PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 5–9

# INDEPENDENT VERTEX SETS IN SOME COMPOUND GRAPHS

**Ivan Gutman**

**Abstract**. Let $G$ be an $n$-vertex graph and $R_1, R_2, \dots, R_n$ distinct rooted graphs. The compound graph $G[R_1, R_2, \dots, R_n]$ is obtained by identifying the root of $R_i$ with the $i$-th vertex of $G$, $i = 1, 2, \dots, n$. We determine the number of independent vertex sets and the independence polynomial of $G[R_1, R_2, \dots, R_n]$. Several special cases of these results are pointed out.

## 1. Introduction

Consider finite graphs without loops. If $G$ is such a graph, then $\boldsymbol{V}(G)$ will denote its vertex set. Any subset of $\boldsymbol{V}(G)$, such that no two elements of it are mutually adjacent, is called an independent vertex set of the graph $G$. Let $\mathrm{Ind}(G)$ be the set of all independent vertex sets of $G$.

The number $\sigma(G)$ of independent vertex sets of the graph $G$, i.e. the cardinality of $\mathrm{Ind}(G)$, has been examined in a number of recent papers [**1–11**]. In particular, Prodinger and Tichy [**7**, **11**] called the quantity $\sigma(G)$ "the Fibonacci number of the graph $G$". The motivation for this was the fact that if $P_n$ is the path-graph with $n$ vertices, then $\sigma(P_n)$ is equal to the $(n+1)$-th Fibonacci number.

In the present paper we determine the number of independent vertex sets of the compound graph $G[R_1, R_2, \dots, R_n]$ constructed in the following manner.

Let $G$ be a graph with the vertex set $\boldsymbol{V}(G) = \{v_1, v_2, \dots, v_n\}$. Let further $R_1, R_2, \dots, R_n$ be distinct rooted graphs; the root of $R_i$ is denoted by $r_i$, $i = 1, 2, \dots, n$. Then $G[R_1, R_2, \dots, R_n]$ is the graph obtained by identifying the vertex $v_i$ of $G$ with the root $r_i$ of $R_i$, simultaneously for $i = 1, 2, \dots, n$ (see Fig. 1).

Denote by $R_i^\circ$ the graph obtained by deleting from $R_i$ the root-vertex $r_i$ and the edges incident to it. Denote by $R_i^\bullet$ the graph obtained by deleting from $R_i$ the root-vertex $r_i$, the vertices adjacent to $r_i$ and all the incident edges. Then the main result of our work can be formulated as follows.

*AMS Subject Classification* (1985): Primary 05 C 70, 05 C 99

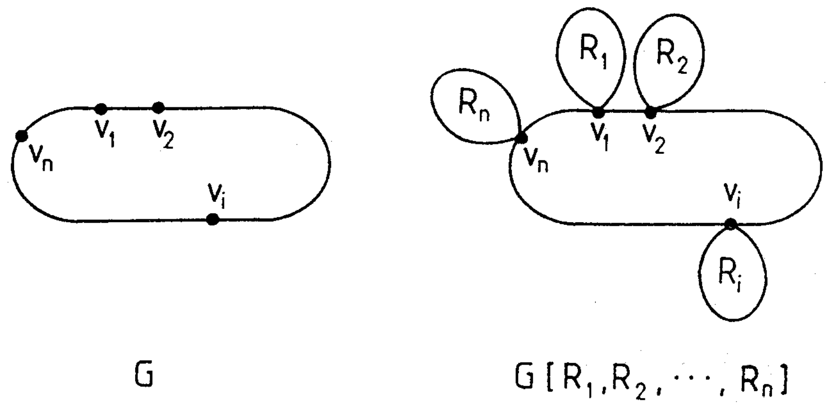


Fig. 1

THEOREM 1. *Let $I$ be an independent vertex set of the graph $G$. Define*

$$\sigma_i(I) = \begin{cases} \sigma(R_i^\circ) & \text{if } v_i \notin I, \\ \sigma(R_i^\bullet) & \text{if } v_i \in I. \end{cases}$$

*Then*

$$\sigma(G[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}(G)} \prod_{i=1}^{n} \sigma_i(I).$$

Instead of Theorem 1 we prove a somewhat stronger result, namely Theorem 2. In order to do this we need some preparations.

## 2. The independence polynomial

Denote by $n(G,k)$ the number of distinct $k$-element independent vertex sets of the graph $G$. Then the polynomial

$$\omega(G) = \omega(G,x) = \sum_{k \geq 0} n(G,k)\, x^k \tag{1}$$

is called the independence polynomial of the graph $G$ [**3**, **5**, **6**]. Evidently, $\omega(G,1) = \sigma(G)$.

The basic properties of the independence polynomial have been determined by Gutman and Harary [**5**] and recently by Hoede and Li [**6**].

Two of these properties will be needed in the subsequent considerations:

(a) If $v$ is a vertex of the graph $G$ and $N_v$ is the set containing $v$ and its first neighbors, then

$$\omega(G) = \omega(G-v) + x\, \omega(G - N_v). \tag{2}$$

(b) If $G_1 \cup G_2$ is the graph composed of components $G_1$ and $G_2$, then

$$\omega(G_1 \cup G_2) = \omega(G_1)\, \omega(G_2). \tag{3}$$

THEOREM 2. *Let $I$ be an independent vertex set of the graph $G$. Define*

$$\omega_i(I) = \begin{cases} \omega(R_i^\circ) & \text{if } v_i \notin I, \\ \omega(R_i^\bullet) & \text{if } v_i \in I. \end{cases} \tag{4}$$

*Then*

$$\omega(G[R_1, R_2, \dots, R_n], x) = \sum_{I \in \mathrm{Ind}(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I) \tag{5}$$

*where $|I|$ stands for the number of elements of $I$.*

Evidently, Theorem 1 is a special case of Theorem 2, obtained by setting $x = 1$ in formula (5).

## 3. Proof of Theorem 2

We demonstrate the validity of Theorem 2 by induction on the number $n$ of vertices of the graph $G$. If $n = 1$, then $\boldsymbol{V}(G) = \{v_1\}$ and therefore $G[R_1, R_2, \dots, R_n]$ coincides with $R_1$. On the other hand, for $n = 1$ the set $\mathrm{Ind}(G)$ consists of only two elements, namely $I_1 = \varnothing$ and $I_2 = \{v_1\}$. Bearing in mind (4) we have $\omega_1(I_1) = \omega(R_1^\circ)$ and $\omega_1(I_2) = \omega(R_1^\bullet)$. Consequently, the right-hand side of (5) is equal to $\omega(R_1^\circ) + x\,\omega(R_1^\bullet)$. Because of (2) this latter expression is equal to $\omega(G)$.

Thus the statement of Theorem 2 is true for $n = 1$. In a similar manner one can check that Theorem 2 is satisfied for $n = 2$ and $n = 3$.

Assume now that Theorem 2 holds for all graphs $G$ with less than $n$ vertices. In order to accomplish the inductive proof we have to show that this assumption implies the validity of Theorem 2 for the graphs $G$ having $n$ vertices.

Suppose that $n \geq 3$ and apply formula (2) to the vertex $v_n$ of the graph $G$. Without loss of generality we may label the vertices of $G$ so that $v_n$ is adjacent to $v_{n-1}, \dots, v_{n-d}$. Then by using (3),

$$\begin{aligned} \omega(G[R_1, R_2, \dots, R_n]) &= \omega(R_n^\circ)\,\omega((G - v_n)[R_1, R_2, \dots, R_n]) \\ &\quad + \omega(R_n^\bullet)\,\omega(R_{n-1}^\circ) \cdot \ldots \cdot \omega(R_{n-d}^\circ)\,\omega((G - N_{v_n})[R_1, R_2, \dots, R_n]). \end{aligned} \tag{6}$$

The subgraphs $G - v_n$ and $G - N_{v_n}$ have $n - 1$ and $n - 1 - d$ vertices, respectively. Therefore according to the induction hypothesis:

$$\omega((G - v_n)[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}(G - v_n)} x^{|I|} \prod_{i=1}^{n-1} \omega_i(I) \tag{7}$$

$$\omega((G - N_{v_n})[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}(G - N_{v_n})} x^{|I|} \prod_{i=1}^{n-d-1} \omega_i(I). \tag{8}$$

The set $\mathrm{Ind}(G)$ can be partitioned into two disjoint subsets $\mathrm{Ind}^\circ(G)$ and $\mathrm{Ind}^\bullet(G)$, such that $\mathrm{Ind}^\circ(G)$ is the set of independent vertex sets of $G$ which do not contain the vertex $v_n$ whereas $\mathrm{Ind}^\bullet(G)$ is the set of those independent vertex sets of $G$ which do contain $v_n$. It is easy to see that

$$\mathrm{Ind}^\circ(G) = \mathrm{Ind}(G - v_n) \tag{9}$$

$$\mathrm{Ind}^\bullet(G) = \big\{ I \cup \{v_n\} \mid I \in \mathrm{Ind}(G - N_{v_n}) \big\}. \tag{10}$$

Bearing in mind (7)–(10), the relation (6) is transformed into

$$\omega(G[R_1, R_2, \dots, R_n]) = \omega(R_n^\circ) \sum_{I \in \mathrm{Ind}^\circ(G)} x^{|I|} \prod_{i=1}^{n-1} \omega_i(I)$$
$$+ x\, \omega(R_n^\bullet)\, \omega(R_{n-1}^\circ) \cdot \ldots \cdot \omega(R_{n-d}^\circ) \sum_{I \in \mathrm{Ind}^\bullet(G)} x^{|I|-1} \prod_{i=1}^{n-d-1} \omega_i(I). \tag{11}$$

For all $I \in \mathrm{Ind}^\circ(G)$, $v_n \notin I$ and therefore $\omega_n(I) = \omega(R_n^\circ)$. For similar reasons, the relations $\omega_n(I) = \omega(R_n^\bullet)$ and $\omega_j(I) = \omega(R_j^\circ)$, $j = 1, \dots, d$, are satisfied for all $I \in \mathrm{Ind}^\bullet(G)$. Consequently, equation (11) becomes

$$\omega(G[R_1, R_2, \dots, R_n]) = \sum_{I \in \mathrm{Ind}^\circ(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I) + \sum_{I \in \mathrm{Ind}^\bullet(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I)$$

and formula (5) follows from the fact that $\mathrm{Ind}^\circ(G) \cup \mathrm{Ind}^\bullet(G) = \mathrm{Ind}(G)$. This completes the proof of Theorem 2.

## 4. Special cases

**4.1. All $R_i$ are isomorphic.** The graph $G[R_1, R_2, \dots, R_n]$ in which all $R_i$, $i = 1, 2, \dots, n$ are isomorphic to the rooted graph $R$ is denoted by $G[R]$. For the compound graphs $G[R]$ formula (5) is much simplified by the fact that the product $\prod_{i=1}^{n} \omega_i(I)$ depends only on the cardinality $k$ of the independent vertex set $I$ and is equal to $\omega(R - r)^{n-k}\, \omega(R - N_r)^k$ where $r$ stands for the root of $R$. Since the number of $k$-element independent vertex sets of the graph $G$ is equal to $n(G, k)$ we further have

$$\omega(G[R]) = \sum_{k \geq 0} x^k\, n(G, k)\, \omega(R - r)^{n-k}\, \omega(R - N_r)^k. \tag{12}$$

This, bearing in mind the definition of $\omega(G)$, immediately leads to Corollary 2.1.

Corollary 2.1. $\omega(G[R], x) = \omega(R - r, x)^n\, \omega(G, \omega(R - N_r)/\omega(R - r))$.

Corollary 2.2. *If $k^*$ is the maximum cardinality of an independent vertex set of the graph $G$, then the polynomial $\omega(R - r)^{n-k^*}$ divides the polynomial $\omega(G[R])$.*

**4.2. The corona.** The corona $G \circ Q$ of the graphs $G$ and $Q$ is obtained from $G$ and $n$ copies of $Q$, so that each vertex of $G$ is joined to all vertices of a copy of $Q$. Whence, $G \circ Q$ is a special case of $G[R]$ when the root $r$ of $R$ is adjacent to all other vertices of $R$. In this notation, $Q = R - r$.

COROLLARY 2.3. $\omega(G \circ Q, x) = \omega(Q, x)^n \, \omega(G, 1/\omega(Q))$.

COROLLARY 2.4. *If $k^*$ is the maximum cardinality of an independent vertex set of the graph $G$, then the polynomial $\omega(Q)^{n-k^*}$ divides the polynomial $\omega(G \circ Q)$.*

**4.3. Some more special cases.** If $G$ is the complete graph $K_n$ then $\mathrm{Ind}(G)$ consists of $n+1$ elements: the empty set and $n$ one-element sets, each containing one vertex of $G$. Formula (5) gives then

$$\omega(K_n[R_1, R_2, \dots, R_n]) = x^0 \prod_{i=1}^{n} \omega_i(\varnothing) + \sum_{j=1}^{n} \prod_{i=1}^{n} \omega_i(\{v_j\}).$$

Bearing in mind (4) we arrive at

COROLLARY 2.5. $\omega(K_n[R_1, R_2, \dots, R_n]) = \left[1 + x \sum_{j=1}^{n} \frac{\omega(R_j^{\bullet})}{\omega(R_j^{\circ})}\right] \prod_{i=1}^{n} \omega(R_i^{\circ})$.

It is easy to deduce combinatorial formulas for the $n(G,k)$-numbers of the path $P_n$ and the circuit $C_n$ [**2**, **5**]. Then equations (12) and (2) lead to

COROLLARY 2.6.

$$\omega(P_n(R)) = \sum_{k \geq 0} \binom{n-1-k}{k} \omega(R-r)^{n-k} [\omega(R) - \omega(R-r)]^k,$$

$$\omega(C_n(R)) = \sum_{k \geq 0} \frac{n}{n-k} \binom{n-k}{k} \omega(R-r)^{n-k} [\omega(R) - \omega(R-r)]^k.$$

## REFERENCES

[**1**] K. Engel, *On the Fibonacci number of an $M \times N$ lattice*, Fibonacci Quart. **28** (1990), 72–78.

[**2**] I. Gutman, *On independent vertices and edges of a graph*, in: R. Bodendiek, R. Henn (Eds.), *Topics in Combinatorics and Graph Theory*, Physica-Verlag, heidelberg 1990, 291–296.

[**3**] I. Gutman, *An identity for the independence polynomials of trees*, Publ. Inst. Math. (Beograd) **50** (1991), 19–23.

[**4**] I. Gutman, *Numbers of independent vertex and edge sets of a graph: some analogies*, Graph Theory Notes New York **22** (1992), 18–22.

[**5**] I. Gutman, F. Harary, *Generalization of the matching polynomial*, Utilitas Math. **24** (1983), 97–106.

[**6**] C. Hoede, X.L. Li, *Clique polynomials and independent set polynomials of graphs*, Memor. Fac. Appl. Math. Univ. Twente **982** (1991), 1–14.

[**7**] P. Kirschenhofer, H. Prodinger, R.F. Tichy, *Fibonacci numbers of graphs. II*, Fibonacci Quart. **21** (1983), 219–229.

[**8**] V. Linek, *Bipartite graphs can have any number of independent sets*, Discrete Math. **76** (1989), 131–136.

[**9**] R.E. Merrifield, H.E. Simmons, *Enumeration of structure-sensitive graphical subsets: theory*, Proc. Nat. Acad. Sci. USA **78** (1981), 692–695.

[**10**] R.E. Merrifield, H.E. Simmons, *Enumeration of structure-sensitive graphical subsets: calculations*, Proc. Nat. Acad. Sci.USA **78** (1981), 1329–1332.

[**11**] H. Prodinger, R.F. Tichy, *Fibonacci numbers of graphs*, Fibonacci Quart. **20** (1982), 16–21.

Prirodno-matematički fakultet
34000 Kragujevac, p.p.60
Jugoslavija

(Received 20 11 1991)