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THERE ARE INFINITELY MANY COUNTABLE MODELS OF STRICTLY STABLE THEORIES WITH NO DENSE FORKING CHAINS

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Abstract. We prove that a countable, complete, strictly stable theory with no dense forking chains has infinitely many pairwise nonisomorphic countable models.

Let T denote a complete countable stable theory and $I(T,\aleph_0)$ the number of its countable pairwise nonisomorphic models. In [**4**] Lachlan proved that if T is superstable then it is either $\aleph_0$-categorical or $I(T,\aleph_0) \geq \aleph_0$. In [**5**] he proved that an $\aleph_0$-categorical superstable theory is $\aleph_0$-stable, and he conjectured that the same is true for stable theories, namely that there is no strictly stable $\aleph_0$-categorical theory. By the time being, it has become clear that the strictly stable theories are much more complicated than superstable ones; Hrushovski has constructed a counterexample to Lachlan's Conjecture.

Some work was done to extract those strictly stable theories that share some of the nice properties of superstables. In [**7**] Pillay proved that if T is strictly stable and 1-based, then $I(T,\aleph_0) \geq \aleph_0$. In [**3**] Hrushovski has introduced theories which admit finite coding and proved that such a T is either $\aleph_0$-categorical or $I(T,\aleph_0) \geq \aleph_0$ holds.

In [**2**] Pillay introduced the class of theories with no dense forking chains, which seems to be a reasonable approximation of superstability; for example, every type in such a theory has a regular decomposition. We prove that the class satisfies Lachlan's conjecture, namely that there are no $\aleph_0$-categorical theories in it; because of the existence of regular decompositions, or just finiteness of the weight, the argument from the superstable case goes through and we get $I(T,\aleph_0) \geq \aleph_0$.

We assume some basic knowledge of stability theory as can be found in [**1**] or [**6**]. Below, we define Pillay's notions of dimension of U_α-rank in terms of partial

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orders in order to prove Proposition 7, which is the main technical result used in the proof of Theorem.

Let $(P, \leq)$ be a partial order. For $p, q \in P$ we denote by $[p,q]_P$ the interval $\{x \in P | p \leq x \leq q\}$ ordered by (the restriction of) $\leq$; $(\leq, p]_P$ denotes $\{x \in P | x \leq p\}$ ordered by $\leq$, and similarly we define $(<, p)_P$, $(p, <)_P$ and $[p, \leq)_P$.

Definition. Let $(P, \leq)$ be a nonempty partial order. Inductively, define the dimension $\dim(P)$ which is an ordinal or ∞:

(1) $\dim(P) \geq 0$

(2) $\dim(P) \geq \alpha + 1$ if there is an infinite decreasing chain $p_0 > p_1 > p_2 > \dots$ such that for every $i \in \omega$ $\dim([p_{i+1}, p_i]_P) \geq \alpha$.

(3) $\dim(P) \geq \lambda$, where λ is a limit ordinal, if $\dim(P) \geq \alpha$ for every $\alpha < \lambda$.

(4) $\dim(P) = \alpha$ iff α is the greatest ordinal for which $\dim(P) \geq \alpha$ holds; $\dim(P) = \infty$ iff $\dim(P) \geq \alpha$ holds for all ordinals α.

Definition. Let α be an ordinal and let $(P, \leq)$ be a partial order. Inductively we define U_α-rank of $(P, \leq)$:

(1) $U_\alpha(P) \geq 0$ if $P \neq \emptyset$.

(2) $U_\alpha(P) \geq \beta + 1$ iff there exists a $p \in P$ such that $U_\alpha((\leq, p]_P) \geq \beta$ and $\dim([p, \leq)_P) \geq \alpha$.

(3) $U_\alpha(P) \geq \lambda$, where λ is a limit ordinal, iff $U_\alpha(P) \geq \beta$ for all ordinals $\beta < \lambda$.

(4) $U_\alpha(P) = \xi$, where ξ is an ordinal, if ξ is the greatest ordinal for which $U_\alpha(P) \geq \xi$. If no such ordinal exists, let $U_\alpha(P) = \infty$

LEMMA 1. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orders.*

(a) *If $f: P \to Q$ is strictly increasing then $\dim(P) \leq \dim(Q)$ and $U_\alpha(P) \leq U_\alpha(Q)$.*

(b) *If $\alpha \geq 1$ then $U_\alpha(P) + U_\alpha(Q) \leq U_\alpha(P \oplus Q)$ where $P \oplus Q$ is the set $P \times \{0\} \cup Q \times \{1\}$ ordered by $\{((p,0),(p',0)) | p \leq p'\} \cup \{((q,1),(q',1)) | q \leq q'\} \cup \{((p,0),(q,1)) | p \in P, q \in Q\}$.*

Proof. The part (a) is an easy induction on $\dim(P)$ and $U_\alpha(P)$; we prove only (b). Q is embedded in $P \oplus Q$, so if $U_\alpha(Q) = \infty$ the conclusion follows by the part (a).

Let $\xi = U_\alpha(Q)$. We use induction on ξ. For $\xi = 0$ it is obvious and for $\xi = 1$ it follows from the definition of U_α. Suppose that $\xi = \eta + 1$ and let $q \in Q$ be such that $U_\alpha((\leq, q]_Q) = \eta$ and $U_\alpha([q, \leq)_Q) = 1$. By the induction hypothesis

$$U_\alpha(P) + \xi = U_\alpha(P) + U_\alpha((\leq, q]_Q) \leq U_\alpha(P \oplus (\leq, q]_Q) = U_\alpha((\leq, q]_{P \oplus Q}).$$

On the other hand $1 = U_\alpha([q, \leq)_Q) = U_\alpha([q, \leq)_{P \oplus Q})$, and from the definition of U_α we get $U_\alpha(P) + U_\alpha(Q) = U_\alpha(P) + \xi + 1 \leq U_\alpha(P \oplus Q)$.

The case when ξ is a limit ordinal is similar.

LEMMA 2. *Let $(P, \leq)$ be a partial order. Then $\dim(P) = \infty$ if and only if there exists an embedding of rationals into $(P, \leq)$.*

Proof. $\leftarrow$ is clear, so we prove only $\rightarrow$. Assume that $\dim(P) = \infty$. Let α be an ordinal such that for all $p, q \in P$ $\dim([p,q]_P) \geq \alpha$ implies $\dim([p,q]_P) = \infty$. Since $\dim(P) \geq \alpha + 1$ there is an infinite decreasing chain $p_0 > p_1 > p_2 > \dots$ such that for all $i \in \omega$ $\dim([p_{i+1}, p_i]_P) \geq \alpha$, thus $\dim([p_{i+1}, p_i]_P) = \infty$. Applying the same reasoning to each $[p_{i+1}, p_i]_P$ for $i \in \omega$ in place of P we get infinite descending chains $p_0^i > p_1^i > p_2^i > \dots$ in $[p_{i+1}, p_i]_P$ so that $\dim([p_{j+1}^i, p_j^i]_P) = \infty$. Continuing in this way we get a chain in P isomorphic to the rationals.

LEMMA 3. *If $(P_i, \leq_i)$ are nonempty partial orders for $i \leq n$, then:*

$$\dim(P_1 \times P_2 \times \dots \times P_n) = \max\{\dim(P_1), \dim(P_2), \dots, \dim(P_n)\}.$$

(Here $P_1 \times P_2 \times \dots \times P_n$ is ordered by the product order, i.e.

$$(p_1, p_2, \dots, p_n) \leq (p_1', p_2', \dots p_n') \text{ iff } p_1 \leq_1 p_1' \;\; p_2 \leq_2 p_2' \dots p_n \leq_n p_n').$$

Proof. Assume $n = 2$. Then $\dim(P_1 \times P_2) \geq \max\{\dim(P_1), \dim(P_2)\}$ follows immediately from Lemma 1, so we prove the reverse inequality. Actually, we show by induction on ordinals α that $\dim(P_1 \times P_2) \geq \alpha$ implies $\max\{\dim(P_1), \dim(P_2)\} \geq \alpha$. For $\alpha = -1$ or 0 the claim is obvious, so we distinguish the following two cases:

Case 1: $\alpha = \beta + 1$. Assume that $\dim(P_1 \times P_2) \geq \beta + 1$. Then there is an infinite decreasing sequence $(p_0, p_0') > (p_1, p_1') > (p_2, p_2') > \dots$ such that for all $i \in \omega$ $\dim([(p_{i+1}, p_i), (p_{i+1}', p_i')]_{P_1 \times P_2}) \geq \beta$. By the induction hypothesis for each $i \in \omega$ either $\dim([p_{i+1}, p_i]_{P_1}) \geq \beta$ or $\dim([p_{i+1}', p_i')]_{P_2}) \geq \beta$ holds. Therefore either for infinitely many $i \in \omega$ $\dim([p_{i+1}, p_i]_{P_1}) \geq \beta$ or for infinitely many $i \in \omega$ $\dim([p_{i+1}', p_i'))]_{P_2}) \geq \beta$. Thus either $\dim(P_1) \geq \beta + 1$ or $\dim(P_2) \geq \beta + 1$ holds.

Case 2: α is a limit ordinal. Let $\alpha = \cup\{\alpha_\xi | \xi < \varkappa\}$ where $\varkappa = \mathrm{cf}(\alpha)$. By the induction hypothesis for each $\xi < \varkappa$ at least one of $\dim(P_1) \geq \alpha_\xi$ and $\dim(P_2) \geq \alpha_\xi$ holds. Thus at least one of the sets $\{\xi < \varkappa | \dim(P_1) \geq \alpha_\xi\}$ and $\{\xi < \varkappa | \dim(P_2) \geq \alpha_\xi\}$ is cofinal in $\varkappa$ and that means that either $\dim(P_1) \geq \alpha$ or $\dim(P_2) \geq \alpha$.

Thus we proved the Lemma for $n = 2$. The general case follows rather easily from this one.

From now on we assume that T is stable, $\mathcal{M}$ is a monster model of T and we operate in $\mathcal{M}^{eq}$. All the sets and tuples mentioned below are 'small' subsets of $\mathcal{M}^{eq}$; models are elementary submodels of $\mathcal{M}$.

Definition. Let $A \subseteq B$, $p \in S(A)$ and $p \subseteq q \in S(B)$.

(a) $\dim(p|q) = \dim([\mathrm{bnd}(q), \mathrm{bnd}(p)]_{o(T)})$.

(b) $U_\alpha(p|q) = U_\alpha([\mathrm{bnd}(q), \mathrm{bnd}(p)]_{o(T)})$.

(c) $\dim(p) = \dim(p|r)$ where r is any algebraic extension of r, and $U_\alpha(p) = U_\alpha(p|r)$.

Further in the text, we will write $\dim(\bar{a}/B)$ instead of $\dim(\mathrm{tp}(\bar{a}/B))$ and $\dim(\bar{a})$ instead of $\dim(\bar{a}/\emptyset)$. Similarly we do for U_α-rank.

If we allow infinitary types, not just types, in the previous definitions, then we get the notions of dim and U_α-rank of infinitary types as well.

Note that U_0 is the usual U-rank and $\dim(p) = 0$ means exactly that p has ordinal U-rank. Also, $U_\alpha(p|q) = 0$ implies $\dim(p|q) < \alpha$.

LEMMA 4. *If $A \subseteq B$ and $C \subseteq \mathrm{acl}(DA)$, then*

$$\dim(C/A|C/B) \leq \dim(D/A|D/B) \quad \textit{and} \quad U_\alpha(C/A|C/B) \leq U_\alpha(D/A|D/B).$$

Proof. By induction on $\dim(C/A|C/B)$. Suppose that $\beta_1 > \beta_2 > \ldots$ is an infinite descending chain between $\mathrm{bnd}(C/B)$ and $\mathrm{bnd}(C/A)$ such that $\dim(C/A|C/B) > \dim([\beta_{i+1}, \beta_i]) = \xi_i$. Pick a sequence $A \subseteq M_1 \subseteq M_2 \subseteq \ldots$ such that for all $i < j$ $\mathrm{bnd}(C/M_i) = \beta_i$, $C \downarrow_{M_1} A$ and $C \downarrow_{M_i} M_j$. Moreover, assume that $\cup M_i \downarrow_{CA} DB$. By the induction hypothesis for all $j < i$:

$$\dim(C/M_i|C/M_j) \leq \dim(D/M_i|D/M_j)$$

Hence $\dim(D/M_i|D/M_j) \geq \xi_i$. From the independence assumptions we derive $D \downarrow_{M_o} A$, $D \downarrow_{M_i} M_j$ and $D \downarrow_B M_i$, for all $i < j$. Therefore

$$\mathrm{bnd}(D/A) \geq \mathrm{bnd}(D/M_0) > \mathrm{bnd}(D/M_1) > \ldots \geq \mathrm{bnd}(D/B).$$

If $\dim(C/A|C/B) = \xi + 1$ then we could choose $\beta_i^{\prime}s$ so that $\xi_i = \xi$, and if $\dim(C/A|C/B)$ is a limit ordinal then it can be chosen so that $\xi_i^{\prime}$ s form a cofinal sequence. In both cases the conclusion follows.

A similar argument works for U_α.

LEMMA 5. *If $p \subseteq q \subseteq r$, then $U_\alpha(q|r) + U_\alpha(p|q) \leq U_\alpha(p|r)$. If r is algebraic, then $U_\alpha(q) + U_\alpha(p|q) \leq U_\alpha(p)$.*

Proof. Follows from Lemma 1 (b).

Definition. T has no dense forking chains if the order type of the rationals can not be embedded into $O(T)$.

As an immediate consequence of Lemma 2 we have that if T has no dense forking chains and $p \subseteq q$, then $\dim(p|q) < \infty$.

THEOREM 6. *Suppose that T has no dense forking chains.*

(a) *For any a, b and $A \subseteq B$ and $\alpha \geq 0$,*

$$\dim(ab/A|ab/B) = \sup\{\dim(b/aA|b/aB), \dim(a/A|a/B)\}.$$

(b) *(U_α-rank inequalities)*

$$U_\alpha(b/aA) + U_\alpha(a/A) \leq U_\alpha(ab/A) \leq U_\alpha(b/aA) \oplus U_\alpha(a/A).$$

(c) *Every type decomposes into a product of regular types.*

Proof. The part (a) is Lemma 10, (b) is Proposition 11 and (c) is Theorem 14 from [**2**].

We note the following instance of Theorem 6 (a) and Lemma 4 that we will use often: if $B \subseteq \mathrm{acl}(C_1C_2\dots C_nA)$ and $\dim(C_i/A) \leq \alpha$ for $1 \leq i \leq n$, then $\dim(B/A) \leq \alpha$.

PROPOSITION 7. *If $A \subseteq B$, $p \in S(A)$ and $p \subseteq q \in S(B)$, then*

$$\dim(p|q) \geq \sup\{\dim(\bar{c}/A) | \bar{c} \in Cb(q)\}.$$

Proof. Without loss of generality, assume that $A = \emptyset$ and we operate in $\mathcal{M}^{eq}$. Let $\bar{c} \in Cb(q)$ and let $I = \bar{a}_1\bar{a}_2\dots\bar{a}_n$ be a Morley sequence in (a stationarization of) q long enough so that $\bar{c} \in \mathrm{dcl}(I)$. Let $C = \mathrm{acl}(\bar{c})$ and we show that $\dim(p|q) \geq \dim(C)$; since $\dim(\bar{c}) \leq \dim(C)$ (by Lemma 4) this will imply the conclusion of the Proposition.

Let $I_k = \bar{a}_1\bar{a}_2\dots\bar{a}_{k-1}$ for $k \leq n$, let $P = \{\beta \in O(T) | \beta < \mathrm{bnd}(C)\}$ and for $\beta \in P$ let D_β be such that $\mathrm{bnd}(C/D_\beta) = \beta$. For $k \leq n$, $\beta \in P$, define $p^k_\beta = \mathrm{bnd}(\bar{a}_k/I_kE_\beta)$ where E_β satisfies $\mathrm{tp}(E_\beta/C) = \mathrm{tp}(D_\beta/C)$ and $E_\beta \downarrow_C I$. We note that p^k_β does not depend on the particular choice of E_β. Actually, since C is algebraically closed, $\mathrm{tp}(D_\beta/C)$ is stationary so it has a unique nonforking extension over CI, thus $\mathrm{tp}(E_\beta/CI)$ is uniquely determined and hence $\mathrm{tp}(I/CE_\beta)$ is uniquely determined, too.

For natural $k \leq n$ let $P_k = \{p^k_\beta | \beta \in P\}$ with the inherited order from $O(T)$. Now we show that $P_k \subseteq [\mathrm{bnd}(q), \mathrm{bnd}(p)]_{O(T)}$. From the definition of p^k_β we have $p^k_\beta \leq \mathrm{bnd}(p)$, and $\mathrm{bnd}(q) \leq p^k_\beta$ follows from:

$$\mathrm{bnd}(q) \leq \mathrm{bnd}(\bar{a}_k/I_kC) = \mathrm{bnd}(\bar{a}_k/I_kCE_\beta) \leq \mathrm{bnd}(\bar{a}_k/I_kE_\beta) = p^k_\beta.$$

The first inequality above is true since I is a Morley sequence in q. From $E_\beta \downarrow_C I$ we have $E_\beta \downarrow_{I_kC} \bar{a}_k$ and the first equality follows. The second inequality is clear and hence $P_k \subseteq [\mathrm{bnd}(q), \mathrm{bnd}(p)]_{O(T)}$.

Further, order $P_1 \times P_2 \times \dots \times P_n$ with the product order and define a mapping $f: P \to P_1 \times P_2 \times \dots \times P_n$ by $f(\beta) = (p^1_\beta, p^2_\beta, \dots, p^n_\beta)$. We show that f is strictly increasing. Assume that $\beta, \gamma \in P$ and $\gamma \leq \beta$. Choose E_β and E_γ such that:

$$\mathrm{tp}(E_\beta/C) = \mathrm{tp}(D_\beta/C),\ \ \mathrm{tp}(E_\gamma/C) = \mathrm{tp}(D_\gamma/C),\ \ E_\beta \downarrow_{E_\gamma} C \ \text{ and } \ E_\beta E_\gamma \downarrow_C I.$$

Then $p^k_\beta = \mathrm{bnd}(\bar{a}_k/I_kE_\beta)$ and $p^k_\gamma = \mathrm{bnd}(\bar{a}_k/I_kE_\gamma)$. From the independence assumptions we derive $I \downarrow_{E_\gamma} E_\beta$, and thus $\bar{a}_k \downarrow_{I_kE_\gamma} E_\beta$. We have:

$$(!)_k \qquad p^k_\beta = \mathrm{bnd}(\bar{a}_k/I_kE_\beta) \geq \mathrm{bnd}(\bar{a}_k/I_kE_\beta E_\gamma) = \mathrm{bnd}(\bar{a}_k/I_kE_\gamma) = p^k_\gamma.$$

Thus $p^k_\beta \geq p^k_\gamma$ and f is increasing. Now, if $\gamma < \beta$ then $C \not\downarrow_{E_\beta} E_\gamma$ and since $C \subseteq \mathrm{acl}(I)$ we have $I \not\downarrow_{E_\beta} E_\gamma$ so for some $j \leq n$ we have $\bar{a}_j \not\downarrow_{I_jE_\beta} E_\gamma$ and $\mathrm{bnd}(\bar{a}_j/I_jE_\beta E_\gamma) < \mathrm{bnd}(\bar{a}_j/I_jE_\beta)$. We conclude that in $(!)_j$ the strict inequality holds and $p^k_\gamma < p^k_\beta$. This proves that f is strictly increasing.

By Lemma 1 we have $\dim(P) \leq \dim(P_1 \times P_2 \times \ldots \times P_n)$ and by Lemma 3 we have $\dim(P_1 \times P_2 \times \ldots \times P_n) = \dim(P_k)$, for some $k \leq n$. Therefore $\dim(P) \leq \dim(P_k)$. But $P_k \subseteq [\mathrm{bnd}(q), \mathrm{bnd}(p)]_{O(T)}$ thus $\dim(P_k) \leq \dim(p|q)$ and we have:

$$\dim(C) = \dim(P) \leq \dim(P_k) \leq \dim(p|q)$$

completing the proof of the Proposition.

From now on we assume that T is strictly stable and has no dense forking chains. Consider all complete types whose domain is finite. Let $\alpha \geq 1$ be the smallest ordinal such that at least one of the types considered has dimension α and let ξ be the smallest possible U_α-rank of such a type. We say that a type is an (α, ξ)-type if its domain is finite, its dimension is α and its U_α-rank is ξ.

LEMMA 8. *If $p = \mathrm{tp}(\bar{a}/B)$ is an (α, ξ)-type, then there is a $\bar{c}$ such that $\bar{c} \in \mathrm{dcl}(\bar{a}B) \backslash \mathrm{acl}(B)$ and $\dim(\bar{c}/B) = 0$. In particular, every (α, ξ)-type is nonorthogonal to a type of dimension 0.*

Proof. Without loss of generality assume that $B = \emptyset$. Since $\alpha > 0$, there exists an infinite sequence $\beta_1 > \beta_2 > \ldots$ below $\mathrm{bnd}(p)$ in $O(T)$. Let $r = \mathrm{tp}(\bar{a}/C)$ be such that $\mathrm{bnd}(r) = \beta_2$. Note that $\beta_3 > \beta_4 > \ldots$ is an infinite descending sequence below $\mathrm{bnd}(r)$ so that $\dim(r) \geq 1$. If we replace C by a large enough finite subset of $Cb(r)$ we can assume that C is finite, r is a forking extension of p and $\dim(r) \geq 1$.

By the minimality assumptions on α and ξ we have $\dim(r) = \alpha$ and $U_\alpha(r) = U_\alpha(p) = \xi$. By Lemma 5 $U_\alpha(r) + U_\alpha(p|r) \leq U_\alpha(p)$ and it follows that $U_\alpha(p|r) = 0$. Thus, $\dim(p|r) < \alpha$. By Proposition 7 we have

$$\sup\{\dim(\bar{d}) | \bar{d} \in Cb(r)\} \leq \dim(p|r).$$

Therefore $\sup\{\dim(\bar{d}) | \bar{d} \in Cb(r)\} < \alpha$ and by the minimality assumption on α we have $\dim(\bar{d}) = 0$ for all $\bar{d} \in Cb(r)$. Let $\bar{d} \in Cb(r)$ be such that $\bar{a} \not\downarrow \bar{d}$ and let $\bar{c}' \in Cb(\bar{d}/\bar{a}) \backslash \mathrm{acl}(\emptyset)$. $\bar{c}'$ is definable in a finite Morley sequence $\bar{d}_1 \bar{d}_2 \ldots \bar{d}_k$ in $\mathrm{stp}(\bar{d}/\bar{a})$. Also $\dim(\bar{d}_i) = 0$, so $\dim(\bar{d}_1 \bar{d}_2 \ldots \bar{d}_k) = 0$ and $\dim(\bar{c}') = 0$. Let c be the name for the set of all $\{\bar{a}\}$-conjugates of $\bar{c}'$. Since $\bar{c}' \in \mathrm{acl}(\bar{a})$ this set is finite, so $c \in \mathrm{dcl}(\bar{a})$; also, every $\{\bar{a}\}$-conjugate of $\bar{c}'$ has the dimension 0 so that $\dim(c) = 0$. Finally, from $\bar{c}' \in \mathrm{acl}(c) \backslash \mathrm{acl}(\emptyset)$ we have $c \notin \mathrm{acl}(\emptyset)$ completing the proof of the Lemma.

THEOREM. *If T is a countable, complete, strictly stable theory with no dense forking chains then $I(T, \aleph_0) \geq \aleph_0$.*

Proof. Let B be finite, let $p = \mathrm{tp}(\bar{a}/B)$ be an (α, ξ)-type, let $A = \{\bar{d} \in \mathrm{dcl}(\bar{a}B) | \dim(\bar{d}/B) = 0\}$ and let $q = \mathrm{tp}(\bar{a}/AB)$. We *claim* that q is nonisolated.

Suppose, on the contrary, that $\varphi(\bar{x}, \bar{b})$ is a formula over AB which isolates q; here $\varphi(\bar{x}, \bar{y})$ is an L-formula $\bar{b} \subseteq AB$ and without any loss of generality we assume that $B \subseteq \bar{b}$. Clearly $\dim(\bar{b}/B) = 0$ holds, so that $\dim(\bar{a}/\bar{b}) > 0$ by Theorem 6 (a). By the minimality assumptions on α and ξ we must have $\dim(\bar{a}/\bar{b}) = \alpha$ and

$U_\alpha(\bar{a}/\bar{b}) = \xi$. By Lemma 8 there exists $\bar{c} \in A \backslash \mathrm{acl}(\bar{b})$. Choose $\bar{a}_1 \vdash \mathrm{stp}(\bar{a}/\bar{b})$ such that $\bar{a}_1 \downarrow_b \bar{c}$. From $\vdash \varphi(\bar{a}_1, \bar{b})$ we get $\mathrm{tp}(\bar{a}_1/AB) = q$, and from the independence assumption on $\bar{a}_1$ we get $\bar{c} \notin \mathrm{acl}(\bar{a}_1\bar{b})$; on the other hand $\bar{c} \in \mathrm{dcl}(\bar{a}\bar{b})$, so that $\mathrm{tp}(\bar{a}_1/\bar{c}B) \neq \mathrm{tp}(\bar{a}/\bar{c}B)$ and $\mathrm{tp}(\bar{a}_1/AB) \neq \mathrm{tp}(\bar{a}/AB) = q$. This is a contradiction and the *claim* is proved.

Continuing the proof of the Theorem, let $r \in S(\mathrm{dcl}(\bar{a}B))$ be a nonforking extension of q. Then, by the Open Mapping Theorem r must be nonisolated too, and $r|B\bar{a}$ is nonisolated as well. We have found a nonisolated type over a finite domain, hence there exists a nonisolated type over $\emptyset$. To complete the proof of the Theorem, we repeat the proof from the superstable case:

Suppose that T is small, otherwise $I(T, \aleph_0) = 2^{\aleph_0}$. Let $\mathrm{tp}(\bar{d})$ be nonisolated and let $\bar{d}_1\bar{d}_2\dots$ be an infinite Morley sequence in $\mathrm{tp}(\bar{d})$. For each n let M_n be prime over $\bar{d}_1\bar{d}_2\dots\bar{d}_n$. By Theorem 6 (c) $m = wt(\bar{d}) < \omega$. We show that in M_n there is no Morley sequence in $\mathrm{tp}(\bar{d})$ of lenght $m \cdot n + 1$, which clearly implies the conclusion of the Theorem.

If $\bar{e} \vdash \mathrm{tp}(\bar{d})$ and $\bar{e} \in M_n$ then $\mathrm{tp}(\bar{e}/\bar{d}_1\bar{d}_2\dots\bar{d}_n)$ is isolated, hence by the Open Mapping Theorem $\bar{e}$ forks with $\bar{d}_1\bar{d}_2\dots\bar{d}_n$. On the other hand $wt(\bar{d}_1\bar{d}_2\dots\bar{d}_n) = m \cdot n$, hence there is no independent set of realizations of $\mathrm{tp}(\bar{d})$ of size $m \cdot n + 1$ in M_n.

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