

ON THE CONVERGENCE RATE ESTIMATES
FOR FINITE DIFFERENCE SCHEMES
APPROXIMATING HOMOGENEOUS INITIAL-BOUNDARY
VALUE PROBLEM FOR HYPERBOLIC EQUATION

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Abstract. Applying the interpolation theory of the function spaces, we obtain a new convergence rate estimate for the weak solution of hyperbolic initial-boundary value problem.

1. Introduction. In the case of elliptic boundary value problem, the convergence rate estimates for finite difference schemes of the form

$$\|u - v\|_{H_h^k} \leq Ch^{s-k} \|u\|_{H^s}, \quad s > k,$$

are said to be compatible with the smoothness of data [3]. Here u denotes the solution of the boundary value problem, v denotes the corresponding discrete approximation, h is the discretisation parameter, H^s denotes the standard Sobolev space and H_h^k is the discrete Sobolev space. The compatible estimates may also be derived in parabolic case [4]. But in the hyperbolic case, the usual estimates are not compatible with the smoothness of data [5]:

$$\|u - v\|_{C_\tau(H_h^k)} \leq Ch^{s-k-1} \|u\|_{H^s(Q)}, \quad s > k + 1,$$

These estimates are usually obtained using the Brumble-Hilbert lemma [2].

A few years ago, Zlotnik [12] applied the interpolation theory to obtain for the hyperbolic projection difference scheme a convergence rate estimate of the order $2(s - k)/3$. Using also the interpolation theory, B.S. Jovanović derived in [6] the convergence rate estimate of the same order for the finite difference schemes in the case of homogeneous hyperbolic equation with constant coefficients. Here we show

how the same estimate can be obtained in the case of homogeneous hyperbolic equation with variable coefficients.

2. Statement of the problem Let $L_q = L_q(0, 1)$ ($1 \leq q \leq \infty$) be Lebesgue spaces of integrable functions, $H^s = H^s(0, 1)$ standard Sobolev spaces, \mathcal{D} the space of infinitely differentiable functions with compact support in $(0, 1)$ and H_0^s is the closure of \mathcal{D} in H^s . (\cdot, \cdot) and $\|\cdot\|$ denote the inner product and the norm in L_2 , respectively. Suppose $a \in L_\infty$ such that

$$(1) \quad a \geq a_0 > 0 \quad \text{in } (0, 1) \text{ a.e.}$$

For the operator $L: H_0^1 \rightarrow H^{-1}$ defined by $Lv = (av)'$ there exist $0 < \lambda_1 < \lambda_2 < \dots$, $\lim_k \lambda_k = \infty$, such that $L\varphi_k = \lambda_k \varphi_k$ ($k \in N$); the sequence of eigenfunctions $(\varphi_k)_{k \in N} \subset H_0^1$ is an orthonormed topological basis of L_2 (see [8]). Introduce the spaces V^α ($\alpha \geq 0$) by $V^\alpha = \{v \in L_2 \mid \|v\|_{V^\alpha}^2 = \sum_{k=1}^{\infty} \lambda_k^\alpha \tilde{v}_k^2 < \infty\}$, where $\tilde{v}_k = (v, \varphi_k)$ are the Fourier coefficients of v in the basis $(\varphi_k)_{k \in N}$.

Consider the initial-boundary value problem for the homogeneous second-order hyperbolic equation (IBVP) in the domain $Q = (0, 1) \times (0, T]$:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x} \left(a(x) \frac{\partial u}{\partial x} \right), & (x, t) \in Q \\ u(0, t) &= u(1, t) = 0 & t \in [0, T] \\ u(x, 0) &= u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x), & x \in (0, 1) \end{aligned}$$

There is the unique weak solution of this problem for $u_0 \in V^1$, $u_1 \in V^0$ (see [10], [12]). It can be represented as the Fourier series

$$(2) \quad u(x, t) = \sum_{k=1}^{\infty} \tilde{u}_k \varphi_k(x),$$

where,

$$(3) \quad \tilde{u}_k(t) = \tilde{u}_k^{(0)} \cos(\sqrt{\lambda_k} t) + \frac{\tilde{u}_k^{(1)}}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t)$$

(here $\tilde{u}_k^{(0)}$, $\tilde{u}_k^{(1)}$ are the Fourier coefficients of the functions u_0 , u_1 , respectively). The relation (3) shows that the series (2) also has meaning for $t < 0$. In such a way, the solution of (IBVP) can be extended in t on $[-T, T]$; this extension we shall also denote by u . If $u_0 \in V^\alpha$, $u_1 \in V^{\alpha-1}$, it satisfies the relation

$$(4) \quad \max_{t \in [-T, T]} \|\partial^l u / \partial t^l\|_{V^{\alpha-1}} \leq C(\|u_0\|_{V^\alpha} + \|u_1\|_{V^{\alpha-1}}),$$

where $l \in Z$, $0 \leq l \leq \alpha$ (see the analogous relation in [9] and the proofs of Propositions 1.1 and 1.3 in [12]). Then, in the perfectly same way as we deduced Theorem 3 in [9], we obtain, applying (4) that for $a \in C^3$ satisfying (1), the following assertion holds:

If $u_0 \in V^\alpha$, $u_1 \in V^{\alpha-1}$ then

$$(5) \quad \max_{t \in [-T, T]} \|\partial^l u / \partial t^l\|_{H^{\alpha-1}} \leq C(\|u_0\|_{V^\alpha} + \|u_1\|_{V^{\alpha-1}}),$$

where $1 \leq \alpha \leq 4$, $l \in Z$, $0 \leq l \leq \alpha$.

3. Discretisation. Lower estimate. Let $\bar{\omega}_h$ be a uniform mesh on $[0, 1]$ with the stepsize $h = 1/n$, $\omega_h = \bar{\omega}_h \cap (0, 1)$ and $\omega_h^- = \omega_h \cup \{0\}$. We set $\overset{0}{H}(\omega)$ to be the space of all functions defined on $\bar{\omega}_h$ vanishing at 0 and 1. Introduce the finite differences in x :

$$v_x = (v(x+h) - v(x))/h, \quad v_{\bar{x}} = (v(x) - v(x-h))/h.$$

We define the following discrete norms

$$\|v\|_h = \left(h \sum_{x \in \omega_h} v^2(x) \right)^{1/2}, \quad \|v\|_h = \left(h \sum_{x \in \omega_h^-} v^2(x) \right)^{1/2}$$

$$\|v\|_{H_h^1} = (\|v\|^2 + \|v_x\|^2)^{1/2}.$$

The operator $L_h: \overset{0}{H}(\omega) \rightarrow \overset{0}{H}(\omega)$ defined by

$$L_h v = \begin{cases} -\frac{1}{2}[(av_x)_{\bar{x}} + (av_{\bar{x}})_x], & x \in \omega_h \\ 0, & x \in \{0, 1\} \end{cases}$$

is positive on $\overset{0}{H}(\omega)$ and satisfies the inequalities

$$(6) \quad c\|v_x\|_h \leq \|v\|_{(L_h)} \leq C\|v_x\|_h.$$

Let $\bar{\omega}_\tau$ be a uniform mesh on $[-\tau/2, T]$ with the stepsize $\tau = T/(m-1/2)$, $\omega_\tau = \bar{\omega}_\tau \cap (0, T)$, and $\omega_\tau^- = \omega_\tau \cup \{-\tau/2\}$ (see [6]). Let us introduce the following notations:

$$v = v(t), \quad \hat{v} = v(t+\tau), \quad \check{v} = v(t-\tau), \quad v^j = v((j-1/2)\tau),$$

$$\bar{v} = (v + \hat{v})/2, \quad v_t = (\hat{v} - v)/\tau, \quad v_{\bar{t}} = (v - \check{v})/\tau.$$

For functions defined on $\bar{\omega}_h \times \bar{\omega}_\tau$ we define the norms

$$\|v\|_{C_\tau(H_h^1)} = \max_{t \in \omega_\tau^-} \|v(\cdot, t)\|_{H_h^1} \quad \text{and} \quad \|v\|_{L_{q,\tau}(L_{2,h})} = \left(\tau \sum_{t \in \omega_\tau} \|v(\cdot, t)\|_h^q \right)^{1/q}.$$

One can easily deduce

LEMMA 1. For $v \in \overset{0}{H}(\omega)$ the inequality $\|v\|_{(I+0.25\tau^2(\sigma-1/4)L_h)} \leq C\|v\|_h$ holds if one of the following two conditions is satisfied:

- (i) If $\sigma > 1/4$, then $\tau/h < C$, where C is an arbitrary constant;
- (ii) If $\sigma < 1/4$, then $\tau/h \leq 4\sqrt{\frac{1-s_0}{(1-4\sigma)c_1^2}}$ for an $s_0 \in (0, 1)$, where c_1 is a constant depending only on the function a . \square

Let S_x and S_t denote the Steklov smoothing operators in x and t :

$$S_x f(x, t) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(s, t) ds, \quad S_t f(x, t) = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} f(x, \eta) d\eta.$$

For the approximation of (IBVP) we shall use a weighted finite difference scheme (FDS) (see [7]):

$$\begin{aligned} v_{t\bar{t}} &= -L_h(\sigma\hat{v} + (1-2\sigma)v + \sigma\check{v}), \\ v(0, t) &= v(1, t) = 0, \quad t \in \bar{\omega}_\tau \\ v^0 &= u_0 - \frac{\tau}{2}S_x^2 u_1, \quad v^1 = u_0 + \frac{\tau}{2}S_x^2 u_1 \end{aligned}$$

Let $z = u - v$ denote the error of the approximation.

Suppose $u_0 \in V^1$, $u_1 \in V^0$, $a \in C^3$ satisfying (1) and that one of conditions in Lemma 1 holds. Then applying the a priori estimate for FDS (see [7]) one obtains

$$(7) \quad N(v) \leq CN(v^0),$$

where $N^2(w) = \|w_t\|_{(I+0.25\tau^2(\sigma-1/4)L_h)}^2 + \|\bar{w}\|_{(L_h)}^2$. Using Lemma 1, we have

$$(8) \quad \|v_t^0\|_{(I+0.25\tau(\sigma-1/4)L_h)} \leq C\|v_t^0\|_h = C\|S_x^2 u_1\|_h \leq C\|u_1\|$$

(the last inequality in (8) follows from the Cauchy-Schwartz inequality). The inequality (6) yields

$$\|\bar{v}^0\|_{(L_h)} = \|u_0\|_{(L_h)} \leq C\|[(u_0)_x]\|_h \leq C\|u_0\|_{H^1}$$

(for the last inequality see [6]). From this, using equivalency of the norms in H^1 and V^1 (see [9]), we obtain

$$\|\bar{v}^0\|_{(L_h)} \leq C\|u_0\|_{V^1}.$$

The estimates (7), (8) and the last inequality yield

$$(9) \quad \max_{t \in \bar{\omega}_t} \|\bar{v}\|_{(L_h)} \leq C(\|u_0\|_{V^1} + \|u_1\|_{V^0}).$$

Further, thanks to (5) (for $l = 0$, $\alpha = 1$), we obtain

$$\|\bar{u}\|_{(L_h)} \leq C\|[(\bar{u})_x]_h\| \leq C \max_{t \in [-T, T]} \|\partial u / \partial x\| \leq C(\|u_0\|_{V^1} + \|u_1\|_{V^0}),$$

whence,

$$(10) \quad \max_{t \in \bar{\omega}_\tau} \|\bar{u}\|_{(L_h)} \leq C(\|u_0\|_{V^1} + \|u_1\|_{V^0}).$$

Finally, from (7), (9), (10) follows the lower estimate

$$(11) \quad \|\bar{z}\|_{C_\tau(H_h^1)} \leq C(\|u_0\|_{V^1} + \|u_1\|_{V^0}).$$

4. Upper estimate. In this section we suppose that $u_0 \in V^4$, $u_1 \in V^3$, $a \in C^3$ satisfying (1) and that one of the conditions in Lemma 1 holds. Then the inequality (5) implies $\partial^2 u / \partial t^2, \partial^2 / \partial x^2 \in H^2(Q)$. Thus, applying the embedding theorem $H^2(Q) \subset C(\bar{Q})$ (see [11]) we conclude that $\partial^2 u / \partial t^2, \partial^2 u / \partial x^2$ are continuous. The error z is the solution of the following finite difference scheme:

$$\begin{aligned} z_{\bar{t}\bar{t}} &= -L_h(\sigma \hat{z} + (1 - 2\sigma)z + \sigma \check{z}) + \psi, \\ z(0, t) &= z(1, t) = 0, \quad t \in \bar{\omega}_\tau \\ z^0 &= u\left(x, -\frac{\tau}{2}\right) - u_0(x) + \frac{\tau}{2} S_x^2 u_1, \quad z^1 = u\left(x, \frac{\tau}{2}\right) - u_0(x) - \frac{\tau}{2} S_x^2 u_1, \end{aligned}$$

where $\psi = u_{\bar{t}\bar{t}} + L_h(\sigma \hat{u} + (1 - 2\sigma)u + \sigma \check{u})$. The application of the a priori estimate to z yields

$$(12) \quad N(z) \leq C(N(z^0) + \frac{1}{\sqrt{c}} \|\psi\|_{L_{1,\tau}(L_{2,h})}),$$

where $c = 1$ if (i) in Lemma 1 is satisfied or $c = s_0$ if (ii) is satisfied. Let us first estimate $N(z^0)$. Decompose the first term in $N(z^0)$:

$$\begin{aligned} z_t^0 &= \frac{u\left(x, \frac{\tau}{2}\right) - u\left(x, -\frac{\tau}{2}\right)}{\tau} - S_x^2 u_1 \\ &= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \frac{\partial u}{\partial t}(x, \eta) d\eta - u_1(x) + u_1(x) - S_x^2 u_1 = g_1(x) + g_2(x). \end{aligned}$$

From

$$\begin{aligned} g_1(x) &= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \frac{\partial u}{\partial t}(x, \eta) d\eta - \frac{\partial u}{\partial t}(x, 0) = -\frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \int_0^\eta (\zeta - \eta) \frac{\partial^3 u}{\partial t^3}(x, \zeta) d\zeta d\eta, \\ (g_1)_x &= -\frac{1}{h\tau} \int_{-\tau/2}^{\tau/2} \int_0^\eta \int_x^{x+h} (\zeta - \eta) \frac{\partial^4 u}{\partial t^3 \partial x}(\xi, \zeta) d\xi d\zeta d\eta, \end{aligned}$$

we have

$$\begin{aligned}
& \|g_1\|_h \leq C \|[(g_1)_x]\|_h \\
(13) \quad & \leq \left[h \sum_{x \in \omega_h} h^{-2} \tau^{-2} \tau^2 \left(\int_x^{x+h} \int_{-\tau/2}^{\tau/2} \int_0^t \left| \frac{\partial^4 u}{\partial t^3 \partial x}(\xi, \zeta) \right| d\zeta dt d\xi \right)^2 \right]^{1/2} \\
& \leq C \tau^2 \max_{t \in [-T, T]} \left\| \frac{\partial^4 u}{\partial t^3 \partial x} \right\|.
\end{aligned}$$

Further, the equality

$$g_2(x) = u_1(x) - S_x^2 u_1 = \frac{1}{2h} \int_{x-h}^{x+h} \int_x^s (\xi - s) u_1''(\xi) d\xi ds,$$

and the equivalence of the norms in H^2 and V^2 imply

$$(14) \quad \|g_2\|_h \leq Ch^2 \|u_1''\| \leq Ch^2 \|u_1\|_{V^2}.$$

Then Lemma 1, together with (13), (5), (14) yields

$$(15) \quad \|z_t^0\|_{(I + \frac{\tau^2}{4}(\sigma - \frac{1}{4})L_h)} \leq C(h^2 + \tau^2)(\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

For

$$\bar{z}^0 = \frac{1}{2} \left(u \left(x, \frac{\tau}{2} \right) + u \left(x, -\frac{\tau}{2} \right) \right) - u_0(x) = \hat{u} \left(x, \frac{\tau}{2} \right) - \hat{u}(x, 0),$$

where $\hat{u}(x, t) = (u(x, t) + u(x, -t))/2$, the identity

$$\bar{z}^0 = \int_0^{\tau/2} \int_0^t \frac{\partial^2 \hat{u}}{\partial t^2}(x, \zeta) d\zeta dt,$$

holds. Hence,

$$\begin{aligned}
\|\bar{z}^0\|_{(L_h)} & \leq C \|[(\bar{z}^0)_x]\|_h \leq C \tau^2 \max_{t \in [0, T]} \left\| \frac{\partial^3 \hat{u}}{\partial t^2 \partial x} \right\| \\
& \leq C \tau^2 \max_{t \in [-T, T]} \left\| \frac{\partial^3 u}{\partial t^2 \partial x} \right\| \leq C \tau^2 (\|u_0\|_{V^4} + \|u_1\|_{V^3}).
\end{aligned}$$

From (15) and the last estimate one obtains

$$(16) \quad N(z^0) \leq C(h^2 + \tau^2)(\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

To estimate ψ , we shall rewrite it in the following manner:

$$\psi = u_{\bar{t}\bar{t}} + L_h u + \sigma \tau^2 L_h u_{\bar{t}\bar{t}} = \alpha + \beta + \gamma,$$

where

$$\alpha = u_{t\bar{t}} - \frac{\partial^2 u}{\partial t^2}, \quad \beta = -Lu + L_h u, \quad \gamma = \sigma\tau^2 L_h u_{t\bar{t}}.$$

Obviously, $\alpha = \alpha_1 + \alpha_2$, where

$$\alpha_1(x, t) = u_{t\bar{t}} - S_x^2 S_t^2 \frac{\partial^2 u}{\partial t^2}, \quad \alpha_2(x, t) = S_x^2 S_t^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t^2}.$$

But,

$$\begin{aligned} \alpha_1(x, t) &= S_t^2 \frac{\partial^2 u}{\partial t^2} - S_x^2 S_t^2 \frac{\partial^2 u}{\partial t^2} \\ &= \frac{1}{h\tau} \int_{x-h}^{x+h} \int_x^s \int_{t-\tau}^{t+\tau} (s-\xi) \left(1 - \frac{|s-x|}{h}\right) \left(1 - \frac{|\zeta-t|}{\tau}\right) \frac{\partial^4 u(\xi, \zeta)}{\partial t^2 \partial x^2} d\zeta d\xi ds \end{aligned}$$

wherefrom,

$$\|\alpha_1(\cdot, t)\|_h \leq Ch^2 \max_{t \in [-T, T]} \left\| \frac{\partial^4 u}{\partial t^2 \partial x^2} \right\|,$$

and consequently, referring to (5), we obtain

$$(17) \quad \|\alpha_1\|_{L_{1,\tau}(L_{2,h})} \leq Ch^2 (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

The term α_2 can be represented in the form

$$\begin{aligned} \alpha_2(x, t) &= -\frac{1}{h\tau} \int_{x-h}^{x+h} \int_x^s \int_{t-\tau}^{t+\tau} (\xi-s) \left(1 - \frac{|s-x|}{h}\right) \left(1 - \frac{|\zeta-t|}{\tau}\right) \frac{\partial^4 u(\xi, t)}{\partial t^2 \partial x^2} d\zeta d\xi ds \\ &\quad - \frac{1}{h\tau} \int_{x-h}^{x+h} \int_{t-\tau}^{t+\tau} \int_t^\eta (\zeta-\eta) \left(1 - \frac{|s-x|}{h}\right) \left(1 - \frac{|\eta-t|}{\tau}\right) \frac{\partial^4 u(s, \zeta)}{\partial t^4} d\zeta d\eta ds, \end{aligned}$$

whence,

$$\|\alpha_2\|_{L_{1,\tau}(L_{2,h})} \leq Ch^2 \max_{t \in [-T, T]} \left\| \frac{\partial^4 u}{\partial t^2 \partial x^2} \right\| + C\tau^2 \max_{t \in [-T, T]} \left\| \frac{\partial^4 u}{\partial t^4} \right\|,$$

and therefore

$$\|\alpha_2\|_{L_{1,\tau}(L_{2,h})} \leq C(h^2 + \tau^2) (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

From (17) and the last estimate we obtain

$$(18) \quad \|\alpha\|_{L_{1,\tau}(L_{2,h})} \leq C(h^2 + \tau^2) (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

Decompose β in the following way: $\beta = \beta_1 + \beta_2 + \beta_3 + \beta_4$, where,

$$\begin{aligned}\beta_1 &= a \left(\frac{\partial^2 u}{\partial x^2} - u_{x\bar{x}} \right), & \beta_2 &= a' \left(\frac{\partial u}{\partial x} - \frac{1}{2}(u_x + u_{\bar{x}}) \right), \\ \beta_3 &= \frac{1}{2}(a' - a_x)(u_x - u_{\bar{x}}), & \beta_4 &= \frac{1}{4} \left(a' - \frac{1}{2}(a_x + a_{\bar{x}}) \right) u_{\bar{x}}.\end{aligned}$$

Combining the estimate $\max_{x \in [0,1]} |a(x)| \leq C$ and the fact that

$$\frac{\partial^2 u}{\partial x^2} - u_{x\bar{x}} = \frac{1}{h} \int_{x-h}^{x+h} \int_x^s (\xi - s) \frac{\partial^4 u(\xi, t)}{\partial x^4} d\xi ds,$$

one obtains

$$(19) \quad \|\beta_1\|_{L_1, \tau(L_2, h)} \leq Ch^2 \max_{t \in [-T, T]} \|\partial^4 u / \partial x^4\|.$$

Using the relation

$$\frac{\partial u}{\partial x} - \frac{1}{2}(u_x + u_{\bar{x}}) = \frac{1}{2h} \int_{x-h}^{x+h} \int_x^s (\xi - s) \frac{\partial^3 u(\xi, t)}{\partial x^3} d\xi ds,$$

and the estimate $\max_{x \in [0,1]} |a'(x)| \leq C$ we have

$$(20) \quad \|\beta_2\|_{L_1, \tau(L_2, h)} \leq Ch^2 \max_{t \in [-T, T]} \|\partial^3 u / \partial x^3\|.$$

Applying Taylor's formula, one has

$$|a'(x) - a_x| \leq h \max_{x \in [0,1]} |a''(x)| \leq Ch.$$

This estimate and the obvious relation $u_x - u_{\bar{x}} = hu_{x\bar{x}} = hS_x^2(\partial^2 u / \partial x^2)$, imply

$$(21) \quad \|\beta_3\|_{L_1, \tau(L_2, h)} \leq Ch^2 \max_{t \in [-T, T]} \|\partial^2 u / \partial x^2\|.$$

From Taylor's formula it follows that

$$|a'(x) - \frac{1}{2}(a_x + a_{\bar{x}})| \leq Ch^2 \max_{x \in [0,1]} |a'''(x)| \leq Ch^2.$$

From this, taking into account that

$$u_{\bar{x}} = \frac{1}{h} \int_{x-h}^x \frac{\partial u(s, t)}{\partial x} ds,$$

we obtain

$$(22) \quad \|\beta_4\|_{L_{1,\tau}(L_{2,h})} \leq Ch^2 \max_{t \in [-T, T]} \|\partial u / \partial x\|.$$

Then (19)–(22), thanks to (5), yield

$$(23) \quad \|\beta\|_{L_{1,\tau}(L_{2,h})} \leq Ch^2 (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

Representing the term γ in the form

$$\gamma = -\frac{\sigma\tau^2}{2} (a_x u_{xt\bar{t}} + a_{\bar{x}} u_{\bar{x}t\bar{t}} + 2a u_{x\bar{x}t\bar{t}}),$$

we easily obtain, using preceding techniques, that

$$\|\gamma\|_{L_{1,\tau}(L_{2,h})} \leq C\tau^2 (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

The last estimate together with (18), (23) yields

$$\|\psi\|_{L_{1,\tau}(L_{2,h})} \leq C(h^2 + \tau^2) (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

From this estimate, (16) and (12) it follows that $N(z) \leq C(h^2 + \tau^2) (\|u_0\|_{V^4} + \|u_1\|_{V^3})$, whence the upper estimate

$$(24) \quad \|\bar{z}\|_{C_\tau(H_h^1)} \leq C(h + \tau)^2 (\|u_0\|_{V^4} + \|u_1\|_{V^3}).$$

5. Interpolation. Now we are going to apply the interpolation theory to our problem. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be two interpolation pairs (see [1]). Then, if L is a continuous linear operator from $A_1 + A_2$ into $B_1 + B_2$ such that its restrictions $L: A_1 \rightarrow B_1$ and $L: A_2 \rightarrow B_2$ are bounded, the inequality

$$(25) \quad \|L\|_{(A_1, A_2)_{\theta, q} \rightarrow (B_1, B_2)_{\theta, q}} \leq \|L\|_{A_1 \rightarrow B_1}^{1-\theta} \|L\|_{A_2 \rightarrow B_2}^{\theta},$$

holds for $0 < \theta < 1$, $1 \leq q \leq \infty$, where $(A_1, A_2)_{\theta, q}$ denotes the interpolation space obtained by the K -method of real interpolation (see [1]).

THEOREM. *Suppose $a \in C^3$ satisfying (1), u is the weak solution of (IBVP), v is the corresponding discrete approximation and let one of the conditions in Lemma 1 is satisfied. Then for the error $z = u - v$ the following estimates hold:*

- (i) $\|\bar{z}\|_{C_\tau(H_h^1)} \leq C(h + \tau)^{2(s-1)/3} (\|u_0\|_{V^s} + \|u_1\|_{V^{s-1}})$, $1 \leq s \leq 4$,
- (ii) $\|\bar{z}\|_{C_\tau(H_h^1)} \leq C(h + \tau)^{2(s-1)/3} (\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}})$, $1 \leq s \leq 4$, $s \neq \text{integer} + 1/2$,

Proof. (i) Let $z^{(0)}$ denote the error in the case when $u_1 = 0$ and $z^{(1)}$ in the case $u_0 = 0$. Define the linear operators R_0, R_1 by $R_0 u_0 = \bar{z}^{(0)}$, $R_1 u_1 = \bar{z}^{(1)}$.

From (11) and (24) it follows that R_0 is a bounded linear operator from V^1 into $D \equiv C_\tau(H_h^1)$ and from V^4 into D . Of course, the corresponding conclusion also holds for R_1 . Therefore, the interpolation inequality (25) yields

$$(26) \quad \|R_0\|_{(V^4, V^1)_{\theta, 2} \rightarrow D} \leq \|R_0\|_{V^4 \rightarrow D}^{1-\theta} \|R_0\|_{V^1 \rightarrow D}^{\theta},$$

$$(27) \quad \|R_1\|_{(V^3, V^0)_{\theta, 2} \rightarrow D} \leq \|R_1\|_{V^3 \rightarrow D}^{1-\theta} \|R_1\|_{V^0 \rightarrow D}^{\theta},$$

Applying the interpolation relation $(V^\alpha, V^\beta)_{\theta, 2} = V^{(1-\theta)\alpha + \theta\beta}$, $\alpha > \beta \geq 0$ (see Proposition 4 in [9]), we have $(V^4, V^1)_{\theta, 2} = V^{4-3\theta}$ and $(V^3, V^0)_{\theta, 2} = V^{3-3\theta}$. Setting $4 - 3\theta = s$, from (11), (24), (26)–(27) one obtains

$$(28) \quad \begin{aligned} \|\bar{z}^{(0)}\|_{C_\tau(H_h^1)} &\leq C(h + \tau)^{2(s-1)/3} \|u_0\|_{V^s} \quad \text{and} \\ \|\bar{z}^{(1)}\|_{C_\tau(H_h^1)} &\leq C(h + \tau)^{2(s-1)/3} \|u_1\|_{V^{s-1}}. \end{aligned}$$

Using $\bar{z} = \bar{z}^{(0)} + \bar{z}^{(1)}$ we finally obtain the desired estimate.

(ii) The continuous injection $H_0^s \subset V^s$, $1 \leq s \leq 4$, $s \neq \text{integer} + 1/2$ (see [9]) applied in (28) implies the estimate (ii). \square

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