1) = \int_0^{+\infty} dt \sum_{k=1}^{n} A_k(t)\frac{\sigma'_{nk}(t)}{k} \tag{30}$$

for $n = 1, 2, \dots$, where c_k are the logarithmic coefficients in (8) (or (23)) of the normalized analytic and univalent functions (1) in $|z| < 1$ of the class S, $A_k(t)$ are the nonnegative Weinstein functions (26) and $\sigma'_{nk}(t)$ are the nonpositive derivatives of the de Branges weight functions $\sigma_{nk}(t)$ in our full notations determined by (20)–(21) and (16), respectively. From our identity (30) the Milin, Robertson and Bieberbach conjectures become evidently true due to either the Askey and Gasper polynomial results or the Legendre polynomial results by means of (22) or (29), respectively. In [14] we showed it with the help of the Chebyshev polynomials of the second kind.

Further in this paper, we will calculate the limit (26).

2. A structural formula for $A_k(t)$

It is necessary for our aim to give the Herglotz representation formula for the function $\varphi(z,t)$ in (11) and (13) (compare with Pommerenke [15, p. 40, Theorem 2.4, ii]):

THEOREM 1. *The function $\varphi(z,t)$ in* (11) *and* (13) *has the following Herglotz representation*

$$\varphi(z,t) = \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta,t), \quad |z| < 1, \quad 0 \le t < +\infty, \tag{31}$$

where $\mu(\theta,t)$ is a probability measure on $\le \theta \le 2\pi$ for each fixed t in $0 \le t < +\infty$.

PROOF. By the Schwarz formula we have

$$\varphi(z,t) = \int_0^{2\pi} \frac{re^{i\theta} + z}{re^{i\theta} - z} d\mu(r,\theta,t), \quad 0 \le |z| < r < 1, \quad 0 \le t < +\infty, \tag{32}$$

where the function

$$\mu(r,\theta,t) = \frac{1}{2\pi}\int_0^{\theta} \operatorname{Re}\varphi(re^{i\tau},t)\, d\tau, \quad 0 \le \theta \le 2\pi, \quad 0 \le t < +\infty, \tag{33}$$

is increasing in $0 \le \theta \le 2\pi$, for each fixed r in $0 < r < 1$ and t in $0 \le t < +\infty$, and satisfies

$$\mu(r, 0, t) = 0, \quad 0 < r < 1, \quad 0 \le t < +\infty, \tag{34}$$

$$\mu(r, 2\pi, t) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} \varphi(re^{i\tau}, t)\, d\tau = \varphi(0, t) = 1 \tag{35}$$

for $0 < r < 1$, $0 \le t < +\infty$. Hence (33)–(35) determine the family of functions $\{\mu(r, \theta, t)\}$ in the interval $0 \le \theta \le 2\pi$ and in the set $0 < r < 1$ for each fixed t in $0 \le t < +\infty$ which are bounded by 1 and their total variations are equal to 1. Then by the Helly selection theorem we can find a sequence $\{0 < r_n < 1\}$, $n = 1, 2, \dots$, with $r_n \to 1$ as $n \to \infty$ such that $\mu(r_n, \theta, t)$ converges to an increasing function $\mu(\theta, t)$ in $0 \le \theta \le 2\pi$ for each fixed t in $0 \le t < +\infty$ for all of the continuity points of $\mu(\theta, t)$ and with total variation equal to 1. Further, since the left-hand side of (32) does not depend on r, the limit process $r = r_n \to 1$ as $n \to \infty$ under the integral sign on the right-hand side of (32) leads us to the formula (31). □

Further, it is necessary for our aim to do the following generalization of the second theorem of Helly for a passage to the limit under the integral sign of an one-dimensional Stieltjes integral (see, for example, Shohat and Tamarkin [16, p. XIII] and Natanson [17, Chapter VIII, p. 219, Theorem 3]).

THEOREM 2. *Given a sequence $\{f_n(x)\}$, $n = 1, 2, \dots$, of continuous functions in the interval $[a, b]$, which converges uniformly to the (continuous) function $f(x)$ in $[a, b]$. Given a sequence $\{g_n(x)\}$, $n = 1, 2, \dots$, of functions with uniformly bounded total variations in $[a, b]$, which converges to the finite function $g(x)$ (with bounded total variation) in $[a, b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dg_n(x) = \int_a^b f(x)\, dg(x). \tag{36}$$

PROOF. We have the identity

(37)

$$\int_a^b f_n(x)\, dg_n(x) - \int_a^b f(x)\, dg(x) = \int_a^b [f_n(x) - f(x)]\, dg_n(x) + \int_a^b f(x)\, d[g_n(x) - g(x)].$$

It follows from the second condition that

$$\overset{b}{\underset{a}{V}} g_n(x) \le K < +\infty, \quad n = 1, 2, \dots, \tag{38}$$

where $V_a^b g_n(x)$ denotes the total variation of the function $g_n(x)$ in $[a, b]$ and K is some positive constant. Having in mind (38), we obtain

$$\overset{b}{\underset{a}{V}} [g_n(x) - g(x)] \le \overset{b}{\underset{a}{V}} g_n(x) + \overset{b}{\underset{a}{V}} g(x) \le K + K = 2K, \quad n = 1, 2, \dots, \tag{39}$$

where the symbol V_a^b denotes the total variations of the corresponding functions in $[a, b]$. It is clear from (39) that the differences $\{g_n(x) - g(x)\}$, $n = 1, 2, \dots$, have uniformly bounded total variations in $[a, b]$. Therefore by the second Helly theorem we have

$$\lim_{n\to\infty}\int_a^b f(x)\,d[g_n(x)-g(x)] = \int_a^b f(x)\,d\lim_{n\to\infty}[g_n(x)-g(x)] = 0. \tag{40}$$

Further, it follows from the first condition that for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$, such that

$$|f_n(x)-f(x)| \le \varepsilon/k, \quad n > N(\varepsilon), \quad \forall x \in [a,b]. \tag{41}$$

From (41) we obtain

$$\left|\int_a^b [f_n(x)-f(x)]\,dg_n(x)\right| < \frac{\varepsilon}{K}\mathop{V}_a^b g_n(x) \le \varepsilon, \quad n < N(\varepsilon), \tag{42}$$

having in mind (38).

Now from (40) and (42) we conclude that the right-hand side of (37) tends to zero as $n \to \infty$. Thus the equation (36) follows. □

THEOREM 3. *The Weinstein functions* (26) *have the following representation formula*

$$A_k(t) = \int_0^{2\pi}\left|2\left(1+\sum_{\nu=1}^k \nu c_\nu(t)e^{i\nu\theta}\right) - kc_k(t)e^{ik\theta}\right|^2 d\mu(\theta,t) \tag{43}$$

for $0 \le t < +\infty$, $k = 1, 2, \ldots$, *where* $c_\nu(t)$ *are the logarithmic coefficients in* (14) *and* $\mu(\theta, t)$ *is a probability measure on* $0 \le \theta \le 2\pi$ *for each fixed* t *in* $0 \le t < +\infty$.

PROOF. Let us set

$$I_k(r,t) = \int_0^{2\pi}\left|2\left(1+\sum_{\nu=1}^k \nu c_\nu(t)r^\nu e^{i\nu\theta}\right) - kc_k(t)r^k e^{ik\theta}\right|^2 d\mu(r,\theta,t) \tag{44}$$

for $0 \le t < +\infty$, $k = 1, 2, \ldots$, $0 < r < 1$. The limit $A_k(t)$ in (26) exists at any $r \to 1$ with $0 < r < 1$ (Weinstein [12] and Hayman [18, pp. 238–242]), i.e., $I_k(r,t) \to A_k(t)$ as $r \to 1-0$.

(i) From (44) and the arguments for (33)–(35) at the selected $r = r_n \to 1$ with $0 < r_n < 1$ as $n \to \infty$ we find that $\mu(r_n, \theta, t) \to \mu(\theta, t)$ where $\mu(\theta, t)$ is that in Theorem 1.

(ii) Let us set

$$M_k(r,\theta,t) = \left|2+\sum_{\nu=1}^k \alpha_{k\nu}\nu c_\nu(t)r^\nu e^{i\nu\theta}\right|^2 \tag{45}$$

for $0 < r < 1$, $0 \le \theta \le 2\pi$, $0 \le t < +\infty$, $k = 1, 2, \ldots$, where

$$\alpha_{k\nu} = 2, \quad 1 \le \nu \le k-1, \quad k \ge 2; \quad \alpha_{kk} = 1, \quad k \ge 1. \tag{46}$$

It follows from (45)–(46) that

$$M_k(r,\theta,t) = 4 + 2\sum_{\nu=1}^{k} \alpha_{k\nu}\nu c_\nu(t) r^\nu e^{i\nu\theta} + 2\sum_{\nu=1}^{k} \alpha_{k\nu}\nu \bar{c}_\nu(t) r^\nu e^{-i\nu\theta} + \sum_{\nu=1}^{k}\sum_{\lambda=1}^{k} \alpha_{k\nu}\alpha_{k\lambda}\nu\lambda c_\nu(t)\bar{c}_\lambda(t) r^{\nu+\lambda} e^{i(\nu-\lambda)\theta}. \tag{47}$$

From (47) we obtain

$$M_k(1,\theta,t) - M_k(r,\theta,t) = 2\sum_{\nu=1}^{k} \alpha_{k\nu}\nu c_\nu(t)(1-r^\nu) e^{i\nu\theta} + 2\sum_{\nu=1}^{k} \alpha_{k\nu}\nu \bar{c}_\nu(t)(1-r^\nu) e^{-i\nu\theta} + \sum_{\nu=1}^{k}\sum_{\lambda=1}^{k} \alpha_{k\nu}\alpha_{k\lambda}\nu\lambda c_\nu(t)\bar{c}_\lambda(t)(1 - r^{\nu+\lambda}) e^{i(\nu-\lambda)\theta}. \tag{48}$$

An upper bound of the modulus of the difference (48) is

$$|M_k(1,\theta,t) - M_k(r,\theta,t)| \leq \left[4\sum_{\nu=1}^{k} \alpha_{k\nu}\nu^2 |c_\nu(t)| + \sum_{\nu=1}^{k}\sum_{\lambda=1}^{k} \alpha_{k\nu}\alpha_{k\lambda}\nu\lambda(\nu+\lambda)|c_\nu(t)||c_\lambda(t)|\right](1-r). \tag{49}$$

The estimate (49) shows that $M_k(r,\theta,t)$ converges uniformly with respect to θ in $0 \leq \theta \leq 2\pi$ for each fixed t in $0 \leq t < +\infty$ to the limit $M_k(1,\theta,t)$ as $r \to 1$ with $0 < r < 1$. Hence, in particular, the sequence $\{M_k(r_n,\theta,t)\}$, $n = 1,2,\dots$, formed by (45) for the selected $r = r_n \to 1$ with $0 < r_n < 1$ as $n \to \infty$ in section (i), converges uniformly with respect to θ in $0 \leq \theta \leq 2\pi$ for each fixed t in $0 \leq t < +\infty$ to the limit $M_k(1,\theta,t)$ as well.

(iii) According to Theorem 2 and the results in the sections (i) and (ii) the sequence $\{I_k(r_n,t)\}$, $n = 1,2,\dots$, formed by (44) for the selected $r = r_n \to 1$ with $0 < r_n < 1$ as $n \to \infty$, converges to the limit $A_k(t)$ represented by the formula (43).

This completes the proof of Theorem 3. □

It is clear from the proof of Theorem 3 that the limit of the integral in (32) for the selected $r = r_n \to 1 - 0$ as $n \to \infty$ can be directly obtained by Theorem 2.

THEOREM 4. *For the Milin functional* (10), *we have the following integral identity*

$$\sum_{k=1}^{n}(k|c_k|^2 - 4/k)(n-k+1) = \int_0^{+\infty} dt \sum_{k=1}^{n} \frac{\sigma'_{nk}(t)}{k} \int_0^{2\pi} \left| 2\left(1 + \sum_{\nu=1}^{k} \nu c_\nu(t) e^{i\nu\theta}\right) - k c_k(t) e^{ik\theta}\right|^2 d\mu(\theta,t) \tag{50}$$

for $n = 1,2,\dots$, *where* c_k *and* $c_\nu(t)$ *are the logarithmic coefficients in* (8) (*or* (23)) *and in* (14), *respectively,* $\sigma'_{nk}(t)$ *are the nonpositive derivatives* (20)–(22) *of the* *de Branges weight functions $\sigma_{nk}(t)$ in* (16) *and $\mu(\theta,t)$ is a probability measure on $0 \le \theta \le 2\pi$ for each fixed t in $0 \le t < +\infty$.*

REMARK. Evidently the Milin functional on the left-hand side of (50) is non-positive since $d\mu(\theta,t) \ge 0$, having in mind (22).

PROOF. The identity (50) follows from (30) and (43). □

COROLLARY. *If the measure $\mu(\theta,t)$ in* (31), (43) *and* (50) *is a step-function, then we have the corresponding representations*

$$\varphi(z,t) = \sum_{p=1}^{q} \mu_p(t) \frac{1 + ze^{-i\theta_p(t)}}{1 - ze^{-i\theta_p(t)}}, \quad |z| < 1, \quad 0 \le t < +\infty, \tag{51}$$

$$A_k(t) = \sum_{p=1}^{q} \mu_p(t) \left| 2\left(1 + \sum_{\nu=1}^{k} \nu c_\nu(t) e^{i\nu\theta_p(t)}\right) - kc_k(t) e^{ik\theta_p(t)} \right|^2 \tag{52}$$

for $0 \le t < +\infty$, $k = 1, 2, \ldots$, and

$$\begin{aligned} &\sum_{k=1}^{n} (k|c_k|^2 - 4/k)(n-k+1) \\ &= \int_0^{+\infty} dt \sum_{k=1}^{n} \frac{\sigma'_{nk}(t)}{k} \sum_{p=1}^{q} \mu_p(t) \left| 2\left(1 + \sum_{\nu=1}^{k} \nu c_\nu(t) e^{i\nu\theta_p(t)}\right) - kc_k(t) e^{ik\theta_p(t)} \right|^2 \end{aligned} \tag{53}$$

for $n = 1, 2, \ldots$, where $\mu_p(t)$ and $\theta_p(t)$ are real continuous functions in $0 \le t < +\infty$ with

$$0 \le \theta_1(t) \le \theta_2(t) \le \cdots \le \theta_q(t) \le 2\pi, \quad 0 \le \mu_p(t) \le 1, \quad \sum_{p=1}^{q} \mu_p(t) = 1, \quad q \ge 1. \tag{54}$$

In particular, for $q = 1$, $\theta(t) := \theta_1(t)$, $0 \le \theta(t) \le 2\pi$, the representations (51)–(54) *are reduced to*

$$\varphi(z,t) = \frac{1 + ze^{-i\theta(t)}}{1 - ze^{-i\theta(t)}}, \quad |z| < 1, \quad 0 \le t < +\infty, \tag{51'}$$

$$A_k(t) = \left| 2\left(1 + \sum_{\nu=1}^{k} \nu c_\nu(t) e^{i\nu\theta(t)}\right) - kc_k(t) e^{ik\theta(t)} \right|^2 \tag{52'}$$

for $0 \le t < +\infty$, $k = 1, 2, \ldots$, and

$$\begin{aligned} &\sum_{k=1}^{n} (k|c_k|^2 - 4/k)(n-k+1) \\ &= \int_0^{+\infty} dt \sum_{k=1}^{n} \frac{\sigma'_{nk}(t)}{k} \left| 2\left(1 + \sum_{\nu=1}^{k} \nu c_\nu(t) e^{i\nu\theta(t)}\right) - kc_k(t) e^{ik\theta(t)} \right|^2 \end{aligned} \tag{53'}$$

for $n = 1, 2, \ldots$, respectively.

The identity (53′) is found by Fitzgerald and Pommerenke [10, p. 686, Identity (3.13)] for the special case (51′) (or equivalently for (24) if we set $\kappa(t) := e^{-i\theta(t)}$).

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