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# ON THE DIFFERENCE BETWEEN THE PRODUCT AND THE CONVOLUTION PRODUCT OF DISTRIBUTION FUNCTIONS

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**Abstract**. If $F$ is a subexponential d.f. it is well known that the tails of the distributions of the partial sums and partial maxima are asymptotically the same. In this paper we analyse the difference between these two d.f. The main part of the paper is devoted to the asymptotic behavior of $F(x)G(x)-F*G(x)$, where $F(x)$ and $G(x)$ are d.f. and where $*$ denotes the convolution product. Under various conditions we obtain a variety of $O$-, $o$- and exact (asymptotic) estimates for $F(x)G(x) - F * G(x)$. Compared to other papers in this field, we don't assume the existence of densities to obtain our estimates.

## 1. Introduction

Let $F$ denote a distribution function (d.f.) with $F(0-) = 0$ and $F(x) < 1$ for all $x$. The d.f. $F$ belongs to the subexponential class (i.e. $F \in S$) if

$$\lim_{\infty} \frac{1 - F^{*2}(x)}{1 - F(x)} = 2 \tag{1.1}$$

where $F^{*2}(x)$ is the convolution of $F$ with itself, i.e. $F^{*2}(x) = P\{X + Y \leq x\}$, where $X$ and $Y$ are i.i.d. with d.f. $F$. If (1.1) holds, then for all $n \leq 2$,

$$\lim_{\infty} \frac{1 - F^{*n}(x)}{1 - F(x)} = n \tag{1.2}$$

where $F^{*n}$ is the $n$-fold convolution of $F$. Many papers have been devoted to properties of functions $F \in S$ and classes of functions related to $S$. Also some papers are devoted to the remainder term in (1.1) and (1.2). More precisely, for $n \geq 2$, let $R_n(x)$ be defined as follows: $R_n(x) = 1 - F^{*n}(x) - n(1 - F(x))$. In the case where $F$ has a regularly varying density $f \in \mathrm{RV}_{-\alpha}$ (defined below) Omey and

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Willekens [**12, 13**] proved that

$$\lim_{\infty} \frac{R_n(x)}{f(x)} = \mu n(n-1) \qquad (\alpha > 2) \tag{1.3}$$

$$\lim_{\infty} \frac{R_n(x)}{(1-F(x))^2} = k(\alpha) n(n-1) \quad (1 \le \alpha < 2) \tag{1.4}$$

$$\lim_{\infty} \frac{R_n(x)}{f(x) \int_0^x (1-F(y))\,dy} = n(n-1) \qquad (\alpha = 2). \tag{1.5}$$

Here $\mu = \int_0^\infty (1-F(y))\,dy$ and $k(\alpha)$ is a constant depending on $\alpha$. Later there have been efforts to remove the condition of a regularly varying density. Geluk and Pakes [**6**] studied the class of d.f. $F$ for which

$$\lim_{\infty} \frac{R_2(x)}{(1-F(x))^2} = -1 \tag{1.6}$$

which corresponds to (1.4) with $\alpha = 1$. In this paper we replace the density condition by a condition on the asymptotic behavior of $F(x+y) - F(x)$ as $x$ tends to infinity. Among others we shall consider the class $D(m, \alpha)$ of d.f. $F(x)$ for which there exists a measurable and positive function $m(x)$ such that for some $\alpha \in R$ and all $y \in R$,

$$\lim_{\infty} \frac{F(x+y) - F(x)}{m(x)} = \alpha y. \tag{1.7}$$

This class (and related classes) of functions has proved to be useful in e.g. local limit theorems in extreme value theory, in the theory of difference equations, convolutions of functions, see e.g. [**14,15**]. In this paper we shall consider the following classes of functions. In each of the definitions $m$ is a positive and measurable function, bounded on bounded intervals. Also without further comments, all $O(1)$, $o(1)$ and other limit statements are considered to hold as $x \to \infty$. For appropriate function $u(x)$ and $v(x)$ we define

$$u * v(x) = \int_0^x u(x-y)\,v(dy), \quad u \otimes v(x) = \int_0^x u(x-y) v(y)\,dy,$$

$$\mathrm{RV}_\alpha = \{m | m(\mathrm{xy})/m(x) \to y^\alpha, \forall y > 0\},$$

$$\mathrm{ORV} = \{m | m(\mathrm{xy}) = O(1) m(x), \forall y > 0\},$$

$$L = \{m | m(\log(x)) \in \mathrm{RV}_0\}, \quad \mathrm{OL} = \{m | m(\log(x)) \in \mathrm{ORV}\}$$

$$\mathrm{SD} = \left\{ m \in L \cap L[0,\infty) | m \otimes m(x)/m(x) \to 2 \int_0^\infty m(y)\,dy \right\},$$

$$\begin{aligned}
\mathrm{OSD} &= \{m \mid m \otimes m(x) = O(1)m(x)\},\\
D(m) &= \{F\text{d.f.} \mid \ \|F\|_m := \sup_x m * F(x)/m(x) < \infty\},\\
D^{\alpha}(m) &= \{F \in D(m) \mid m * F(x)/m(x) \to \alpha\},\\
\mathrm{OD}(m) &= \{F\text{d.f.} \mid \ | F(x+y) - F(x) | = O(1)m(x), \forall y \in R\},\\
D(m,\alpha) &= \{F\text{d.f.} \mid (F(x+y) - F(x))/m(x) \to \alpha y, \forall y \in R\},\\
\Pi_{\beta}(m) &= \{f : R^{+} \to R^{+} \mid (f(\mathrm{xt}) - f(x))/m(x) \to \beta \log t, \forall t > 0\}.
\end{aligned}$$

Some of these classes of functions have been used before in connection with asymptotic expansions for convolutions of functions and sequences. Obviously if $m(x) = 1 - F(x)$, then $F \in D^1(m)$ iff $R_2(x)/\bar{F}(x) \to 0$ iff $F \in S$. If $F$ has a density $f \in \mathrm{SD}$, then $F \in D(f(x), 1) \cap D^2(f)$. It can be shown that for all classes where the quantors $\forall y \in R$ or $\forall y > 0$ appear, the limit relation holds locally uniformly in $y$, (see e.g. [**2,5**]). We summarize some more relations between the previous function classes:

(i) $1 - F(x) \in \mathrm{ORV} \cap L$ implies $F(x) \in S$ and $F(x) \in S$ implies $1 - F(x) \in L$;
(ii) $m(x) \in \mathrm{ORV} \cap L[0,\infty)$ implies $m(x) \in \mathrm{OSD}$;
(iii) $m(x) \in L \cap \mathrm{ORV} \cap L[0,\infty)$ implies $m(x) \in \mathrm{SD}$;
(iv) for $m(x) \in L \cap \mathrm{ORV}$: $m(x) \in L[0,\infty)$ iff $m \in \mathrm{OSD}$;
(v) if $F(x) \in D(m,\alpha)$ with $m(x) = (1 - F(x))/x$, then $\bar{F}(x) \in \mathrm{RV}_{-\alpha}$.

Relations (i)–(iv) are well known; relation (v) is proved in [**15**]; see also section 3 below. Being interested in d.f. $F(x) \in S \subset L$, the auxiliary function $m(x)$ in $\mathrm{OD}(m)$ or $D(m,\alpha)$ should satisfy $m(x) = o(1 - F(x))$. If $F(x) \in D(m,\alpha)$ with $\alpha \neq 0$, then $\bar{F}(\log x) \in \Pi_{-\alpha}(m(\log x))$ and automatically $m(x) = o(1 - F(x))$ holds (cf. [**5,8**]).

*Remark.* In several of the results below, the condition that $F$ is a d.f. can be omitted. Often it is sufficient to assume that $F$ is bounded on bounded intervals.

The paper is organized as follows. In section 2 we consider various closure properties of the classes $D(m)$ and $D^{\alpha}(m)$. Section 2.2 provides useful $O$-estimate of $R_n$. We also give a number of sufficient conditions on $F$ and $m$ to conclude $F \in D(m)$ or $D^{\alpha}(m)$. In section 3 we discuss closure properties of $D(m,\alpha)$ and in section 4 we discuss the asymptotic behavior of the convolution of d.f. in $\mathrm{OD}(m)$ or $D(m,0)$ and we obtain several estimates of $F(x)G(x) - F * G(x)$ under various assumptions on $F$ and $G$. In Corollary 4.6 we obtain asymptotic estimates of the form $R_n(x) = O(1)\bar{F}(x)/x$ or $R_n(x) = O(1)\bar{F}^2(x)$. In section 5 we discuss $F(x)G(x) - F * G(x)$ for the classes $D(m,\alpha)$ and this section is divided into three parts. In part 1 we assume $F$ and $G$ have a finite mean and obtain estimates of the form $R_n(x) \sim \text{constant} \cdot m(x)$; the infinite-mean case in part 2 is more complicated and the mixed case (i.e. $F$ has a finite mean and $G$ an infinite mean) is a combination of parts 1 and 2. In Corollary 5.7 we summarize the conditions under which $R_n(x) \sim \binom{n}{2} R_2(x)$ as $x \to \infty$. In the final section 6 we consider some examples and give two applications.

## 2. Closure properties of $D(m), D^\alpha(m)$ and $\mathrm{OD}(m)$.

**2.1. Closure properties.** In this section, we consider closure properties of the classes $D(m)$ and $D^\alpha(m)$ introduced in section 1. In the first proposition, no extra conditions on $m$ are used.

PROPOSITION 2.1. (i) *$F, G \in D(m)$ implies $F * G \in D(m)$;* (ii) *$F, G \in \mathrm{OD}(m) \cap D(m)$ implies $F * G \in \mathrm{OD}(m) \cap D(m)$;* (iii) *$F, G \in D(m,0) \cap D(m)$ implies $F * G \in D(m,0) \cap D(m)$;* (iv) *Suppose $U(x)$ is locally bounded and $U(x)/m(x) \to \beta$. If $F \in D^\alpha(m) \cap D(m,0)$ then $U * F(x)/m(x) \to \alpha\beta$.* (v) *$F \in D^\alpha(m) \cap D(m,0)$ and $G \in D^\beta(m) \cap D(m,0)$ imply $F * G \in D^{\alpha\beta}(m) \cap D(m,0)$.*

*Proof.* (i) Obviously we have

$$m * F * G(x) \le \|F\|_m m * G(x) \le \|F\|_m \|G\|_m m(x).$$

(ii) For $h > 0$ and some fixed $x_0 (0 < x_0 < x)$ we decompose $F * G(x+h) - F * G(x)$ as follows:

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = \int_0^{x-x_0} (F(x+h-y) - F(x-y))\, dG(y)$$
$$+ \int_{x-x_0}^{x} (F(x+h-y) - F(x-y))\, dG(y) + \int_x^{x+h} F(x+h-y)\, dG(y).$$

Now

$$0 \le \mathrm{I} \le \sup_{z \ge x_0} \frac{F(z+h) - F(z)}{m(z)} \int_0^{x-x_0} m(x-y)\, dG(y)$$
$$\le m * G(x) \sup_{z \ge x_0} \frac{F(z+h) - F(z)}{m(z)}$$

and $0 \le \mathrm{II} + \mathrm{III} \le G(x+h) - G(x-x_0)$. It follows that

$$\limsup_{\infty} \frac{F * G(x+h) - F * G(x)}{m(x)} \le \|G\|_m \sup_{z \ge x_0} \frac{F(z+h) - F(z)}{m(z)} + \limsup_{\infty} \frac{G(x+h) - G(x-x_0)}{m(x)} \tag{2.1}$$

The result (ii) follows.

(iii) Similarly to (ii) we arrive at (2.1) which simplifies to:

$$\limsup_{\infty} \frac{F * G(x+h) - F * G(x)}{m(x)} \le \|G\|_m \sup_{z \ge x_0} \frac{F(z+h) - F(z)}{m(z)}.$$

Now let $x_0 \uparrow \infty$ to obtain result (iii).

(iv) For each $\varepsilon > 0$ we can find $x_0$ such that $| U(x-y)/m(x-y) - \beta | \leq \varepsilon$, for $x - y \geq x_0$. Now let $V(x) = U * F(x) - \beta m * F(x)$ and write

$$V(x) = \int_0^{x-x_0} (U(x-y) - \beta m(x-y))\, dF(y) + \int_{x-x_0}^{x} (U(x-y) - \beta m(x-y))\, dF(y).$$

It follows that

$$|V(x)| \leq \varepsilon \int_0^{x-x_0} m(x-y)\, dF(y) + k(F(x) - F(x-x_0))$$

where $k = \sup_{0 \leq z \leq x_0} |U(z) - \beta m(z)|$. Hence $\limsup_\infty |V(x)|/m(x) \leq \varepsilon\alpha$. Now let $\varepsilon \downarrow 0$.

(v) Apply (iv) with $U = m * G$. □

*Remark.* In (ii) we proved that $\mathrm{OD}(m) \cap D(m)$ is closed under $*$; if we only consider functions in the class $\mathrm{OD}(m)$, more has to be assumed about $m$ to conclude that $\mathrm{OD}(m)$ is closed under $*$. See section 3 below.

**2.2. First estimates of $R_n(x)$.** The previous result can be applied to obtain a first useful estimate of $R_n$.

COROLLARY 2.2. *Assume $F \in D^\alpha(m) \cap D(m, 0)$. If $R_2(x)/m(x) \to \beta$, then for all $n$, $\frac{R_n(x)}{m(x)} \to \beta_n$ where $\beta_n = \beta\binom{n}{2}$ if $\alpha = 1$ and $\beta_n = \beta\frac{(\alpha^n-1)-n(\alpha-1)}{(\alpha-1)^2}$ if $\alpha \neq 1$.*

*Proof.* It is easy to prove that

$$R_{n+1}(x) = nR_2(x) + R_n * F(x). \tag{2.2}$$

It follows from Proposition 2.1 (iv) and by induction on $n$, that $\beta_n < \infty$ and that $\beta_{n+1} = n\beta + \beta_n\alpha$ $(\beta_2 = \beta)$. This shows that $\beta_n = \beta\sum_{k=1}^{n-1} k\alpha^{n-1-k}$ and the result follows. □

In the next proposition we obtain $O$-estimates of $R_n$.

PROPOSITION 2.3. *Suppose $\beta_2 = \sup_{x\geq 0} |R_2(x)|/m(x) < \infty$.*

(i) *If $F \in D(m)$, then for $n \geq 2$, $\beta_n := \sup_{x \geq 0} \frac{|R_n(x)|}{m(x)} \leq \beta_2\binom{n}{2}\max(\|F\|_m^n, 1)$.*

(ii) *If $F \in D^\alpha(m) \cap \mathrm{OD}(m)$, then for each $\varepsilon > 0$ there exist constants $x_0$ and $k = k(x_0)$ such that for $n \geq 2$, $\gamma_n(x_0) := \sup_{x\geq x_0} \frac{|R_n(x)|}{m(x)} \leq k(\alpha+\varepsilon)^n\binom{n}{2}$.*

*Proof.* (i) Using (2.2) we obtain that $\beta_{n+1} \leq n\beta_2 + \beta_n\|F\|_m$. By induction on $n$ it follows that for all $n \geq 2$, $\beta_n < \infty$ and that $\beta_n \leq \beta_2\sum_{k=1}^{n-1} k\|F\|_m^{n-1-k}$. Hence result (i) follows.

(ii) For $n \geq 2$ and $x_0$ to be fixed later, define

$$\gamma_n(x_0) = \sup_{x \geq x_0} \frac{|R_n(x)|}{m(x)}, \quad G(x_0) = \sup_{x\geq x_0} \frac{|F(x) - F(x-x_0)|}{m(x)},$$

and $D(x_0) = \sup_{x \geq x_0} \frac{m*F(x)}{m(x)}$.

Using

$$R_n * F(x) = \int_0^{x-x_0} R_n(x-y)\,dF(y) + \int_{x-x_0}^{x} R_n(x-y)\,dF(y)$$

we obtain

$$\begin{aligned}|R_n * F(x)| \leq& \gamma_n(x_0) \int_0^{x-x_0} m(x-y)\,dF(y) + \sup_{0\leq z\leq x_0} |R_n(z)| G(x_0) m(x) \\ \leq& \gamma_n(x_0) m * F(x) + (n+1) G(x_0) m(x)\end{aligned}$$

and hence $\sup_{x\geq x_0} |R_n * F(x)|/m(x) \leq \gamma_n(x_0) D(x_0) + (n+1)G(x_0)$. Using the relation (2.2) again we obtain

$$\begin{aligned}\gamma_{n+1}(x_0) &\leq n\gamma_2(x_0) + \gamma_n(x_0)D(x_0) + (n+1)G(x_0) \\ &\leq n \max(\gamma_2(x_0), 3/2 \cdot G(x_0)) + \gamma_n(x_0)D(x_0) \\ &=: nE(x_0) + \gamma_n(x_0)D(x_0).\end{aligned}$$

Hence $\gamma_n(x_0) \leq E(x_0)\sum_{k=1}^{n-1} kD^{n-k-1}(x_0)$. As in part (i) we obtain $\gamma_n(x_0) \leq E(x_0)\binom{n}{2}\max(1, D^{n-2}(x_0))$ and by an appropriate choice of $x_0$ the result follows.

*Remarks.* (1) From (2.2) it follows that $R_n(x) = \sum_{k=1}^{n-1} kF^{*n-k-1} * R_2(x)$ and this equality makes the previous results more transparant.

(2) If $\alpha < 1$, in Proposition 2.3(ii) we can choose $\varepsilon$ such that $\gamma_n(x_0) \leq k\binom{n}{2}$.

(3) For sets of conditions under which $R_2(x)/m(x) \to \beta$ we refer to [**9,12, 11,14**]. See also section 5 below.

(4) For the class $S$ with remainder term, Corollary 2.2 and Proposition 2.3 are only useful if $m(x) = o(1 - F(x))$.

In [**6**] the authors consider the class $S^2(m)$ of d.f. for which $R_2(x) + \bar{F}^2(x) = o(m(x))$, where $m(x) = o(1 - F(x))$. Obviously $S^2(m) \subset S$. If also $m(x) = O(\bar{F}^2(x))$, more can be said.

COROLLARY 2.4. *Suppose* $F \in S^2(m)$ *with* $m(x) = O(\bar{F}^2(x))$. *Then*
(i) $F \in D^1(\bar{F}^2(x)) \cap D(\bar{F}^2(x), 0)$ *and* $R_2(x)/\bar{F}^2(x) \to -1$.
(ii) *For all* $n \geq 2$, $R_n(x)/\bar{F}^2(x) \to -\binom{n}{2}$.
(iii) $\bar{F}(x) \in \mathrm{RV}_0$ *and* $-R_n(x) \in \mathrm{RV}_0$ *for all* $n \geq 2$.

*Proof.* (i) Using the inequality

$$R_2(x) + \bar{F}^2(x) = \int_0^x (F(x) - F(x-y))\,dF(y) \geq (F(x) - F(A))(F(x) - F(x-A))$$

we obtain that $F \in D(m, 0)$. By taking $A = x/2$ the inequality also shows that

$$(F(x) - F(x/2))^2 = o(m(x)). \tag{2.3}$$

Now

$$0 \le \bar{F}^2 * F(x) - \bar{F}^2(x)F(x) = \int_0^x (F(x) - F(x-y))(2 - F(x) - F(x-y))\, dF(y).$$

Since $0 \le F(x) + F(x-y) \le 2$, we obtain

$$\begin{aligned} 0 \le \bar{F}^2 * F(x) - \bar{F}^2(x)F(x) &\le 2\int_0^x (F(x) - F(x-y))\, dF(y) \\ &= 2(R_2(x) + \bar{F}^2(x)) = o(m(x)). \end{aligned}$$

Using $m(x) = O(\bar{F}^2(x))$, this implies that $F \in D^1(\bar{F}^2(x))$. By the definition of $S^2(m)$, the proof of (i) is complete.

Part (ii) follows from (i) and Corollary 2.2. Finally, part (iii) follows from part (ii), (2.3) and $m(x) = O(\bar{F}^2(x))$. □

**2.3. Necessary and sufficient conditions.** To see the implications of the assumption $F \in D(m)$ or $F \in D^1(m)$ we assume that $m$ is nonincreasing. For a fixed number $A \le x/2$ we have :

$$\begin{aligned} m * F(x) &= \left( \int_0^A + \int_A^{x-A} + \int_{x-A}^x \right) m(x-y)\, dF(y) \\ &\ge m(x)F(A) + m(x-A)(F(x-A) - F(A)) + m(A)(F(x) - F(x-A)). \end{aligned}$$

Using this inequality we immediately obtain the following result.

PROPOSITION 2.5. *Suppose $m$ is nonincreasing. Then*
(i) *Allways* $\liminf m * F(x)/m(x) \ge 1$*;*
(ii) *If $F \in D(m)$, then* $\limsup (F(x) - F(x-A))/m(x) < \infty$*;*
(iii) *If $F \in D^1(m)$ then $m \in L$ iff $F \in D(m, 0)$.* □

In the next result we collect sufficient conditions to conclude $F \in D(m)$ or $F \in D^\alpha(m)$.

PROPOSITION 2.6. *Suppose $m$ is bounded. Then*
(i) *If $m \in$ ORV and $F(x/2) - F(x) = O(m(x))$, then $F \in D(m) \cap$ OD$(m)$.*
(ii) *If $m \in$ ORV$\cap L$ and $F(x/2) - F(x) = o(m(x))$, then $F \in D^1(m) \cap D(m, 0)$.*
(iii) *If $m \in$ OSD $\cap L$ and $F \in D(m, 0)$, then $F \in D^1(m)$.*
(iv) *Suppose $m \in$ SD and $H := 1 - m \in S$. If $F \in D(m, \alpha)$, then $F \in D^\beta(m)$, where $\beta = 1 + \alpha \int_0^\infty m(s)\, ds$.*

*Proof.* (i) We have $m * F(x) = \left( \int_0^{x/2} + \int_{x/2}^x \right) m(x-y)\, dF(y) =:$ I+II. Obviously I $\le \sup_{x/2 \le z \le x} m(z)F(x/2) = O(m(x))$ and, since $m$ is bounded by, say $K$, we also have $\mathrm{II} \le K(F(x) - F(x/2)) = O(m(x))$. Hence the first result follows. To prove that $F \in \mathrm{OD}(m)$, take $y > 0$ and $x > y$. Since $F(x+y) - F(x) \le F(2x) - F(x) = O(m(2x))$ and $m \in \mathrm{ORV}$, the result follows.

(ii) As in part (i) we have $m * F(x) = \mathrm{I} + \mathrm{II}$. Clearly, $\mathrm{II} = o(m(x))$ and using $m \in \mathrm{ORV} \cap L$ and Lebesgue's theorem, we obtain $\mathrm{I}/m(x) \to 1$. Hence $F \in D^1(m)$. The second result follows as in (i).

(iii) Let $x_0 \in \mathbf{N}$ be fixed and write

$$m * F(x) = \left\{ \int_0^{x_0} + \int_{x_0}^{[x-x_0]} + \int_{[x-x_0]}^{x} \right\} m(x-y)\, dF(y) =: \mathrm{I} + \mathrm{III} + \mathrm{II}.$$

Since $m \in L$, we have

$$m(x-y)/m(x) \to 1, \text{ locally uniformly in } y; \tag{2.4}$$

and for $x \ge x_0$,

$$1/2 \le m(x-y)/m(x) \le 2 \text{ uniformly in } 0 \le y \le 1. \tag{2.5}$$

From (2.4) it follows that

$$\mathrm{I}/m(x) \to F(x_0). \tag{2.6}$$

Also, since $m$ is bounded by, say $K$, we have $\mathrm{III} \le K(F(x) - F([x-x_0])$, and using $F \in D(m,0)$ it follows that

$$\mathrm{III} = o(m(x)). \tag{2.7}$$

As to II, note that $\mathrm{II} = \sum_{k=x_0}^{[x-x_0]} \int_k^{k+1} m(x-y)\, dF(y)$; using (2.5) we obtain $\mathrm{II} \le 2\sum_{k=x_0}^{[x-x_0]} m(x-k-1)(F(k+1) - F(k))$. By using $F \in D(m,0)$ again, for arbitrary $\varepsilon > 0$, we can choose $x_0$ so that $(F(k+1) - F(k)) \le \varepsilon m(k)$, for all $k \ge x_0$. Hence $\mathrm{II} \le 2\varepsilon \sum_{k=x_0}^{[x-x_0]} m(x-k-1)m(k)$. Using (2.5) and standard arguments we can find a constant $C$ (independent of $x_0$) such that $\mathrm{II} \le \varepsilon C m \otimes m(x)$. Since $m \in \mathrm{SD}$, it follows that

$$\limsup \mathrm{II}/m(x) \le \varepsilon K. \tag{2.8}$$

Now combine (2.6)-(2.8) to obtain $\limsup |m * F(x)/m(x) - 1| \le 1 - F(x_0) + \varepsilon K$. By letting $x_0 \uparrow \infty$ and $\varepsilon \downarrow 0$ we obtain the desired result.

(iv) Since $F \in D(m,\alpha)$ and $m \in L$, we have $(F(x+h) - F(x))/m(x) \to \alpha h$ l.u. in h. If we define $R(x) := \int_0^1 (F(x+h) - F(x))\, dh$ this implies $R(x)/m(x) \to \alpha/2$. Now let $G(x) := \int_0^1 F(x+h)\, dh$; obviously we have $R(x) = G(x) - F(x)$, $R(0) = G(0)$, $R(\infty) = 0$, and $m * F(x) = m * G(x) - m * R(x)$. Moreover $G'(x) \sim \alpha m(x)$. First we estimate $m * R$. Since $m(x) = 1 - H(x)$ we have $m * R(x) = R(x) - R(0)m(x) - \int_0^x R(x-y)\, dH(y)$. Using $H \in S$ we have $H \in D^1(m) \cap D(m,0)$ and an application of Proposition 2.1(iv) yields $m * R(x)/m(x) \to \alpha/2 - R(0) - \alpha/2 = -R(0)$. Next we consider $m * G(x)$ and for fixed $x_0$ write $m * G(x) =$

$\left(\int_0^{x_0} + \int_{x_0}^{x}\right) m(x-y)\, dG(y) =: I + II$. Using $m \in L$, we have $I/m(x) \to G(x_0) - G(0)$. As to $II$ we have $II = \int_{x_0}^{x} m(x-y)G'(y)\, dy$. Using $G'(x) \sim \alpha m(x)$, for each $\varepsilon > 0$ we can find $x_0$ so that

$$(\alpha - \varepsilon) \int_{x_0}^{x} m(x-y)m(y)\, dy \leq II \leq (\alpha + \varepsilon) \int_{x_0}^{x} m(x-y)m(y)\, dy.$$

Using $m \in \mathrm{SD}$ this implies that

$$(\alpha - \varepsilon) \left\{ 2 \int_0^{\infty} m(y)\, dy - \int_0^{x_0} m(y)\, dy \right\}$$
$$\leq \lim \begin{matrix} \sup \\ \inf \end{matrix} \frac{II}{m(x)} \leq (\alpha + \varepsilon) \left\{ 2 \int_0^{\infty} m(y)\, dy - \int_0^{x_0} m(y)\, dy \right\}.$$

Now combine the two estimates and let $x_0 \uparrow \infty$, $\varepsilon \downarrow 0$ to obtain

$$m * G(x)/m(x) \to G(\infty) - G(0) + \alpha \int_0^{\infty} m(y)\, dy = 1 - R(0) + \alpha \int_0^{\infty} m(y) dy.$$

The result follows. □

*Remarks.* (1) If in Proposition 2.6(i), $m(x)$ satisfies $m(x) = o(\bar{F}(x))$, then automatically $\bar{F}(x) \in RV_0$ with remainder term $m(x)/\bar{F}(x)$. (2) The conditions of Proposition 2.6(i) and (ii) show that $R_2(x) + \bar{F}^2(x) = O(1)m(x)$ (resp. $o(1)m(x)$). To see this it is sufficient to rewrite

$$R_2(x) + \bar{F}^2(x) = 2 \int_0^{x/2} (F(x) - F(x-y))dF(y) + (F(x) - F(x/2))^2.$$

## 3. The classes $\mathrm{OD}(m)$ and $D(m, \alpha)$

**3.1. Preliminaries.** In this subsection we study into further detail, classes of functions related to $\mathrm{OD}(m)$ and $D(m, \alpha)$. More precisely, we shall consider the classes of positive, measurable functions $a(x)$, $f(x)$ for which one of the relations below holds.

$$f \in \mathrm{OD}_+(a) \quad \text{iff} \quad \limsup_{t \to \infty} \frac{|f(t+x) - f(t)|}{a(t)} < \infty, \qquad \forall x \geq 0;$$
$$f \in D_+(a, c) \quad \text{iff} \quad \limsup_{t \to \infty} \frac{f(t+x) - f(t)}{a(t)} = cx, \qquad \forall x \geq 0.$$

Note that these classes are defined in general and not for distribution functions only. These classes of functions were studied in **[15]** and in order to state the main results of [**15**], we need some more definitions and notations. Recall that for $f \in \mathrm{ORV}$, the upper and lower Matuszewska indices $\alpha(f)$ and $\beta(f)$ are defined as follows :

$$\alpha(f) = \lim_{y\to\infty} \frac{\log \limsup_{x\to\infty} f(xy)/f(x)}{\log y}, \quad \beta(f) = \lim_{y\to\infty} \frac{\log \liminf_{x\to\infty} f(xy)/f(x)}{\log y}.$$

It is well known that $f \in \mathrm{ORV}$ if and only if both $\alpha(f)$ and $\beta(f)$ are finite.

A positive function has bounded increase (BI) if $\alpha(f) < \infty$ and has bounded decrease (BD) if $\beta(f) > -\infty$. The function $f$ has positive increase (PI) if $\beta(f) > 0$ and it has positive decrease (PD) if $\alpha(f) < 0$. In studying $\mathrm{OD}_+(a)$ and $D_+(a,c)$, the results of Bingham et al. [**2**] provided the necessary framework. In [**2**] the authors study classes of functions satisfying general asymptotic relations of the following form. For $a(x) \in \mathrm{RV}_\alpha$, the class $\Pi_a$ is the class of measurable $f$ satisfying

$$\forall x \geq 1, \ \lim_{t\to\infty} \frac{f(tx) - f(t)}{a(t)} = ck_\alpha(x), \ \text{ where } k_\alpha(x) = \int_1^x u^{\alpha-1} du.$$

The constant $c$ is called the a-index of $f$. For $a(x) \in \mathrm{BI}$, the class $\mathrm{O}\Pi_a$ (resp. $\mathrm{o}\Pi_a$) is the class of measurable $f$ that satisfy:

$$\forall x \geq 1, \ f(tx) - f(t) = O(a(t)) \ \text{(resp. } o(a(t))) \ \text{(as } t \to \infty).$$

If $A(x) := a(\log(x)) \in \mathrm{BI}$ and $f(x) \in \mathrm{OD}_+(a(x))$ it follows immediately that the function $F(x) := f(\log x)$ belongs to $\mathrm{O}\Pi_A$ and if $a(t) = o(1)f(t)$ we have $D_+(a,c) \subset \mathrm{OD}_+(a) \subset L$. In [**15**] the following two results were proved.

PROPOSTION 3.1.1. *Suppose $f \in OD_+(a)$ (resp. $f \in D_+(a,0)$) and let $A(x) := a(\log(x))$.*

(i) *If $A(x)$ is of bounded increase, then*

$$f(x) = C + n(x)a(x) + \int_X^x m(z)a(z)\, dz, \ \forall x \geq X$$

*where $C, X$ are constants and $n(x)$ and $m(x)$ are measurable and bounded functions (resp. $n(x)$ and $m(x)$ are measurable and $o(1)$).*

(ii) *If $a(x) \in \mathrm{BI}$, then $f(x) \in \mathrm{O}\Pi_b$ (resp. $\mathrm{o}\Pi_b$) with $b(x) = xa(x)$;*

(iii) *If $a(x) \in \mathrm{BD}$ and $A(x) \in \mathrm{BI}$, then for each $c$ $(0 \leq c < 1)$ there exists constants $K$ (resp. $\forall \varepsilon > 0$) and $X$ such that for all $y$ with $0 \leq y \leq cx$,*

$$|f(x) - f(x-y)| \leq K(1+y)a(x), \quad (\textit{resp. } \varepsilon(1+y)a(x)), \ \forall x \geq X.$$

(iv) *If $xa(x) \in \mathrm{PD}$, then $f(x) = C + O(1)xa(x)$ (resp. $C + o(1)xa(x)$) and $f(x) \to C$ as $x \to \infty$.*

(v) *If $xa(x) \in \mathrm{PI} \cap \mathrm{BI}$, then $f(x) = O(1)xa(x)$ (resp. $o(1)xa(x)$).*

(vi) *If $f \in \mathrm{OD}_+(a)$ with $a(x) = f(x)/x$, then $f \in \mathrm{ORV}$.*

PROPOSITION 3.1.2. *Assume that $f$ is locally integrable.*

(i) *If $f(x) \in D_+(a,c)$ with $a(x) \in \mathrm{RV}_\beta$, then:*

(1) *if $\beta > -1$, then $f(x)/xa(x) \to c/(\beta+1)$;*

(2) *If $\beta < -1$, then $f(x) \to f(\infty)$ exists, finite and* $(f(\infty) - f(x))/xa(x) \to -c/(\beta+1)$.

(3) *If $\beta = -1$, then $f(x) \in \Pi_c(xa(x))$.*

(ii) *If $f(x) \in D_+(a,c)$ with $a(x) = f(x)/x$, then $f(x) \in \mathrm{RV}_c$.*

**3.2. Distribution functions in $\mathrm{OD}(m)$ and $D(m,\alpha)$.** In this section we analyse in further detail the classes $D(m,\alpha)$ and $\mathrm{OD}(m)$. In many cases we shall assume that $m(x) \in L$ and/or $m(x) \in \mathrm{ORV}$. In the first proposition we connect $1 - F(x)$ with $m(x)$ and integrals of $m(x)$.

PROPOSITION 3.2.1. (i) *If $F(x) \in D(m,\alpha)$ with $\alpha \neq 0$, then $m(x) \in L \cap L[0,\infty)$ and*

$$\bar{F}(x) = (\alpha + o(1)) \int_x^\infty m(s)\,ds. \tag{3.2.1}$$

(ii) *If $F(x) \in D(m,0)$ and $m(x) \in L \cap L[0,\infty)$, then (3.2.1) holds.*

(iii) *If $F(x) \in \mathrm{OD}(m)$ and $m(x) \in \mathrm{OL} \cap L[0,\infty)$, then $\bar{F}(x) = O(1)\int_x^\infty m(s)\,ds$.*

(iv) *If $F(x) \in \mathrm{OD}(m)$ and $xm(x) \in \mathrm{PD}$, then $\bar{F}(x) = O(1)xm(x)$.*

*Proof.* (i) Since $F(x) \in D(m,\alpha)$ we have

$$\begin{aligned}
\alpha(y+z) &= \lim_\infty \frac{F(x+y+z) - F(x)}{m(x)} \\
&= \lim_\infty \frac{F(x+y+z) - F(x+y)}{m(x+y)} \frac{m(x+y)}{m(x)} + \lim_\infty \frac{F(x+y) - F(x)}{m(x)} \\
&= \alpha z \lim_\infty \frac{m(x+y)}{m(x)} + \alpha y.
\end{aligned}$$

Since $\alpha \neq 0$ it follows that $m(x) \in L$. To prove (3.2.1), note that for $\varepsilon > 0$ we can find $n_0$ so that $(\alpha - \varepsilon)m(n) \le F(n+1) - F(n) \le (\alpha+\varepsilon)m(n)$, $n \ge n_0$. Hence for $N \ge M \ge n_0$ we obtain

$$(\alpha - \varepsilon) \sum_{n=M}^N m(n) \le F(N+1) - F(M) \le (\alpha+\varepsilon) \sum_{n=M}^N m(n).$$

It follows that

$$\sum_{n=M}^\infty m(n) < \infty \text{ and } (\alpha - \varepsilon) \sum_{n=M}^\infty m(n) \le 1 - F(M) \le (\alpha + \varepsilon) \sum_{n=M}^\infty m(n).$$

Using standard arguments we obtain (3.2.1).

(ii) and (iii) The proof is similar and omitted.

(iv) This is Proposition 3.1.1(iv). □

*Remarks.* (1) If $F(x) \in D(m,\alpha)$ with $\alpha \neq 0$, then the mean $E(X)$ is finite iff $m(x) \in L_1[0,\infty)$.

(2) If $F(x) \in \mathrm{OD}(m)$ with $m(x) \in \mathrm{OL} \cap L_1[0,\infty)$, then $E(X) < \infty$.

A similar result can be proved for $F_1(x) := \int_0^x y\,dF(y)$. Obviously $F(x) \in \mathrm{OD}(m(x))$ iff $F_1(x) \in \mathrm{OD}(xm(x))$ and $F(x) \in D(m(x),\alpha)$ iff $F_1(x) \in D(xm(x),\alpha)$. Also note that in case the mean $E(X)$ is finite, it equals $E(X) = F_1(\infty)$.

PROPOSITION 3.2.2. (i) *Suppose $m(x) \in L_1[0,\infty)$. Then*

(a) *If $F(x) \in \mathrm{OD}(m)$ and $m(x) \in \mathrm{OL}$, then $E(X) < \infty$ and $F_1(\infty) - F_1(x) = O(1)\int_x^\infty ym(y)\,dy$; if also $x^2m(x) \in \mathrm{PD}$, then $F_1(\infty) - F_1(x) = O(1)x^2m(x)$.*

(b) *If $F(x) \in D(m,\alpha)$ and $m(x) \in L$, then $E(X) < \infty$ and $F_1(\infty) - F_1(x) = (\alpha + o(1))\int_x^\infty ym(y)\,dy$;*

(ii) *Suppose $\int_0^\infty ym(y)\,dy = \infty$. Then:*

(a) *If $F \in D(m,\alpha)$ and $m(x) \in L$, then $F_1(x) = (\alpha + o(1))\int_0^x ym(y)\,dy$.*

(b) *If $F \in \mathrm{OD}(m)$ and $m(x) \in \mathrm{OL}$, then $F_1(x) = O(1)\int_0^x ym(y)\,dy$; if also $x^2m(x) \in \mathrm{PI} \cap \mathrm{BI}$, then $E(X) = \infty$ and $F_1(x) = O(1)x^2m(x)$.*

**3.3. Closure properties of $\mathrm{OD}(m)$ and $D(m,\alpha)$.** Now we prove the analogue of Proposition 2.1 for $\mathrm{OD}(m)$ and $D(m,\alpha)$. The main result of this section is the following:

PROPOSITION 3.3.1. *Suppose $F \in D(m,\alpha)$ and $G \in D(n,\beta)$. Then:*

(i) *If $m,n \in \mathrm{BD} \cap L$ (e.g. $m,n \in \mathrm{ORV} \cap L$), then*

$$F * G(x+h) - F * G(x) = \alpha hm(x) + \beta hn(x) + o(m(x)) + o(n(x));$$

(ii) *If $m \in \mathrm{SD}$ and $n(x)/m(x) \to C$, then $F * G \in D(m, \alpha + \beta C)$.*

*Proof.* (i) We have

$$F * G(x) = \int_0^{x/2} F(x-y)\,dG(y) + \int_0^{x/2} G(x-y)dF(y) - F(x/2)G(x/2)$$

so that (for $h > 0$):

$$F * G(x+h) - F * G(x) = \int_0^{x/2} (F(x+h-y) - F(x-y))dG(y)$$

$$+ \int_0^{x/2} (G(x+h-y) - G(x-y))\,dF(y) + \int_{x/2}^{(x+h)/2} (F(x+h-y) - F(x/2))\,dG(y)$$

$$+\int_{x/2}^{(x+h)/2} (G(x+h-y) - G((x+h)/2))\, dF(y) = I_1 + I_2 + I_3 + I_4.$$

As to $I_1$, use $F \in D(m, \alpha)$ and fix $\varepsilon > 0$ and $x_0$ to see that

$$(\alpha h - \varepsilon) \int_0^{x/2} m(x-y)\, dG(y) \le I_1 \le (\alpha h - \varepsilon) \int_0^{x/2} m(x-y)\, dG(y), \ \ \forall x \ge x_0.$$

Using $m \in \mathrm{BD} \cap L$ and dominated convergence we have:

$$\int_0^{x/2} m(x-y)\, dG(y)/m(x) \to 1.$$

It readily follows that

$$\lim I_1/m(x) = \alpha h. \tag{3.3.1}$$

In a similar way we find

$$\lim I_2/n(x) = \beta h. \tag{3.3.2}$$

To estimate $I_3$ and $I_4$ note that for $\varepsilon > 0$ we can find $x_0$ so that

$$\begin{aligned} 0 \le I_3 + I_4 &\le \varepsilon(G((x+h)/2) - G(x/2) + F((x+h)/2) - F(x/2)) \\ &\le \varepsilon((\beta h/2 + \varepsilon) n(x/2) + (\alpha h/2 + \varepsilon) m(x/2)). \end{aligned}$$

Since $n, m \in \mathrm{BD}$, this yields

$$0 \le I_3 + I_4 \le \varepsilon K_1 n(x) + \varepsilon K_2 m(x). \tag{3.3.3}$$

Combining (3.3.1)–(3.3.3) gives the proof of (i).

(ii) To prove (ii) we fix $x_0 > h$ and write for $x \ge x_0$,

$$\begin{aligned} F * G(x) - F * G(x-h) &= \left\{ \int_0^{x-x_0} + \int_{x-x_0}^{x-h} \right\} (F(x-y) - F(x-y-h))\, dG(y) \\ &\quad + \int_{x-h}^{x} F(x-y)\, dG(y) = I + II + III \end{aligned}$$

First consider $III$; we have

$$\begin{aligned} III = \int_{y=x-h}^{x} \int_{z=0}^{x-y} dF(z)\, dG(y) &= \int_{z=0}^{h} \int_{y=x-h}^{x-z} dG(y)\, dF(z) \\ &= \int_{z=0}^{h} (G(x-z) - G(x-h))\, dF(z). \end{aligned}$$

Using $G \in D(n,\beta)$ it follows that

$$\lim_{\infty} \frac{III}{n(x)} = \beta \int_{z=0}^{h} (h-z)\, dF(z). \tag{3.3.4}$$

Next consider $II$; using partial integration we have

$$II = \int_{y=x-x_0}^{x-h} \int_{z=x-y-h}^{x-y} dF(z)\, dG(y)$$

$$= \left( \int_{z=0}^{h} \int_{y=x-z-h}^{x-h} + \int_{z=h}^{x_0-h} \int_{y=x-z-h}^{x-z} + \int_{z=x_0-h}^{x_0} \int_{y=x-x_0}^{x-z} \right) dG(y) dF(z)$$

$$= \int_{z=0}^{h} (G(x-h) - G(x-z-h))\, dF(z) + \int_{z=h}^{x_0-h} (G(x-z) - G(x-z-h))\, dF(z)$$

$$+ \int_{z=x_0-h}^{x_0} (G(x-z) - G(x-x_0))\, dF(z).$$

Using $G \in D(n,\beta)$ again we obtain

$$\frac{II}{n(x)} \to \beta \int_{z=0}^{h} z\, dF(z) + \beta \int_{z=h}^{x_0-h} h\, dF(z) + \beta \int_{z=x_0-h}^{x_0} (x_0 - z) dF(z). \tag{3.3.5}$$

Combining the estimates (3.3.4) and (3.3.5), we obtain that

$$\frac{(II + III)}{n(x)} \to \beta \int_{x_0-h}^{x_0} F(z)\, dz.$$

It follows that

$$\limsup \left| \frac{II + III}{m(x)} - C\beta h \right| \le C\beta h(1 - F(x_0 - h)). \tag{3.3.6}$$

Finally we consider $I$; since $F \in D(m,\alpha)$ and $m \in L$, for each $\varepsilon > 0$ we can find $x_0$ such that

$$(\alpha h - \varepsilon) \int_{0}^{x-x_0} m(x-y)\, dG(y) \le I \le (\alpha h + \varepsilon) \int_{0}^{x-x_0} m(x-y)\, dG(y).$$

To estimate the integral term we write

$$\int_{0}^{x-x_0} m(x-y)\, dG(y) = \left( \int_{0}^{[x_0]} + \int_{[x_0]}^{[x-x_0]} + \int_{[x-x_0]}^{x-x_0} \right) m(x-y) dG(y) = I_1 + I_2 + I_3.$$

Using (2.4) we have

$$I_1/m(x) \to G([x_0]). \tag{3.3.7}$$

In $I_3$ we have $x_0 \le x - y \le x - [x - x_0] \le x_0 + 1$; using (2.5) we obtain $m(x-y) \le 2m(x_0)$ and hence $I_3 \le 2m(x_0)(G(x-x_0) - G([x-x_0]))$. We conclude

$$\limsup I_3/m(x) \le 2m(x_0)C\beta. \tag{3.3.8}$$

For $I_2$ we write (using a constant $K$ which may be different in each inequality)

$$\begin{aligned} I_2 &= \sum_{k=[x_0]}^{[x-x_0]-1} \int_k^{k+1} m(x-y)dG(y) \\ &\le K \sum_{k=[x_0]}^{[x-x_0]-1} m(x-k)(G(k+1)-G(k)) \ \ (\text{since } m \in L) \\ &\le K \sum_{k=[x_0]}^{[x-x_0]-1} m(x-k)m(k) \ \ (\text{since } G \in D(n,\beta) \text{ and } n/m \to C) \\ &\le K \sum_{k=[x_0]}^{[x-x_0]-1} \int_k^{k+1} m(x-y)m(y)dy \ \ (\text{since } m \in L) \\ &\le K \int_{[x_0]}^{[x-x_0]} m(x-y)m(y)\,dy. \end{aligned}$$

Since $m \in \mathrm{SD}$ it follows that

$$\limsup \frac{I_2}{m(x)} \le K \int_{[x_0]}^{\infty} m(y)\,dy. \tag{3.3.9}$$

Combining the estimates (3.3.7)–(3.3.9) we obtain

$$\limsup \left| \frac{\int_0^{x-x_0} m(x-y)\,dG(y)}{m(x)} - 1 \right| \le (1 - G([x_0])) + 2\beta m(x_0) + K \int_{[x_0]}^{\infty} m(y)\,dy.$$

This shows that

$$\limsup |I/m(x) - \alpha h| \le K_1\varepsilon + K_2(1 - G([x_0]) + K_3 m(x_0) + K_4 \int_{[x_0]}^{\infty} m(y)\,dy.$$

Now combine this last estimate with (3.3.6). Let $x_0 \uparrow \infty$ and $\varepsilon \downarrow 0$ to obtain the desired result. □

Specializing the previous result, we obtain the following

COROLLARY 3.3.2. *Suppose that $F, G \in D(m,\alpha)$. Then*

(i) *If $m(x) \in \mathrm{BD} \cap L$ or $m(x) \in \mathrm{SD}$, then $F * G \in D(m, 2\alpha)$ and $F^{*n} \in D(m, n\alpha)$.*

(ii) *If $m(x) = 1 - H(x) \in \mathrm{SD}$ and $H(x) \in S$, then $F * G \in D^{\gamma}(m)$ with $\gamma = 1 + 2\alpha \int_0^\infty m(y)\, dy$ and $F^{*n} \in D^{\beta}(m)$ with $\beta = 1 + n\alpha \int_0^\infty m(y)\, dy$.*

*Proof.* (i) Follows immediately from Proposition 3.3.1.

(ii) Combine (i) with Proposition 2.6(iv). □

The proof of Proposition 3.3.1 can also be used to obtain the following closure properties for $\mathrm{OD}(m)$.

COROLLARY 3.3.3. (i) *If $F \in \mathrm{OD}(m)$ with $m \in \mathrm{BD}$, then $F^{*2} \in \mathrm{OD}(m)$ and $F^{*n} \in \mathrm{OD}(m)$.*

(ii) *If $F \in D(m,0)$ with $m \in \mathrm{BD}$, then $F^{*2} \in D(m,0)$ and $F^{*n} \in D(m,0)$.*

*Remarks.* (1) Comparing with Proposition 2.1 we see that the condition $m \in \mathrm{BD}$ replaces the condition $F \in D(m)$. (2) A result similar to Proposition 3.3.1(i) has been proved by Frenk [**4**, Lemma 4.1.20].

## 4. Asymptotic behavior of convolution products: the classes $\mathrm{OD}(m)$ and $D(m,0)$.

**4.1. The asymptotic behavior of $F * G(x)$.** In the sequel we shall estimate the asymptotic behavior of $F(x)G(x) - F * G(x)$ under various conditions on $F$ and $G$. A partial result has been obtained already in Proposition 2.6. There we proved

PROPOSITION 4.1. *Suppose $F \in D(m,\alpha)$ with $m(x) = 1 - H(x)$. Then*

(i) *If $H \in S$ and $m \in \mathrm{SD}$, then $\lim \frac{F(x)H(x) - F*H(x)}{1-H(x)} = \alpha \int_0^\infty (1 - H(x))dx$.*

(ii) *If $\alpha = 0$ and $m \in \mathrm{OSD} \cap L$, then $\lim \frac{F(x)H(x)-F*H(x)}{1-H(x)} = 0$.* □

In the following discussion we shall frequently use the following identity: for two d.f. $F(x)$ and $G(x)$ we have

$$
\begin{aligned}
&F(x)G(x) - F * G(x) = \\
&\int_0^{x/2} (F(x) - F(x-y))\, dG(y) + \int_0^{x/2} (G(x) - G(x-y))dF(y) \qquad (4.1)\\
&+ (F(x) - F(x/2))(G(x) - G(x/2)) =: I + II + III.
\end{aligned}
$$

In Propositions 4.2 and 4.3, we consider the classes $\mathrm{OD}(m)$ and $D(m,0)$.

PROPOSITION 4.2. (a) *Suppose $F \in \mathrm{OD}(m)$, $G \in \mathrm{OD}(n)$ with $m, n \in \mathrm{ORV}$.*

(i) *There holds: $F(x)G(x) - F * G(x) = O(1)m(x)G_1(x) + O(1)n(x)F_1(x)$.*

(ii) *If* $E(X) + E(Y) < \infty$, *then* $F(x)G(x) - F * G(x) = O(1)m(x) + O(1)n(x)$.

(iii) *If* $\beta(m) > -2$ *and* $\beta(n) > -2$, *then* $E(X) = E(Y) = \infty$ *and* $F(x)G(x) - F * G(x) = O(1)x^2 m(x)n(x)$.

(b) *If* $F \in D(m, 0)$, $G \in D(n, 0)$ *with* $m, n \in \mathrm{ORV}$, *then the results of* (a) *hold with the* $O(1)$*-terms replaced by* $o(1)$*-terms.*

*Proof of Proposition* 4.2 (a). (i) Consider the first term $I$ in (4.1); using Proposition 3.1.1 (iii) we obtain $0 \le I \le Km(x)\int_0^{x/2} \max(1, y)\, dG(y) \quad (x \ge x_0)$. Hence $I = O(1)m(x)G_1(x)$. Similarly we have $II = O(1)n(x)F_1(x)$. Using Proposition 3.1.1(iii) again, we have $III = O(1)xm(x)(G(x) - G(x/2))$. Since $G(x) - G(x/2) \le (2/x)G_1(x)$ we obtain $III = O(1)m(x)G_1(x)$. This proves (i).

(ii) This follows from (i).

(iii) This follows from (i) and Propositon 3.2.2 (ii, b).

*Proof of Proposition* 4.2 (b). Similar. □

Proposition 4.2 is useful to estimate the difference between $F^n(x)$ and $F^{*n}(x)$.

COROLLARY 4.3. *Suppose* $m \in \mathrm{ORV}$. *Then*

(i) *If* $F \in \mathrm{OD}(m)$, *then for all* $n \ge 2$, $F^n(x) - F^{*n}(x) = O(1)m(x)F_1(x)$.

(ii) *If* $F \in D(m, 0)$, *then for all* $n \ge 2$, $F^n(x) - F^{*n}(x) = o(1)m(x)F_1(x)$.

*Proof.* (i) Let $G(x) := F^{*n-1}(x)$; from Corollary 3.3.3 we see that $G \in \mathrm{OD}(m)$. Applying Proposition 4.2 we obtain $F^n(x) - F^{*n}(x) = O(1)m(x)G_1(x) + O(1)m(x)F_1(x)$. Since $G_1(x) = (n-1)F^{*n-2} * F_1(x) \le (n-1)F_1(x)$ we obtain the desired result.

(ii) Similar. □

Proposition 4.2 will not be useful if $m(x)G_1(x) \to \infty$ or $n(x)F_1(x) \to \infty$. The terms $F_1(x)$ and $G_1(x)$ were obtained by applying Proposition 3.1.1. If, as in Proposition 2.6, we strengthen the conditions on $F, G$ then we obtain an estimate which is independent of $F_1(x)$ and $G_1(x)$. The result is similar to [**6**, Theorem 1] and [**7**, Lemma 3.1].

PROPOSITION 4.4. (i) *Suppose* $F(x) - F(x/2) = O(m(x))$ *and* $G(x) - G(x/2) = O(n(x))$. *Then* $F(x)G(x) - F * G(x) = O(1)m(x) + O(1)n(x)$.

(ii) *If also* $F \in D(m, 0)$ *and* $G \in D(n, 0)$, *then* $F(x)G(x) - F * G(x) = o(1)m(x) + o(1)n(x)$.

*Proof.* (i) Use (4.1).

(ii) Use (4.1) and Lebesgue's theorem on dominated convergence. □

*Remark.* The class of d.f. considered in Proposition 4.4 (i) is connected with the class $\mathrm{O}\Pi(m)$ (cf. [**2**, Chapter 3]). If for example $F \in \Pi_\alpha(m)$ and $G \in \Pi_\beta(n)$, then Proposition 4.4 (ii) applies.

If in Proposition 4.4 we also assume $m \in \mathrm{ORV}$, then Propositions 2.1 and 2.6 can be used to obtain the following result (cf. [**6**, Theorem 4] and [**7**, Corollary 3.2]).

COROLLARY 4.5. *Suppose* $F(x) - F(x/2) = O(m(x))$ *and* $m \in \mathrm{ORV}$.

(i) *For all* $n \geq 2$ *we have* $F^n(x) - F^{*n}(x) = O(1)m(x)$.

(ii) *If also* $F \in D(m,0)$, *then* $F^n(x) - F^{*n}(x) = o(1)m(x)$. □

As to $R_n(x)$ (cf. Introduction), using Corollary 4.3 (i) we obtain

$$R_n(x) = 1 - F^{*n}(x) - n(1 - F(x)) = O(1)m(x)F_1(x) + \left(-\binom{n}{2} + o(1)\right)\bar{F}^2(x). \tag{4.2}$$

In terms of the class $S$ with remainder term, (4.2) is only useful if $m(x)F_1(x) = o(1)\bar{F}(x)$. In general it is not clear which term ($m(x)F_1(x)$ or $\bar{F}^2(x)$) is dominant. If $E(X) < \infty$, then (4.2) reduces to

$$R_n(x) = O(1)m(x) + \left(-\binom{n}{2} + o(1)\right)\bar{F}^2(x). \tag{4.3}$$

If $E(X) = \infty$ and $\beta(m) > -2$, Proposition 3.2.2 (ii) shows that $F_1(x) = O(1)x^2m(x)$. The estimate (4.2) then reduces to

$$R_n(x) = O(1)x^2m^2(x) + \left(-\binom{n}{2} + o(1)\right)\bar{F}^2(x). \tag{4.4}$$

If also $\alpha(m) < -1$, then Proposition 3.2.1 (iv) and (4.4) yield the estimate

$$R_n(x) = O(1)x^2m^2(x). \tag{4.5}$$

A very useful situation appears when $m(x) = \bar{F}(x)/x$.

COROLLARY 4.6. *Suppose* $m(x) = \bar{F}(x)/x$.

(i) *If* $F \in \mathrm{OD}(m)$, *then* $\bar{F} \in \mathrm{ORV}$ *and:*

(a) *if* $E(X) < \infty$, *then* $R_n(x) = O(1)\bar{F}(x)/x$.

(b) *if* $\beta(\bar{F}) > -1$, *then* $R_n(x) = O(1)\bar{F}^2(x)$.

(ii) *If* $F \in D(m,0)$ *then* $\bar{F} \in \mathrm{RV}_0$ *and* $R_n(x) = \left(-\binom{n}{2} + o(1)\right)\bar{F}^2(x)$.

*Proof.* (i) From Proposition 3.1.1 (vi) it follows that $\bar{F} \in \mathrm{ORV}$. Hence also $m \in \mathrm{ORV}$ and if $E(X) < \infty$, (4.3) yields the result a). If $\beta(\bar{F}) > -1$, (4.5) gives the second result.

(ii) From Proposition 3.1.2 (ii) it follows that $\bar{F} \in \mathrm{RV}_0$ and hence that $\beta(\bar{F}) > -1$. In this case we have $F_1(x) = o(1)x\bar{F}(x)$ and using (4.2) we obtain the result (ii).□

*Remarks.* (1) The second result should be compared with Corollary 2.4.

(2) The slow variation of $1 - F(x)$ alone is not sufficient to obtain the conclusion of Corollary 4.6 (ii). The example $1 - F(x) = 1/[\log(x)]$ $(x \geq e)$ is an example for which the conclusion is false, cf. [**6**, p.83].

(3) For a refinement of Corollary 4.6 we refer to Section 5 below.

(4) Let $R(x) = -\log(1 - F(x))$ and assume that $R(x)$ has a derivate $R'(x)$ which is eventually nonincreasing and $R'(x) \to 0$. Then $F \in D(m(x) =$

$(1-F(x))/x,0)$ holds iff $xR'(x)\to 0$. Geluk and Pakes [**6**] prove that the condition $xR'(x)\to 0$ is fully equivalent with the conclusion of Corollary 4.6 (ii). This remark shows that the conditions on $F$ are almost necessary and sufficient.

**4.2. Stability.** If the d.f. $F$ is not in $\mathrm{OD}(m)$ or $D(m,0)$, then Corollary 4.3 will not be applicable. The following two propositions may be used to transfer properties from $F$ to $G$ if the d.f. $G$ is a suitable approximation of $F$. A related result was proved in [**6**, Theorem 3].

PROPOSITION 4.7. (i) *If $F(x)$ satisfies $F^2(x)-F^{*2}(x)=O(1)W(x)$ and if $G(x)$ is a d.f. such that $G(x)-F(x)=O(1)\bar F^2(x)$, then $G^2(x)-G^{*2}(x)=O(1)W(x)+O(1)\bar F^2(x)$.*

(ii) *Suppose $1-F(x)\in L$ and $F^2(x)-F^{*2}(x)=o(1)W(x)$. If $G(x)$ is a d.f. such that $G(x)-F(x)=(c+o(1))\bar F^2(x)$ $(c\in\mathbf{R})$, then $G^2(x)-G^{*2}(x)=o(1)W(x)+o(1)\bar F^2(x)$.*

*Proof.* (i) Define $U(x):=G(x)-F(x)$ so that by assumption $U(x)=O(1)\bar F^2(x)$. Without loss of generality we shall assume that $|U(x)|\le K\bar F^2(x)$ for all $x\ge 0$. First consider $\bar F^2 * F(x)$; we have

$$\begin{aligned}0\le \bar F^2 * F(x)-\bar F^2(x)F(x) &= \int_0^x (\bar F(x-y)-\bar F(x))(\bar F(x-y)+\bar F(x))dF(y)\\ &\le 2\int_0^x (\bar F(x-y)-\bar F(x))dF(y)\\ &\le 2(F^2(x)-F^{*2}(x)).\end{aligned}\tag{4.6}$$

Hence

$$\bar F^2 * F(x)=O(1)W(x)+\bar F^2(x)F(x).\tag{4.7}$$

Next consider $\bar F^2 * G(x)$; as in (4.6) we have

$$0\le \bar F^2 * G(x)-\bar F^2(x)G(x)\le 2(F(x)G(x)-F*G(x)).\tag{4.8}$$

Now use (4.7) and the assumptions about $F$ and $U$ to see that

$$\begin{aligned}F*G(x)&=U*F(x)+F*F(x)\\ &\le K\bar F^2 * F+F*F(x)\\ &=O(1)W(x)+O(1)\bar F^2(x)+F^2(x)\end{aligned}\tag{4.9}$$

and similarly

$$F(x)G(x)=O(1)\bar F^2(x)+F^2(x).\tag{4.10}$$

Using (4.9) and (4.10) in (4.8) we obtain

$$\bar F^2 * G(x)=O(1)W(x)+O(1)\bar F^2(x).\tag{4.11}$$

Now we consider $U*U(x)=U*G(x)-U*F(x)$. Using $|U(x)|\le K\bar F^2(x)$, (4.7) and (4.11) we obtain

$$U*U(x)=O(1)W(x)+O(1)\bar F^2(x).\tag{4.12}$$

On the other hand we have $G * G(x) = U{*}U(x) + 2U{*}F(x) + F{*}F(x)$. Using (4.7), (4.12) and the assumptions about $U$ and $F$, we obtain

$$G * G(x) - F * F(x) = O(1)W(x) + O(1)\bar{F}^2(x). \tag{4.13}$$

Since also $G^2(x) - F^2(x) = U(x)(G(x)+F(x)) = O(1)\bar{F}^2(x)$, using (4.13) we finally obtain $G^2(x) - G * G(x) = O(1)W(x) + O(1)\bar{F}^2(x)$.

(ii) By assumption we have $U(x) = (c + o(1))\bar{F}^2(x)$. From (4.6) now we deduce

$$\bar{F}^2 * F(x) = o(1)W(x) + \bar{F}^2(x)F(x). \tag{4.14}$$

Now we consider $U * F(x)$ and $U{*}G(x)$. Choose $\varepsilon > 0$ and $x_0$ so that $c - \varepsilon \le U(x)/\bar{F}^2(x) \le c + \varepsilon, \ \forall x \ge x_0$. We have

$$\begin{aligned} U * F(x) &= \int_0^{x-x_0} U(x-y)\, dF(y) + \int_{x-x_0}^{x} U(x-y)\, dF(y) \\ &\le (c+\varepsilon) \int_0^{x-x_0} \bar{F}^2(x-y) dF(y) + O(1)(F(x) - F(x-x_0)) \\ &\le (c+\varepsilon)\bar{F}^2 * F(x) + O(1)(F(x) - F(x-x_0)). \end{aligned}$$

Since for each $A > 0$, $F^2(x) - F^{*2}(x) \ge (F(x) - F(x-A))(F(x) - F(A))$, by assumption we have $F(x) - F(x-A) = o(1)W(x)$. Hence $U * F(x) \le (c+\varepsilon)\bar{F}^2 * F(x) + o(1)W(x)$. Similarly we obtain $U * F(x) \ge (c-\varepsilon)\bar{F}^2 * F(x) + o(1)W(x)$. Using (4.14) we arrive at

$$U * F(x) = o(1)W(x) + c\bar{F}^2(x) + o(1)\bar{F}^2(x). \tag{4.15}$$

Next consider $\bar{F}^2 * G(x)$; using (4.8), (4.15) and $F{*}G(x) = U{*}F(x) + F{*}F(x)$ we obtain $\bar{F}^2 * G(x) = o(1)W(x) + o(1)\bar{F}^2 + \bar{F}^2 G$. To estimate $U{*}G(x)$ we proceed as in the proof of (4.15) now using $1 - F(x) \in L$ and

$$G(x) - G(x-A) = U(x) - U(x-A) + F(x) - F(x-A) = o(1)\bar{F}^2(x) + o(1)W(x).$$

We obtain

$$U * G(x) = o(1)W(x) + c\bar{F}^2(x) + o(1)\bar{F}^2(x). \tag{4.16}$$

Results (4.15) and (4.16) imply that $U * U(x) = o(1)W(x) + o(1)\bar{F}^2(x)$. On the other hand we have $G{*}G(x) = U{*}U(x) + 2U{*}F(x) + F{*}F(x)$. Using (4.15) and the estimate for $U{*}U$, we obtain $G{*}G(x) - F{*}F(x) = o(1)W(x) + o(1)\bar{F}^2(x) + 2c\bar{F}^2(x)$. Since also $G^2(x) - F^2(x) = U(x)(G(x) + F(x)) = 2c\bar{F}^2(x) + o(1)\bar{F}^2(x)$ we finally obtain

$$G^2(x) - G * G(x) = o(1)W(x) + o(1)\bar{F}^2(x). \ \square$$

*Remark.* If $G(x)$ is such that $\bar{G}(x) - a\bar{F}(x) = O(1)\bar{F}^2(x)$ $(a \ne 0)$ (or $\bar{G}(x) - a\bar{F}(x) = (c + o(1))\bar{F}^2(x)$), the conclusions of Proposition 4.7 remain valid.

A result similar to Proposition 4.7 is obtained in the next proposition. In the proposition we analyse stability if $G(x)$ is a d.f. such that $(1-G(x))/(1-F(x))$ converges to 1 with a certain rate.

PROPOSITION 4.8. (i) *Suppose* $1-F(x) \in \mathrm{ORV}$ *and that* $F(x)$ *satisfies*

$$F^2(x) - F^{*2}(x) = O(1)W(x). \tag{4.17}$$

*Suppose* $G(x)$ *is a d.f. such that*

$$1 - G(x) = (1 + r(x))(1 - F(x)) \tag{4.18}$$

*where* $r(x)$ *satisfies* $|r(x)| \leq s(x)$ *and* $s(x) \in \mathrm{ORV}$ *with* $s(x) \to 0$. *Then*

$$G^2(x) - G^{*2}(x) = O(1)W(x) + O(1)s(x)\bar{F}(x) + O(1)\bar{F}^2(x). \tag{4.19}$$

(ii) *Suppose* $1 - F(x) \in \mathrm{ORV}$ *and that* $F(x)$ *satisfies* $F^2(x) - F^{*2}(x) = o(1)W(x)$. *Suppose* $G(x)$ *is a d.f. such that* (4.18) *holds with* $r(x)/s(x) \to c \in \mathbf{R}$ *where* $s(x) \in \mathrm{ORV} \cap L$ *and* $s(x) \to 0$. *Then*

$$G^2(x) - G^{*2}(x) = o(1)W(x) + o(1)s(x)\bar{F}(x) + o(1)\bar{F}(x)\sqrt{W(x)}. \tag{4.20}$$

*Moreover, if also* $|r(x)| \leq K\bar{F}(x)$ *and* $r(x)/\bar{F}(x) \to c \in \mathbf{R}$ *with* $1-F(x) \in \mathrm{ORV} \cap L$, *then* (4.20) *can be replaced by*

$$G^2(x) - G^{*2}(x) = o(1)W(x) + o(1)\bar{F}^2(x). \tag{4.21}$$

*Remarks.* (1) Proposition 4.8 and its proof remain valid if we start from the assumption $\bar{G}(x) = a\bar{F}(x) + r(x)\bar{F}(x)$ where $a > 0$.

(2) If $r(x) = O(1)(1 - F(x))$, Proposition 4.7 shows that the assumption $1 - F(x) \in \mathrm{ORV}$ is superfluous.

*Proof.* Before proving the results we rewrite $G^2(x) - G^{*2}(x)$ in a more suitable form. We have

$$G^2(x) - G^{*2}(x) = 2\int_0^{x/2} (\bar{G}(x-y) - \bar{G}(x))dG(y) + (G(x) - G(x/2))^2 =: I + T_1. \tag{4.22}$$

Using (4.18) we have

$$\begin{aligned} I = {} & 2\int_0^{x/2} (1 + r(x-y))(\bar{F}(x-y) - \bar{F}(x))dG(y) + 2\bar{F}(x)\int_0^{x/2} r(x-y)dG(y) \\ & - 2\bar{F}(x)r(x)G(x/2) =: II + T_2 - T_3. \end{aligned}$$

Since by assumption, $r(x) = o(1)$, $II$ can be replaced by

$$II = 2(1 + o(1))\int_0^{x/2} (\bar{F}(x-y) - \bar{F}(x))dG(y) =: 2(1 + o(1))III.$$

Using partial integration and then (4.18) we obtain

$$
\begin{aligned}
III &= \int_{x/2}^{x} (G(x/2) - G(x-z))\, dF(z) \\
&= \int_{x/2}^{x} (F(x/2) - F(x-z))\, dF(z) \\
&\quad + \int_{x/2}^{x} r(x-z)\bar{F}(x-z) dF(z) - r(x/2)\bar{F}(x/2) \int_{x/2}^{x} 1\, dF(z).
\end{aligned}
$$

Using partial integration in the first term and then using the analogue of (4.22) for $F(x)$, this term can be replaced by $1/2(F^2(x) - F^{*2}(x)) - 1/2(F(x) - F(x/2))^2$. Hence $II$ can be replaced by $II = T_4 - T_5 + T_6 - T_7$, where

$$
T_4 = (1+o(1))(F^2(x) - F^{*2}(x)), \; T_6 = 2(1+o(1)) \int_{x/2}^{x} r(x-z)\bar{F}(x-z)\, dF(z)
$$

$$
T_5 = (1+o(1))(F(x) - F(x/2))^2, \; T_7 = 2(1+o(1))r(x/2)\bar{F}(x/2)(F(x) - F(x/2)).
$$

Combining the different terms we have $G^2(x) - G^{*2}(x) = T_1 + T_2 - T_3 + T_4 - T_5 + T_6 - T_7$.

*Proof.* (i) By assumption we have $T_4 = O(1)W(x)$ and $|T_3| \leq 2s(x)\bar{F}(x)$. Since $|r(x)| \leq s(x) \in$ ORV, we also obtain $|T_2| = O(1)\bar{F}(x)s(x)$. By using the $O$-regular variation of $s(x)$ and $1 - F(x)$ we obviously also have $|T_7| = O(1)s(x)\bar{F}^2(x)$. Using (4.18) we obtain $T_1 = O(1)\bar{F}^2(x)$. To analyse $T_5$ we use the inequality

$$
F^2(x) - F^{*2}(x) \geq (F(x) - F(x/2))^2 \tag{4.23}
$$

and (4.17) to obtain $T_5 = O(1)W(x)$. Finally consider $T_6$; using the boundedness of $r(x)$ we have

$$
\begin{aligned}
|T_6| &\leq K \int_{x/2}^{x} \bar{F}(x-z)\, dF(z) \\
&\leq K \left( \int_{x/2}^{x} (\bar{F}(x-z) - \bar{F}(x)) dF(z) + \bar{F}(x)(F(x) - F(x/2)) \right) \\
&\leq K(F^2(x) - F^{*2}(x)) + K\bar{F}(x)(F(x) - F(x/2)).
\end{aligned}
$$

Hence $T_6 = O(1)W(x) + O(1)\bar{F}^2(x)$. Combining these 7 estimates we obtain (4.19).

(ii) By assumption $T_4 = o(1)W(x)$ and as in part (i) we have $T_5 = o(1)W(x)$. By the conditions on $s(x)$ and $r(x)$ we have $T_2/(s(x)\bar{F}(x)) \to 2c$ and

$T_3/(s(x)\bar{F}(x)) \to 2c$. Also, since $\bar{F} \in \text{ORV}$ we have $T_7 = o(1)\bar{F}(x)s(x)$. Now consider $T_6$; as in part (i) we have $|T_6| \le K(F^2(x) - F^{*2}(x)) + K\bar{F}(x)(F(x) - F(x/2))$. Using (4.23) and $T_5 = o(1)W(x)$ we obtain $T_6 = o(1)W(x) + o(1)\bar{F}(x)\sqrt{W(x)}$. Finally consider $T_1$; using (4.18) we obtain

$$\begin{aligned}
T_1 &= ((1 + r(x))(F(x) - F(x/2)) + ((r(x) - r(x/2))\bar{F}(x/2))^2 \\
&= O(1)T_5 + O(1)(r(x) - r(x/2))(F(x) - F(x/2))\bar{F}(x/2) \\
&\quad + (r(x) - r(x/2))^2\bar{F}^2(x/2) \\
&= o(1)W(x) + o(1)\bar{F}^2(x) + o(1)s^2(x)\bar{F}^2(x).
\end{aligned}$$

Combining these 7 estimates we obtain (4.20).

In order to prove (4.21) we reconsider $T_6$; in the case where $|r(x)| \le K\bar{F}(x)$, we have

$$T_6 = O(1)\int_{x/2}^{x} \bar{F}^2(x - z)\, dF(z) = O(1)\left\{\bar{F}^2 * F(x) - \int_0^{x/2} \bar{F}^2(x - z) dF(z)\right\}.$$

First consider $\bar{F}^2 * F(x)$; we have

$$\begin{aligned}
\bar{F}^2 * F(x) - \bar{F}^2(x)F(x) &= \int_0^x (\bar{F}^2(x - z) - \bar{F}^2(x))\, dF(z) \\
&= O(1)\int_0^x (\bar{F}(x - z) - \bar{F}(x))\, dF(z) \\
&= O(1)(F^2(x) - F^{*2}(x)) = o(1)W(x)
\end{aligned}$$

As to the second term, since $1 - F(x) \in \text{ORV} \cap L$, an application of Lebesque's theorem yields $\int_0^{x/2} \bar{F}^2(x - z)\, dF(z)/\bar{F}^2(x) \to 1$. Combining these estimates we obtain $T_6 = o(1)W(x) + o(1)\bar{F}^2(x)$. Now (4.21) follows as before. □

Using a similar method of proof we also obtain

COROLLARY 4.9. *Under the conditions of Proposition* 4.8 *we have*

$$F(x)G(x) - F * G(x) = O(1)W(x) + O(1)s(x)\bar{F}(x) + O(1)\bar{F}^2(x).$$

*resp.*

$$F(x)G(x) - F * G(x) = o(1)W(x) + o(1)s(x)\bar{F}(x) + o(1)\bar{F}\sqrt{W(x)}.$$

*resp.*

$$F(x)G(x) - F * G(x) = o(1)W(x) + o(1)\bar{F}^2(x).\ \square$$

## 5. Asymptotic behavior of convolution products: the class $D(m,\alpha)$

In our next results we discuss the case where $F \in D(m,\alpha)$ and $G \in D(n,\beta)$ where $\alpha \geq 0$ and $\beta \geq 0$. If $\alpha$ is positive then automatically $m \in L$. If we define $U(x) = F(x) - \alpha M(x)$ where $M(x) = \int_0^x m(y)\,dy$, it follows that $U(x) \in D(m,0)$. Now consider $I = \int_0^{x/2}(F(x) - F(x-y))\,dG(y)$. Obviously we have

$$I = \int_0^{x/2} (U(x) - U(x-y))\,dG(y) + \alpha \int_0^{x/2} (M(x) - M(x-y))dG(y) = I_1 + \alpha I_2.$$

Using the approach of Proposition 4.2 we obtain $|I_1| \leq \varepsilon m(x)G_1(x)$ for $x \geq x_0$. Similarly we can write

$$II = \int_0^{x/2} (V(x) - V(x-y))\,dF(y) + \beta \int_0^{x/2} (N(x) - N(x-y))\,dF(y) = II_1 + \beta II_2.$$

where $V(x) = G(x) - \beta N(x)$ and $N(x) = \int_0^x n(y)\,dy$. Obviously $V \in D(n,0)$ and we obtain $|II_1| \leq \varepsilon n(x)F_1(x)$ for $x \geq x_0$. It remains to analyse $I_2$, $II_2$ and $III$. If $E(X)$ and $E(Y)$, the means corresponding to the distribution functions $F(x)$ and $G(x)$, are finite, the analysis is easy.

**Part 1: Finite-means case.** THEOREM 5.1. *Suppose $F(x) \in D(m,\alpha)$, $G(x) \in D(n,\beta)$, $E(X) + E(Y) < \infty$ and $m,n \in L \cap$ ORV. Then $F(x)G(x) - F{*}G(x) = \alpha E(Y)m(x) + \beta E(X)n(x) + o(m(x)) + o(n(x))$.*

*Proof.* Let us consider $I_2$; using partial integration we have

$$\begin{aligned} I_2 &= \int_0^{x/2}\int_{x-y}^{x} m(z)\,dz\,dG(y) = \int_{z=x/2}^{x}\int_{y=x-z}^{x/2} dG(y)m(z)\,dz \\ &= \int_{z=0}^{x/2} (\bar{G}(z) - \bar{G}(x/2))m(x-z)dz = \int_0^{x/2} \bar{G}(z)m(x-z)\,dz - \bar{G}(x/2)\int_{x/2}^{x} m(z)\,dz. \end{aligned}$$

Since $m \in L \cap$ ORV we obtain

$$\begin{aligned} \lim \frac{I_2}{m(x)} &= \int_0^{\infty} \bar{G}(z)\,dz - \lim x\bar{G}(x/2)\lim\int_{1/2}^{1} \frac{m(xz)}{m(x)}dz \\ &= E(Y) - 0\cdot O(1) = E(Y). \end{aligned}$$

Similarly we have $\lim II_2/n(x) = E(X)$.

Next consider $III = (F(x) - F(x/2))(G(x) - G(x/2))$. Using Proposition 3.1.1(iii) and the finiteness of $E(X)$ and $E(Y)$ we have

$$III = O(1)xm(x)(G(x) - G(x/2)) = o(1)m(x);$$

also $III = O(1)xn(x)(F(x) - F(x/2)) = o(1)n(x)$ holds. Combining these estimates, we obtain the desired result. □

COROLLARY 5.2. *Suppose $F \in D(m,\alpha)$, $E(X) < \infty$ and $m \in L \cap \mathrm{ORV}$. Then for all $n \geq 2$,*

$$\lim \frac{F^n(x) - F^{*n}(x)}{m(x)} = 2\alpha \binom{n}{2} E(X). \tag{5.1}$$

*Moreover, if $\alpha(m) < -1$, then for all $n \geq 2$,*

$$\lim \frac{R_n(x)}{m(x)} = \lim \frac{1 - F^{*n}(x) - n(1 - F(x))}{m(x)} = 2\alpha \binom{n}{2} E(X). \tag{5.2}$$

*Proof.* The result for $n = 2$ follows from Theorem 5.1. For $n > 2$, we use Corollary 3.3.2 to obtain $F^{*n} \in D(m, n\alpha)$. Since $\int_0^\infty (1 - F^{*n}(x))\,dx = nE(X)$, Theorem 5.1 applies again and we obtain

$$\lim \frac{F^{*n}(x)F(x) - F^{*n+1}(x)}{m(x)} = n\alpha E(X) + n\alpha E(X) = 2n\alpha E(X).$$

Finally note that

$$F^{n+1}(x) - F^{*n+1}(x) = F(x)(F^n(x) - F^{*n}(x)) + F(x)F^{*n}(x) - F^{*n+1}(x),$$

so that

$$\lim \frac{F^{n+1}(x) - F^{*n+1}(x)}{m(x)} = \lim \frac{F^n(x) - F^{*n}(x)}{m(x)} + 2n\alpha E(X).$$

The result (5.1) now follows by induction on $n$.

To prove (5.2) we write

$$R_n(x) = F^n(x) - F^{*n}(x) + \frac{1 - F^n(x) - n(1 - F(x))}{(1 - F(x))^2}(1 - F(x))^2.$$

Using (5.1) we obtain

$$R_n(x) = 2\alpha \binom{n}{2} E(X)m(x) - \binom{n}{2} \bar{F}^2(x) + o(1)m(x) + o(1)\bar{F}^2(x). \tag{5.3}$$

To prove (5.2) we use Proposition 3.2.1 (iv) to obtain $1 - F(x) = O(1)xm(x)$. Using $E(X) < \infty$ this gives $\bar{F}^2(x) = O(1)m(x)x\bar{F}(x) = o(1)m(x)$. Relation (5.2) now follows from (5.3). □

*Remarks.* (1) Since $R_{n+1}(x) = nR_2(x) + R_n * F(x)$ the previous result implies that $\lim(R_n * F(x))/m(x) = 2\alpha\binom{n}{2}E(X)$. If $\alpha > 0$, this implies that $F \in D(m)$ and Proposition 2.3 applies.

(2) If $\alpha > 0$ and $m(x) = o(\bar{F}^2(x))$ or $\alpha = 0$ and $m(x) = O(\bar{F}^2(x))$, it follows from (5.1) with $n = 2$ that $\lim(F^2(x) - F^{*2}(x))/\bar{F}^2(x) = 0$. Using the inequality

$F^2(x) - F^{*2}(x) \geq (F(x) - F(x/2))^2$ we obtain $\lim \bar{F}(x/2)/\bar{F}(x) = 1$. This implies $1 - F(x) \in \mathrm{RV}_0$ and hence $E(X) = \infty$ which we excluded here. This remark shows that in many cases $m(x)$ will be the dominant term in (5.3).

(3) Under the conditions of Corollary 5.2, the special choice $G(x) = 1 - \bar{F}^2(x)$ yields $G(x) \in D(m, 0)$. An application of Theorem 5.1 yields

$$\lim \frac{\bar{F}^2(x) * F(x) - \bar{F}^2(x)F(x)}{m(x)} = \alpha \int_0^\infty \bar{F}^2(s)\, ds.$$

This result should be compared with Corollary 2.4, Proposition 4.1 and the proof of Proposition 4.7.

The special choice $m(x) = (1 - F(x))/x$ in Corollary 5.2 yields

COROLLARY 5.3. *Suppose* $F \in D(m, \alpha)$ *with* $m(x) = (1 - F(x))/x$*, and assume* $E(X) < \infty$*. Then for all* $n \geq 2$*,* (5.1) *and* (5.2) *hold.*

*Proof.* From Proposition 3.1.2 it follows that $1 - F(x) \in \mathrm{RV}_{-\alpha}$, $\alpha \geq 1$ and consequently that Corollary 5.2 is applicable. □

**Part 2: Infinite-means case.** If $E(X) = E(Y) = \infty$ we shall assume $F \in D(m, \alpha)$ and $G \in D(n, \beta)$ where the auxiliary functions $m(x)$, $n(x)$ are regularly varying. Recall the following results of Proposition 3.1.2.

PROPOSITION 5.4 *Suppose* $F \in D(m, \alpha)$, $G \in D(n, \beta)$ *with* $m \in \mathrm{RV}_\delta$ *and* $n \in \mathrm{RV}_\rho$.

(i) *If* $\delta + 1 < 0$, *(resp.* $= 0$, *resp.* $> 0$*), then* $\bar{F}(x)/xm(x) \to -\alpha/(1+\delta)$ *(resp.* $F \in \Pi_\alpha(xm(x))$*, resp.* $\alpha = 0$*).*

(ii) *If* $\rho + 1 < 0$, *(resp.* $= 0$, *resp* $> 0$*), then* $\bar{G}(x)/xn(x) \to -\beta/(1+\rho)$ *(resp.* $G \in \Pi_\beta(xn(x))$*, resp.* $\beta = 0$*).* □

Using the decompositions (4.1) and $I = I_1 + \alpha I_2$ of the beginning of Section 5, we proceed with estimating $I_2$. Using partial integration we have

$$\begin{aligned} I_2 &= \int_{y=0}^{x/2} \int_{z=x-y}^{x} m(s)\, ds\, dG(y) = \int_{s=x/2}^{x} \int_{y=x-s}^{x/2} dG(y)\, ds \\ &= \int_0^{x/2} (\bar{G}(s) - \bar{G}(x/2)) m(x-s) ds \\ &= \int_0^{x/2} \bar{G}(s) m(x-s)\, ds - \bar{G}(x/2) \int_0^{x/2} m(x-s)\, ds =: A - B. \end{aligned}$$

We consider the cases $\rho + 1 < 0$ and $\rho + 1 = 0$ separately.

*Case* 1: $\rho + 1 < 0$. For $\varepsilon > 0$ we write

$$A = x\int_0^{1/2} \bar{G}(xs)m(x(1-s))\,ds = x\left\{\int_0^{\varepsilon} + \int_{\varepsilon}^{1/2}\right\}\bar{G}(xs)m(x(1-s))\,ds =: A_1 + A_2.$$

Using the regular variation of $m$ and $n$ and uniform convergence, we have

$$\lim \frac{A_2}{x^2 n(x)m(x)} = -\frac{\beta}{1+\rho}\int_{\varepsilon}^{1/2} s^{1+\rho}(1-s)^{\delta}ds. \tag{5.4}$$

Next consider $A_1$; using uniform convergence we have $m(x(1-t))/m(x) \to (1-t)^{\delta}$ uniformly in $0 \le t \le \varepsilon$. For $x$ sufficiently large, this implies

$$(1-\varepsilon)xm(x)\int_0^{\varepsilon} \bar{G}(xs)\,ds \le A_1 \le (1+\varepsilon)xm(x)\int_0^{\varepsilon}\bar{G}(xs)ds.$$

Now we estimate $x\int_0^{\varepsilon}\bar{G}(xs)\,ds = \int_0^{\varepsilon x}\bar{G}(s)ds$. Since $E(Y) = \infty$ we have

$$\int_0^{\infty} yn(y)\,dy = \infty$$

and $\lim \int_0^{\varepsilon x}\bar{G}(t)\,dt/\int_0^{\varepsilon x} yn(y)dy = -\beta/(1+\rho)$. If $\rho + 1 > -1$, this shows that $\int_0^{\varepsilon x}\bar{G}(t)\,dt \sim \frac{-\beta}{1+\rho}x^2 n(x)\varepsilon^{2+\rho}$ and hence that $A_1 = O(1)\varepsilon^{2+\delta}$. Now combine the estimates for $A_1$ and $A_2$ and let $\varepsilon \to 0$ to obtain

$$\lim \frac{A}{x^2 m(x)n(x)} = -\frac{\beta}{1+\rho}\int_0^{1/2} s^{1+\rho}(1-s)^{\delta}ds. \tag{5.5}$$

If $\rho+1 = -1$, we have $\int_0^{\varepsilon x}\bar{G}(t)\,dt \sim \beta\int_0^{\varepsilon x} yn(y)\,dy$. Since in this case $\int_0^x yn(y)\,dy \in \Pi(x^2 n(x))$ we have $x^2 n(x) = o(1)\int_0^x yn(y)\,dy$ and we obtain

$$(1-\varepsilon)\beta \le \lim\begin{matrix}\sup\\ \inf\end{matrix}\frac{A_1}{m(x)\int_0^x yn(y)\,dy} \le (1+\varepsilon)\beta.$$

Moreover, from (5.4) we have $A_2 = o(m(x)\int_0^x yn(y)\,dy)$. Combining the two estimates and then letting $\varepsilon \to 0$, we obtain

$$A = (\beta + o(1))(m(x)\int_0^x yn(y)\,dy). \tag{5.6}$$

Next we consider $B$; using Proposition 5.4 and the regular variation of $m$ and $n$ we obtain

$$B \sim \frac{-\beta}{1+\rho}x^2 n(x)m(x)\left(\frac{1}{2}\right)^{1+\rho}\int_0^{1/2}(1-t)^{\delta}dt. \tag{5.7}$$

If $\rho+1=-1$, this yields

$$B = o(m(x)\int_0^x yn(y)\,dy). \tag{5.8}$$

Combining (5.5) and (5.7) or (5.6) and (5.8) we obtain

$$(5.9.a)\qquad \begin{cases} \text{if } \rho+1>-1: & \lim I_2/(x^2m(x)n(x)) = \beta \int_{t=0}^{1/2}\int_{z=t}^{1/2} z^\rho(1-t)^\delta dz dt \\ \text{if } \rho+1=-1: & \lim I_2/(m(x)\int_0^x yn(y)\,dy) = \beta. \end{cases}$$

Note that $\rho+1<-1$ is not possible here since we assumed $E(Y)=\infty$.

*Case* 2: $\rho+1=0$. In this case for $\varepsilon>0$ we have

$$\begin{aligned} I_2 &= x\int_0^{1/2}(\bar{G}(xs)-\bar{G}(x/2))m(x(1-s))ds \\ &= x\left\{\int_0^{\varepsilon}+\int_{\varepsilon}^{1/2}\right\}(\bar{G}(xs)-\bar{G}(x/2))m(x(1-s))ds =: A+B. \end{aligned}$$

Using uniform convergence, we obtain

$$\lim B/(x^2m(x)n(x)) = -\beta\int_{\varepsilon}^{1/2}\log(2s)(1-s)^\delta ds.$$

As to $A$, we have $m(x(1-s))/m(x)\to(1-s)^\delta$ uniformly in $0\le s\le\varepsilon$. Hence

$$0\le\frac{A}{m(x)}\le Kx\int_0^{\varepsilon}(\bar{G}(xs)-\bar{G}(x/2))ds =: KC(x).$$

Now $C(x)=\varepsilon xG(x/2)-\int_0^{\varepsilon x}G(t)\,dt$ and $C(x)/x\varepsilon = G(x/2)-G(x\varepsilon)+G(x\varepsilon)-\int_0^{\varepsilon x}G(t)\,dt$. Since $G\in\Pi_\beta(xn(x))$ we obtain (cf. [**5**,**8**]) $\lim G(x)/(x\varepsilon)^2n(x\varepsilon) = \beta-\beta\log(2\varepsilon)$ and hence $0\le\limsup A/x^2n(x)m(x)\le K\varepsilon(\beta-\beta\log(2\varepsilon))$. Now combine the estimates for $A$ and $B$ and let $\varepsilon\to0$ to obtain

$$\lim\frac{I_2}{x^2m(x)n(x)} = -\beta\int_0^{1/2}\log(2s)(1-s)^\delta ds = \beta\int_0^{1/2}\int_t^{1/2}z^{-1}dz(1-t)^\delta dt. \tag{5.9.b}$$

As a second step, we estimate $I_1$. At the beginning of Section 5 we obtained $|I_1|\le\varepsilon m(x)G_1(x)$. The classical properties of regularly varying functions show that for $-1<\rho+1\le0$, $G_1(x)/x^2n(x)\to\beta/(2+\rho)$ and for $\rho+1=-1$, that

$G_1(x)/\left(\int_0^x yn(y)\,dy\right) \to \beta$. Since $\varepsilon$ was arbitrary, using (5.9.a) and (5.9.b) we obtain

$$(5.10)\quad \begin{cases} \text{if } -1<\rho+1\le 0: & \lim I/x^2m(x)n(x) = \alpha\beta \int\limits_{t=0}^{1/2}\int\limits_{t=z}^{1/2} z^\rho(1-t)^\delta\,dz\,dt \\ \text{if } \rho+1=-1: & \lim I/(m(x)\int\limits_0^x yn(y)\,dy) = \alpha\beta. \end{cases}$$

For the second term II in (4.1), we obtain in a similar way that

$$(5.11)\quad \begin{cases} \text{if } -1<\delta+1\le 0: & \lim II/x^2m(x)n(x) = \alpha\beta \int\limits_{t=0}^{1/2}\int\limits_{z=t}^{1/2} z^\delta(1-t)^\rho\,dz\,dt \\ \text{if } \delta+1=-1: & \lim II/(n(x)\int\limits_0^x ym(y)\,dy) = \alpha\beta. \end{cases}$$

As to the third term III in (4.1), using Proposition 5.4 we have

$$\begin{aligned} \lim \frac{III}{x^2m(x)n(x)} &= \lim \frac{(F(x)-F(x/2))(G(x)-G(x/2))}{xm(x)\ \ xn(x)} \\ &= \alpha\beta \int\limits_{1/2}^{1}\int\limits_{1/2}^{1} s^\delta t^\rho\,ds\,dt,\quad \delta+1\le 0, \rho+1\le 0. \end{aligned} \tag{5.12}$$

Combining (5.10) up to (5.12) we summarize our findings in the following

THEOREM 5.5. *Suppose* $F\in D(m,\alpha)$, $G\in D(n,\beta)$, $E(X)=E(Y)=\infty$ *and* $m\in \mathrm{RV}_\delta$, $n\in \mathrm{RV}_\rho$ *with* $\delta+1\le 0$ *and* $\rho+1\le 0$. *Let* $R(x)=F(x)G(x)-F*G(x)$.

(i) *If* $-1<\delta+1$, $\rho+1\le 0$, *then*

$$\frac{R(x)}{x^2m(x)m(x)} \to \alpha\beta\left\{\int\limits_0^{1/2}\int\limits_t^{1/2}(z^\rho(1-t)^\delta + z^\delta(1-t)^\rho)\,dz\,dt + \int\limits_{1/2}^{1}\int\limits_{1/2}^{1} z^\delta t^\rho\,dz\,dt\right\}.$$

(ii) *If* $\delta+1=\rho+1=-1$, *then*

$$R(x) = (\alpha\beta+o(1))m(x)\int\limits_0^x yn(y)\,dy + (\beta\alpha+o(1))n(x)\int\limits_0^x ym(y)\,dy.$$

(iii) *If* $\delta+1=-1$, $-1<\rho+1\le 0$ *(resp.* $\rho+1=-1$, $-1<\delta+1\le 0$*), then*

$$R(x) = (\alpha\beta+o(1))n(x)\int\limits_0^x ym(y)\,dy\left(\textit{resp. } m(x)\int\limits_0^x yn(y)dy\right). \ \square$$

*Remarks.* (1) In Theorem 5.5(i), the limit can be rewritten as

$$\alpha\beta C(\rho,\delta) = \alpha\beta \int_{t=0}^{1} \int_{z=0}^{t} (1-z)^\rho t^\delta \, dz \, dt.$$

Using the hypergeometric function (cf. [**1**]) we have

$$\begin{aligned}(\rho+1)(\delta+1)C(\rho,\delta) &= 1 - F(\delta+1; -\rho-1; \delta+2; z=1) \\ &= 1 - \frac{\Gamma(\delta+2)\Gamma(\rho+2)}{\Gamma(\delta+\rho+3)}(-1 < \rho+1, \delta+1 < 0).\end{aligned} \tag{5.12}$$

For $\rho = \delta = -1$, we have $C(-1,-1) = \pi^2/6$ and for $\rho = -1$; $-1 < \delta + 1 < 0$, we have $C(-1,\delta) = \gamma + \Gamma'(\delta+2)/\Gamma(\delta+2)$ where $\gamma$ is Eulers' constant.

(2) If $m(x) = n(x) \in \mathrm{RV}_\delta$, $-1 \le \delta + 1 \le 0$ we have $x^2 m(x) \sim (\delta+2) \times \int_0^x ym(y)\,dy$ and Theorem 5.5 yields

$$R(x) = (\alpha\beta K(\delta) + o(1)) \left( m(x) \int_0^x ym(y)dy \right) \tag{5.13}$$

where $K(\delta) = 2$ if $\delta = -2$ and $K(\delta) = (2+\delta)C(\delta,\delta)$ if $-1 < \delta + 1 \le 0$.

In the case where $F(x) = G(x)$ we obtain:

COROLLARY 5.6. *Suppose $F \in D(m,\alpha)$, $E(X) = \infty$ and $m(x) \in \mathrm{RV}_\delta$, $-2 \le \delta \le -1$. Then for all $n \ge 2$,*

$$\lim \frac{F^n(x) - F^{*n}(x)}{A(x)} = \alpha^2 K(\delta) \binom{n}{2} \tag{5.14}$$

*where $A(x) = m(x) \int_0^x ym(y)\,dy$, and*

$$\lim \frac{R_n(x)}{A(x)} = \alpha^2 \binom{n}{2} \left( K(\delta) - \frac{\delta+2}{(\delta+1)^2} \right), \qquad \text{if } \delta + 1 < 0 \tag{5.15a}$$

$$\lim \frac{R_n(x)}{\bar{F}^2(x)} = -\binom{n}{2} \qquad \text{if } \delta + 1 = 0. \tag{5.15b}$$

*Proof.* If $n = 2$, (5.12) is the content of Theorem 5.5, cf. (5.13). For $n \ge 2$, we can use Corollary 3.3.2 to see that $F^{*n}(x) \in D(m, n\alpha)$. Another application of (5.13) yields

$$F^{*n}(x)F(x) - F^{*n+1}(x) = (n\alpha^2 K(\delta) + o(1))A(x).$$

Since $F^{n+1}(x) - F^{*n+1}(x) = F(x)(F^n(x) - F^{*n}(x)) + F^{*n}(x)F(x) - F^{*n+1}(x)$, we obtain

$$\lim \frac{F^{n+1}(x) - F^{*n+1}(x)}{A(x)} = \lim \frac{F^n(x) - F^{*n}(x)}{A(x)} + n\alpha^2 K(\delta).$$

By induction on $n$, we obtain (5.14). To prove (5.15), we rewrite $R_n(x)$ as follows:

$$R_n(x) = F^n(x) - F^{*n}(x) + \frac{1 - F^n(x) - n(1 - F(x))}{\bar{F}^2(x)} \bar{F}^2(x). \tag{5.16}$$

If $\delta+1 < 0$, Proposition 5.4 gives $\lim \bar{F}^2(x)/x^2m^2(x) = \alpha^2/(1+\delta)^2$ and this implies $\bar{F}^2(x)/A(x) \to \alpha^2(\delta+2)/(1+\delta)^2$. Combining this with (5.16) and (5.14) we obtain (5.15a). If $\delta + 1 = 0$, Proposition 5.4 implies $\lim x^2m^2(x)/\bar{F}^2(x) = 0$ and now (5.15b) follows again from (5.14) and (5.16). □

*Remark.* Using (5.12) and the definition of $K(\delta)$, the relation (5.15a) can be rewritten as

$$\lim \frac{R_n(x)}{A(x)} = \alpha^2 \binom{n}{2} L(\delta) \tag{5.17}$$

where $L(\delta) = 2$ if $\delta = -2$ and $L(\delta) = \frac{(2+\delta)(\Gamma(\delta+2))^2}{(1+\delta)^2\Gamma(2\delta+3)}$ if $-1 < \delta + 1 < 0$. Note that the limit in (6.17) is zero only if $\alpha = 0$ and/or if $\delta = -3/2$.

A combination of Corollary 5.2 and Corollary 5.6 yields the following corollary, which unifies the results of this section.

COROLLARY 5.7. *Suppose $F \in D(m, \alpha)$, $\alpha \neq 0$ and suppose that:* (i) $E(X) < \infty$ *and* $m \in L \cap \mathrm{ORV}$ *or* (ii) $E(X) = \infty$ *and* $m \in \mathrm{RV}_\delta$, $\delta \neq -3/2$. *Then for all $n \geq 2$, $\lim R_n(x)/R_2(x) = \binom{n}{2}$.* □

In the next corollary we complete Corollary 5.3.

COROLLARY 5.8. *Suppose $F \in D(m, \alpha)$ with $m(x) = (1 - F(x))/x$ and $E(X) = \infty$. Then $1 - F(x) \in \mathrm{RV}_{-\alpha}$ and the results of Corollary 5.6 hold with $\delta = -\alpha - 1$.* □

**Part 3: mixed case.** If $E(X) = \infty$ and $E(Y) < \infty$ we can combine the efforts of part 1 and part 2 to estimate $F(x)G(x) - F{*}G(x)$. We shall prove

THEOREM 5.9. *Suppose $F \in D(m, \alpha)$, $G \in D(n, \beta)$, $E(X) = \infty$, $E(Y) < \infty$. Also assume $m \in \mathrm{RV}_\delta$, $-1 \leq \delta + 1 \leq 0$ and $n \in \mathrm{RV}_\rho$, $\rho < -2$. Then*

$$\lim(F(x)G(x) - F * G(x))/m(x) = \alpha E(Y).$$

*If $\rho = \delta = -2$ and $m(x) = (c + o(1))n(x) \int_0^x ym(y)\,dy$ $(0 \leq c < \infty)$, then*

$$\lim \frac{(F(x)G(x) - F * G(x))}{n(x) \int_0^x ym(y)\,dy} = \alpha\beta + c\alpha E(Y).$$

*Proof.* Using the methods and notations of Parts 1 and 2, we readily obtain the following estimates:

(1) $\lim I/m(x) = \alpha E(Y)$;

(2) $\lim II/(x^2m(x)n(x)) = \alpha\beta$ Const, if $-1 < \delta + 1 \leq 0$, and

$\lim II \Big/ \left(n(x) \int_0^x ym(y)\,dy\right) = \alpha\beta$ if $\beta = -2$;

(3) $\lim III/x^2m(x)n(x) = \alpha\beta$ Const.

If $\rho = \delta = -2$, the desired result follows; if $\rho < -2$ and $-1 < \delta + 1 \leq 0$ we have $x^2m(x)n(x)/m(x) = x^2n(x) \to 0$ and the desired result follows. □

## 6. Examples and applications

**6.1. Examples.** 6.1.1. Assume that $F(x)$ has a density $f(x) = O(1)m(x)$ (resp. $o(1)m(x)$) where $m(x) \in \mathrm{ORV}$. In this case $F \in \mathrm{OD}(m)$ and Corollary 4.3 applies. Now assume $G(x)$ is the d.f. defined by $\bar{G} = [1/\bar{F}(x)]^{-1}$, where as usual $[x]$ denotes the integer part of $x$. It is readily seen that $\bar{G}(x) = \bar{F}(x) + (1+o(1))\bar{F}^2(x)$ and if $1 - F(x) \in L$, Proposition 4.7 yields $G^2(x) - G^{*2}(x) = O(1)m(x)F_1(x) + O(1)\bar{F}^2(x)$ (resp. $o(1)m(x)F_1(x) + o(1)\bar{F}^2(x)$).

6.1.2. If $\delta = -3/2$, then (5.17) shows that $R_n(x) = o(1)A(x)$ and it seems to be necessary to consider a third-order approximation here. A partial answer to this has been given in [**12**, Lemma 2.5] where we considered stable distributions on $\mathbf{R}_+$. If $F$ is stable with index $\beta = 1/2$, then $F \in D(m(x) = x^{-3/2}, \alpha)$ and (5.17) shows that $R_n(x) = o(1)\bar{F}^2(x)$. In [**12**, Lemma 2.5] we proved that $\lim R_n(x)/R_2(x) = \binom{n+1}{3}$. If $F$ is stable with index $\beta$, $0 < \beta < 1$, $\beta \neq 1/2$ we also showed [**12**, p. 349] that for some constant $k(\beta)$,

$$\lim \frac{R_n(x) - \binom{n}{2}R_2(x)}{\bar{F}^3(x)} = \binom{n}{3}k(\beta),$$

which gives a rate of convergence result in Corollary 5.7.

6.1.3. Theorem 5.9 is applicable in the following situation. Suppose that $F(x) \in D(\bar{F}^p(x), \alpha)$ $(p > 1)$. In this case it is easily verified that the d.f. $G(x)$ where $G(x) := 1 - \bar{F}^p(x)$, belongs to the class $D(\bar{F}^{2p-1}(x), p\alpha)$. If $\alpha \neq 0$, Proposition 3.2.1 implies that $\bar{F}(x)/\int_x^\infty \bar{F}^p(y)\,dy \to \alpha$ and it easily follows that $\bar{F}(x) \sim Cx^{-1/(p-1)}$ where $C$ denotes some positive constant. If $p > 2$, we can apply Theorem 5.9 to obtain

$$\lim \frac{(F(x)G(x) - F * G(x))}{m(x)} = \alpha \int_0^\infty \bar{F}^p(y)dy$$

and hence

$$\lim \frac{\bar{F}^p * F(x)}{\bar{F}^p(x)} = 1 + \alpha \int_0^\infty \bar{F}^p(y)\,dy.$$

**6.2. Subordinate probability distributions.** Suppose $F$ is a d.f.on $\mathbf{R}$ and $\{p_n\}_N$ the probability distribution of an integer-valued r.v. $N$. The d.f. $H(x)$ defined by $H(x) = \sum_{n=0}^\infty p_n F^{*n}(x)$ is called subordinated to $F$ with subordinator $\{p_n\}_N$. Such type of d.f. arise in many stochastic models, see e.g. [**18**] and the references given there. In studying the asymptotic behavior of $1 - H(x)$ it is wellknown that subexponential d.f. play an essential role. Here we focus on the second-order behavior of $1 - H(x)$. Clearly we have

$$R(x) := 1 - H(x) - E(N)(1 - F(x)) = \sum_{n=2}^\infty p_n R_n(x) \tag{6.1}$$

and Proposition 2.3 will allow us to use Lebesgue's theorem. Under appropriate conditions on the sequence $\{p_n\}_N$ it follows from Proposition 2.3 and (6.1) that

$$R(x) = O(1)m(x). \tag{6.2}$$

Using Corollary 5.7 we obtain an asymptotic equality instead of the $O$-estimate (6.2).

THEOREM 6.1. *Suppose $F(x)$ satisfies the conditions of Corollary 5.7 and assume $\sum_0^\infty p_n x^n$ is analytic at $x = 1$. Then* $\lim R(x)/R_2(x) = E\binom{N}{2}$. □

This result unifies the results of [**12, 13, 6**].

Classical examples are the Compound Poisson and Compound Geometric d.f.

**6.3. Infinitely divisible d.f.** Our next application is devoted to the relation between the tail of an infinitely divisible d.f. (i.d.) and its Levy measure $\nu$. If $F$ is i.d. with Levy measure $\nu$ we set $\lambda = \nu([1,\infty))$ and define the d.f. $Q(x)$ as $Q(x) = \lambda^{-1}\nu([1,x])$ $(x \geq 1)$. It is known (see e.g. [**3**]) that $F$ can be written as

$$F(x) = U{*}V(x) \tag{6.3}$$

where $V(x) = e^{\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}Q^{*n}(x)$ and where $U(x)$ is a d.f. satisfying for all $\varepsilon > 0$,

$$1 - U(x) = o(1)e^{-\varepsilon x}. \tag{6.4}$$

The d.f. $V(x)$ being compound Poisson, Theorem 6.1 applies and we obtain

$$\lim_{\infty}\frac{1 - V(x) - \lambda(1 - Q(x))}{R_{2,Q}(x)} = \frac{\lambda^2}{2} \tag{6.5}$$

where $R_{2,Q}(x) = 1 - Q^{*2}(x) - 2(1 - Q(x))$. To obtain an asymptotic result for $F$, we first estimate $U(x)V(x) - U{*}V(x)$. Using (4.1) we have

$$\begin{aligned} U(x)V(x) - U * V(x) &= \int_0^{x/2} (U(x) - U(x-y))\,dV(y) \\ &+ \int_0^{x/2} (V(x) - V(x-y))\,dU(y) \\ &+ (U(x) - U(x/2))(V(x) - V(x/2)) =: I + II + III. \end{aligned}$$

Using (6.4), for each $\varepsilon > 0$ we have $I = o(1)e^{-\varepsilon x/2}$ and $III = o(1)e^{-\varepsilon x/2}$. To estimate $II$ we shall apply Lebesgue's theorem. Under the conditons of Corollary 5.7 (for $Q(x)$) we have $Q(x) \in D(R_{2,Q}(x); 1/2E(Z))$, where $E(Z) = \int_0^\infty \bar{Q}(y)\,dy$ and $E(Z) \leq \infty$. Also $R_{2,Q}(x) \in L$. Using (6.5) we also have $V(x) \in D(R_{2,Q}(x); \lambda/(2E(Z)))$. Now we apply Lebesgue's theorem and obtain

$$\lim \frac{II}{R_{2,Q}(x)} = \frac{\lambda}{2E(Z)}\int_0^\infty y\,dU(y).$$

Hence

$$U(x)V(x) - U(x) * V(x) = \left( \frac{\lambda}{2E(Z)} \int_0^\infty y\, dU(y) + o(1) \right) R_{2,Q}(x).$$

Using this estimate and (6.5) we obtain

$$\lim \frac{1 - F(x) - \lambda(1 - Q(x))}{R_{2,Q}(x)} = \frac{\lambda^2}{2} \left( 1 + \frac{1}{\lambda E(Z)} \int_0^\infty y\, dU(y) \right). \tag{6.6}$$

If $E(Z) < \infty$ we have $R_{2,Q}(x) \sim 2m_Q(x)\alpha E(Z)$. In this case (6.6) can be replaced by

$$\lim \frac{1 - F(x) - \lambda(1 - Q(x))}{m_Q(x)} = \lambda\alpha \left( \lambda E(Z) + \int_0^\infty y\, dU(y) \right) = \lambda\alpha \int_0^\infty y\, dF(y).$$

Summarizing our results, we have proved:

THEOREM 6.3. (i) *Suppose $F$ is i.d. with Levy measure $\nu$. Suppose that $Q(x) \in D(m_Q, \alpha)$ where $Q(x) := \lambda^{-1}\nu([1, x])$ and where $m_Q(x) \in L \cap \mathrm{ORV}$. If the mean of $F$ is finite, then* $\lim(1 - F(x) - \nu([x, \infty)))/m_Q(x) = \lambda\alpha \int_0^\infty y\, dF(y)$.

(ii) *If the mean of $F$ is infinite and if $Q(x)$ satisfies the conditions of Corollary* 5.7(ii), *then* $\lim(1 - F(x) - \nu([x, \infty)))/R_{2,Q}(x) = \lambda^2/2$. □

**6.4. Concluding remarks.** 6.4.1. To obtain third order results we can analyse $Q_n(x)$ defined by $Q_n(x) = R_n(x) - \binom{n}{2}R_2(x)$ $(n \geq 3)$. It is not hard to show that $Q_{n+1}(x) = \binom{n}{2}Q_n(x) + Q_n * F(x)$ so that depending on $Q_3(x)$ several parts of Section 2 can be applied here. If $F(x) \in D(m)$ and $Q_3(x) = O(1)m(x)$, then Proposition 2.3 gives $Q_n(x) = O(1)m(x)$ for all $n \geq 3$. The exact asymptotic behaviour of $Q_3$ (and $Q_n$) has been analysed in [**9,18**] in the case where $F$ has a finite mean and a differentiable density. The infinite mean case will be treated in a forthcoming paper.

6.4.2. Several results of Sections 4 and 5 also hold for the convolution of densities. In [**14**] the behaviour of $f \otimes g(x) - f(x)g(x)$ has been analysed for $L^1[0, \infty)$-functions. The infinite mean case is treated in [**19**].

6.4.3. It seems to be of interest to study the class of d.f. $F(x)$ for which $R_n(x)/R_2(x)$ exists for all $n \geq 3$.

## REFERENCES


[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation*, Cambridge University Press, 1987.

[3] W. Feller, *An Introduction to Probability Theory and its Applications II*, J. Wiley, New York, 1971.

[4] J.B.G. Frenk, *On Renewal Theory, Banach Algebras and Functions of Bounded Increase*, Ph.D. thesis, Erasmus University Rotterdam, The Netherlands, 1983.

[5] J. Geluk and L. de Haan, *Regular Variation, Extensions and Tauberian Theorems*, CWI tract 40, Centre for Math. and Comp. Sc., M.C. Amsterdam, 1987.

[6] J.L. Geluk and A.G. Pakes, *Second-order subexponential distributions*, J. Austral. Math. Soc. (series A) **51** (1991), 73–87.

[7] J.L. Geluk, *Second-order tailbehaviour of a subordinated probability distribution*, Stochastic Processes Appl. **40** (1992), 325–337.

[8] L. de Haan, *On Regular Variation and its Applications to the Weak Convergence of Sample Extremes*, Mathematical Centre tracts 32, M.C. Amsterdam, 1970.

[9] E. Omey and E. Willekens, *Second-order behaviour of the the tail of a subordinated probability distribution*, K.U. Leuven Communications Wisk. Inst. 178, 1984.

[10] E.Omey, *Asymptotic properties of convolution products of sequences*, Publ. Inst. Math. (Beograd) (N.S.) **36(50)** (1984), 67–78.

[11] E. Omey and E. Willekens, *On the difference between distributions of sums and maxima*, in *Stability Problems for Stochastic Processes*, Lecture Notes in Mathematics 1233, Springer-Verlag, 1985, pp. 103–113.

[12] E. Omey and E. Willekens, *Second order behaviour of the tail of a subordinated probability distribution*, Stochastic Processes Appl. **21** (1986), 339–353.

[13] E. Omey and E. Willekens, *Second order behaviour of distributions subordinate to a distribution with finite mean*, Comm. Statist. Stoch. Models **3** (1987), 311–342.

[14] E. Omey, *Asymptotic properties of convolution products of functions*, Publ. Inst. Math. (Beograd) (N.S.) **43(57)** (1988), 41–57.

[15] E. Omey, *On a Subclass of Regularly Varying Functions*, J. Statist. Plann Inferenc, 1995, to appear.

[16] E.J.G. Pitman, *Subexponential distribution functions*, J. Austr. Math. Soc. **A29** (1980), 337-347.

[17] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics 508, Springer-Verlag, 1976.

[18] E. Willekens, *Hogere orde theorie voor subexponentiele verdelingen*, Ph. D. thesis (in Dutch), K.U. Leuven, 1986.

[19] E. Omey, *The difference between the convolution product and the sum of density functions,* In: V.M. Zolotarev, V.M. Krugov, V.Y. Korolov, (Eds), *Stability Problems for Stochastic Models, Proceedings the 15th Perm Seminar*, Moscow, TVP, 1993.

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