PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 50 (64), 1991, 135–140

TRACE FORMULA FOR NUCLEAR PERTURBATIONS OF DISCRETE NONSELFADJOINT OPERATORS

Darko Milinković

Abstract. It is shown that, if T is a discrete nonselfadjoint operator, and P is nuclear, than, under some condition, $T+P$ is discrete. A regularized trace formula is given. It is shown that this result is applicable to differential operators given by regular boundary conditions.

1. Introduction

In [**3**] it was shown that, if T is a discrete selfadjoint operator with distribution of eigenvalues $N(T,r)=O(r^p)$, $0<p<1<2$ ($N(T,r)$ denotes the number of eigenvalues of the operator T in the circle with center at 0 and radius r), and if P is a nuclear operator, then there exists an increasing sequence of natural numbers k_n such that $k_n\to\infty$ and

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}(\mu_k-\lambda_k)=\operatorname{Sp}P,$$

(Sp denotes the trace of an operator) where μ_k and λ_k denote the eigenvalues of the operators $T+P$ and T. This result is applicable to a selfadjoint differential operator T given by a differential expression of order greater than 2 and with regular boundary conditions.

In this paper the case of a nonselfadjoint operator is treated. The result obtained is applicable to nonselfadjoint differential operators of order greater than 6 given by regular boundary conditions.

2. Main result

The main result of this paper is the following theorem.

THEOREM. *Let H be a separable complex Hilbert space with inner product $(\cdot,\cdot)$, P a nuclear operator in H and T an operator such that*

AMS Subject Classification (1985): Primary 47 B 10, 47 A 55

1) $T^{-1} \in \sigma_q$ *for some* $q \geq 1$ (σ_q *is defined in* [**1**]);

2) *all generalized eigenspaces* $\mathcal{N}((\lambda_i I - T)^{m_i})$ (m_i *is the ascent of the operator* $\lambda_i I - T$) *except finitely many contain only eigenvectors*;

3) $|\arg(Tf, f)| < \pi/2q$, *for all* $f \in H$;

4) $N(T, r) = O(r^p)$ ($N(T, r)$ *denotes the number of eigenvalues of the operator* T, *including multiplicities, in the circle with center at* 0 *and radius* r), *for some* p, $0 < p < 1/6$;

5) $\left\|\sum_{i=1}^{k} P_i\right\| < M$, *for* $k = 1, 2, \ldots$, *where* P_i *denotes the projection of* H *onto the generalized eigenspace* $\mathcal{N}((\lambda_i I - T)^{m_i})$ *along* $\mathcal{R}((\lambda_i I - T)^{m_i})$, *where* m_i *is the ascent of the operator* $\lambda_i I - T$.

Then, the operator $T + P$ *is discrete and there exists an increasing sequence of natural numbers* k_n *such that* $\lim_{n\to\infty} k_n = \infty$ *and the following formula holds*:

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \operatorname{Sp} P,$$

where μ_k *and* λ_k *denote the eigenvalues of operators* $T + P$ *and* T, *enumerated* (*including multiplicities*) *in the non-decreasing order of absolute value.*

The proof of the theorem will be exposed in several lemmas. In the following, the conditions of the theorem hold, with the notation introduced above.

LEMMA 1. *There exists an unbounded increasing sequence* r_n *such that*:

1) $\lim_{n\to\infty} r_n / d_n^{6/5} = 0$

2) $\sum_{n=1}^{\infty} r_n^{1+p} / d_n^{7/5} < \infty$ *where* $d_n = d(\sigma(T), \{z : |z| = r_n\})$.

Proof. Let $\delta > 0$ and $c > 0$ (c will be chosen later). Let

$$K_n = \{z \in \mathbf{C} : n^c < |z| < (n+1)^c\}.$$

The width of the ring K_n is a value of order n^{c-1}. In the ring K_n there exist $N(T, (n+1)^c) - N(T, n^c)$ eigenvalues of the operator T, and by condition 4) of the theorem, this is equal to $O(n^{cp})$. If we divide the ring K_n into n^{cp} rings of equal width, there will be no eigenvalue of the operator T in at least one of the new rings. Let us denote by r_n the radius of the circle which divides that ring into two rings of equal width. Then the value $r_n d_n^{-6/5}$ is of order $n^{-c(1/5-6/5p)+6/5(1+\delta)}$. The value $r_n^{1+p} d_n^{-7/5}$ is of order $n^{-c(2/5-12/5p)+7/5(1+\delta)}$. From the fact $0 < p < 1/6$ it follows that we can pick a c such that 1)–3) are fulfilled. □

Remark. From the proof of Lemma 1 it is evident that on the circle $\{z \in \mathbf{C} : |z| = r_n\}$ there are no eigenvalues of the operator T. In the following we shall denote the distance of a point to the spectrum of the operator T by $d(\lambda)$, and $\{z \in \mathbf{C} : |z| = r_n\}$ by Γ_n.

LEMMA 2. *The series* $\sum_{i=1}^{\infty} \dfrac{d(\lambda)^{3/5}}{\lambda - \lambda_i} \cdot P_i$ *converges uniformly in the space* $B(H)$ *of bounded operators, for* $\lambda \in \bigcup_{n=1}^{\infty} \Gamma_n$.

Proof. From Abel's formula it follows that

$$\sum_{k=m}^{n} \frac{d(\lambda)^{3/5}}{\lambda-\lambda_k} P_k = \sum_{k=m}^{n} \Big(\sum_{j=1}^{k} P_j\Big)\Big(\frac{d(\lambda)^{3/5}}{\lambda-\lambda_k} - \frac{d(\lambda)^{3/5}}{\lambda-\lambda_{k+1}}\Big) - \Big(\sum_{j=1}^{m-1} P_j\Big)\cdot\frac{d(\lambda)^{3/5}}{\lambda-\lambda_m} + \Big(\sum_{j=1}^{n} P_j\Big)\frac{d(\lambda)^{3/5}}{\lambda-\lambda_m}.$$

Thus, by virtue of condition 5) of the theorem, it follows that it is sufficient to show that the sequence $\dfrac{d(\lambda)^{3/5}}{|\lambda-\lambda_j|}$ and the series $\sum\limits_{k=1}^{\infty}\Big|\dfrac{d(\lambda)^{3/5}}{\lambda-\lambda_k} - \dfrac{d(\lambda)^{3/5}}{\lambda-\lambda_{k+1}}\Big|$ converges uniformly for $\lambda\in\bigcup_{n=1}^{\infty}\Gamma_n$.

Let $\varepsilon>0$ and $r_k\le|\lambda_i|\le r_{k+1}$. Then,

$$\sup_{\lambda\in\cup\Gamma_n}\frac{d(\lambda)^{3/5}}{|\lambda-\lambda_i|}\le\frac{1}{d_k^{2/5}}<\varepsilon,$$

for sufficiently large k, e.g. i and thus the sequence $d(\lambda)^{3/5}/|\lambda-\lambda_i|$ converges uniformly. For $r_k\le|\lambda_{i+1}|\le r_{k+1}$,

$$\Big|\frac{d(\lambda)^{3/5}}{\lambda-\lambda_k} - \frac{d(\lambda)^{3/5}}{\lambda-\lambda_{k+1}}\Big| \le \frac{|\lambda_i|+|\lambda_{i+1}|}{|\lambda-\lambda_i|\cdot|\lambda-\lambda_{i+1}|}\cdot d(\lambda)^{3/5}\le\frac{2r_{k+1}}{d_{k+1}^{7/5}}.$$

Since in the ring $\{r_k\le|z|\le r_{k+1}\}$ there are $O(r_k^p)$ eigenvalues of the operator T it follows that

$$\sum_{k=1}^{\infty}\Big(\sum_{r_k\le|\lambda_k|\le r_{k+1}}\frac{r_{k+1}}{d_{k+1}^{7/5}}\Big)\le \text{const}\sum_{k=1}^{\infty}\frac{r_{k+1}^{1+p}}{d_{k+1}^{7/5}}<\infty,$$

by Lemma 1. □

Let N_i denote the restriction of $\lambda_i I-T$ to $\mathcal{N}((\lambda_i I-T)^{m_i})$, $R_\lambda^0=(T-\lambda I)^{-1}$, $R_\lambda=(T+P-\lambda I)^{-1}$.

LEMMA 3. $\sup\big\{(d(\lambda))^{3/5}\|R_\lambda^0\|:\lambda\in\bigcup_{n=1}^{\infty}\Gamma_n\big\}<C<\infty$.

Proof. In [**2**] it is shown that the resolvent of a discrete operator which satisfies condition 5) of the theorem has a representation

$$R_\lambda^0=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\frac{(-N_i)^{j-1}P_i}{(\lambda-\lambda_i)^j}+\sum_{j=0}^{\infty}(\lambda_0-\lambda)^j R_{\lambda_0}(T_\infty)^{j+1}(I-\mathbf{P}_\infty)\quad\text{in } B(H),$$

where $\mathbf{P}_\infty=\sum_{i=1}^{\infty}P_i$ (strong convergence), and $R_{\lambda_0}(T_\infty)$ is an operator without significance to us.

From conditions 1) and 3) of the theorem and the theorem of Lidsky [**1**, ch. V, §6, T 6.1] it follows that $\mathbf{P}_\infty=I$[1)] and so

$$d(\lambda)^{3/5}R_\lambda^0=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\frac{d(\lambda)^{3/5}}{|\lambda-\lambda_{ij}|}(-N_i)^{j-1}P_i+\sum_{i=m+1}^{\infty}\frac{d(\lambda)^{3/5}P_i}{\lambda-\lambda_i}$$

[1)] $\mathbf{P}_\infty$ is a projection onto $S_\infty=\{x\in H: x=\sum_{i=1}^{\infty}x_i,\ x_i\in\mathcal{N}((T-\lambda_i I)^{m_i})\}=\overline{S}_\infty$ [**2**, p. 20], and $S_\infty=H$.

(from the condition 2) of the Theorem it follows that $N_i \equiv 0$ for $i > m$).

The proof follows immediately from Lemma 2. □

COROLLARY 1. *The operator $T + P$ is discrete.*

Proof. By virtue of Lemma 3 $\|PR_\lambda^0\| < 1$, for λ sufficiently distant from $\sigma(T)$, so the series

$$R_\lambda = \sum_{k=0}^{\infty} (-1)^k R_\lambda^0 (PR_\lambda^0)^k \tag{1}$$

converges, and thus R_λ is compact. □

COROLLARY 2. *For n sufficiently large, $\Gamma_n \cap \sigma(T + P) = \varnothing$.*

Proof.

$$\|P\| C d(\lambda)^{-3/5} < \|P\| C d_n^{-3/5} \tag{2}$$

for $\lambda \in \Gamma_n$, and $d_n \to \infty$ (follows from Lemma 1). Thus, since for every $\lambda \in \Gamma_n$ there exists R_λ^0 (see the remark after Lemma 1) from (2) it follows that R_λ exists too. □

COROLLARY 3. $\|R_\lambda - R_\lambda^0\|_1 \le k d_n^{-6/5}$ *if $\lambda \in \Gamma_n$, where $k > 0$ and $\|\cdot\|_1$ denotes the nuclear norm of the operator.*

Proof. From (1) it follows that $\|R_\lambda - R_\lambda^0\|_1 \le \sum_{k=1}^{\infty} \|R_\lambda^0\|^{k+1} \|P\|_1^k$. It remains only to apply Lemma 3. □

LEMMA 4. *For n large enough, in the circle $\{z : |z| < r_n\}$ the number of eigenvalues of the operator T is equal to the number of eigenvalues of the operator $T + P$, including multiplicities.*

Proof. Let $N_1(r)$ be the number of eigenvalues of the operator $T + P$, and $N_2(r)$ the number of eigenvalues of the operator T, in the circle $\{z : |z| < r\}$. Then, by the properties of Riesz projectors:

$$N_1(r_n) - N_2(r_n) = \frac{1}{2\pi i} \int_{\Gamma_n} \mathrm{Sp}(R_\lambda - R_\lambda^0)\, d\lambda. \tag{3}$$

Since $|\mathrm{Sp}(R_\lambda - R_\lambda^0)| \le \|R_\lambda - R_\lambda^0\|_1$, it follows that $|N_1(r_n) - N_2(r_n)| = O(r_n d_n^{-6/5})$, and that is, by virtue of Lemma 1, less than 1 for n large enough. □

Let k_n be the number of eigenvalues of the operators T and $T + P$ in the circle $\{z : |z| < r_n\}$.

LEMMA 5. $\lim_{n\to\infty} \left(\sum_{k=1}^{k_n} (\mu_k - \lambda_k) - \operatorname{Res}_{\lambda_k}(\mathrm{Sp}(PR_\lambda^0)) \right) = 0$, *where $\operatorname{Res}_{\lambda_k}$ is the residuum of a function at point λ_k.*

Proof. By the properties of Riesz projectors:

$$\sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \frac{1}{2\pi i} \int_{\Gamma_n} \lambda\, \mathrm{Sp}(R_\lambda - R_\lambda^0)\, d\lambda. \tag{4}$$

On the other hand, from the property of the trace $\operatorname{Sp}(R_\lambda^0 P R_\lambda^0) = \operatorname{Sp}(P(R_\lambda^0)^2)$ and the fact that $dR_l^0/d\lambda = (R_\lambda^0)^2$, after partial integration, it follows that

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda \operatorname{Sp}(R_\lambda^0 P R_\lambda^0) = \frac{-1}{2\pi i}\int_{\Gamma_n} \operatorname{Sp}(P R_\lambda^0)\, d\lambda$$

so

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda \operatorname{Sp}(R_\lambda^0 P R_\lambda^0) = -\sum_{k=1}^{k_n} \operatorname{Res}_{\lambda_k}(\operatorname{Sp}(P R_\lambda^0)). \tag{5}$$

By adding (5) and (4), we obtain

$$\sum_{k=1}^{k_n}\Big(\mu_k - \lambda_k - \operatorname{Res}_{\lambda_k}(\operatorname{Sp}(P R_\lambda^0))\Big) = \frac{1}{2\pi i}\int_{\Gamma_n} \lambda \operatorname{Sp}(R_\lambda - R_\lambda^0 + R_\lambda^0 P R_\lambda^0)\, d\lambda. \tag{6}$$

From (2) it follows that

$$R_\lambda - R_\lambda^0 + R_\lambda^0 P R_\lambda^0 = \sum_{k=2}^{\infty} (-1)^k R_\lambda^0 (P R_\lambda^0)^k \tag{7}$$

and since $\operatorname{Sp}\big(d(PR_\lambda^0)^k/d\lambda\big) = \operatorname{Sp}\big(kR_\lambda^0(PR_\lambda^0)^k\big)$, after substituting (7) into (6) and applying partial integration to each member of the uniformly convergent series (7) we obtain

$$\sum_{k=1}^{k_n}\Big(\mu_k - \lambda_k - \operatorname{Res}_{\lambda_k}(\operatorname{Sp}(P R_\lambda^0))\Big) = -\frac{1}{2\pi i}\int_{\Gamma_n} \operatorname{Sp}\Big(\sum_{k=2}^{\infty}\frac{(-1)^k}{k}(P R_\lambda^0)^k\Big)\, d\lambda. \tag{8}$$

By applying the inequality $|\operatorname{Sp}(P R_\lambda^0)| \le \|R_\lambda^0\|^k \cdot \|P\|_1^k$ and Lemma 3 to each member of the series (8) we obtain

$$\begin{aligned} \Big|\sum_{k=1}^{k_n}\Big(\mu_k - \lambda_k - \operatorname{Res}_{\lambda_k}(\operatorname{Sp}(P R_\lambda^0))\Big)\Big| &\le r_n \sum_{k=2}^{\infty}\frac{\|P\|_1^k}{k}\cdot\Big(\frac{c}{d_n^{3/5}}\Big)^k \\ &= \frac{r_n}{d_n^{6/5}}\sum_{k=2}^{\infty}\Big(\frac{\|P\|_1 C}{d_n^{3/5}}\Big)^{k-2}\cdot\big(\|P\|_1 C\big)^2 = O\Big(\frac{r_n}{d_n^{6/5}}\Big) \end{aligned}$$

since for n large enough, $\|P\|_1 C d_n^{-3/5} < 1$. The last expression converges to zero, by virtue of Lemma 1. □

Proof of the Theorem. From [**2**] and the fact that $\mathbf{P}_\infty = I$ which has already been proved, it follows that

$$P R_\lambda^0 = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}\frac{1}{(\lambda-\lambda_i)^j} P(-N_i)^{j-1} P_i$$

and thus $\operatorname{Res}_{\lambda_k} P R_\lambda^0 = P R_k$. This proves the theorem, because

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\operatorname{Res}_{\lambda_k}(\operatorname{Sp}(P R_\lambda^0)) = \lim_{n\to\infty} \operatorname{Sp} P \sum_{k=1}^{k_n} P_k = \operatorname{Sp} P.$$

3. Applications

The theorem proved above can be applied to differential operators of order not less than 7, as in the following example:

Let $H = L^2[0,1]$ be a complex-valued continuous function defined on $[0,1]$ with values in the angle $\{z : |\operatorname{Arg} z| < \pi/4\}$ and T a differential operator in H given by the differential expression $l(y) = y^{\mathrm{VIII}} + q(x)y$ and boundary conditions of Sturm type

$$y^{(j)}(0) = 0, \quad y^{(j)}(1) = 0, \qquad j = 0,1,2,3,$$

which are regular [**4**, II, §4.8].

Let K be a nuclear integral operator in the space H:

$$(\mathbf{K}f)(x) = \int_0^1 K(x,t)f(t)\,dt.$$

The conditions 1) and 4) of the theorem are fulfilled by virtue of the theorem on the asymptotic behaviour of the eigenvalues of regular boundary problem [**4**, II, §4.9].

The condition 5) of the theorem is fulfilled since the union of bases of the spaces $\mathcal{N}((\lambda_i I - T)^{m_i})$ form a Riesz basis in H [**4**, II, §5.3] and [**1**, VI, §5].

The condition 1) is fulfilled because the Green function of the operator T is continuous and thus belongs to $L^2(0,1)^2$, so $T^{-1} \in \sigma_2$.

Since

$$(Ty,y) = \int_0^1 (y^{(\mathrm{VIII})} + q(x)y)\overline{y}\,dy = \int_0^1 |y^{(\mathrm{IV})}|^2\,dx + \int_0^1 q(x)|y|^2\,dx$$

(after 4 partial integrations), the condition 1) of the theorem is fulfilled, by virtue of the assumption $|\arg q(x)| < \pi/4$ and thus, with the notation introduced before

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \int_0^1 K(x,x)\,dx,$$

for some increasing unbounded sequence of natural numbers k_n.

REFERENCES

[**1**] И. Ц. Гохберг, М. Г. Крейн, *Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве*, Наука, Москва, 1965.

[**2**] P. Lang, J. Locker, *Spectral representation of the resolvent of a discrete operator*, J. Funct. Anal. **79** (1988), 18–31.

[**3**] В. А. Любишкин, И. Д. Цопанов, *Регуљризованные следы интегро-дифференциальных операторов*, Мат. заметки **43** (1988), 786–793.

[**4**] М. А. Наймарк, *Линейные дифференциальные операторы*, Наука, Москва, 1969.

Matematički fakultet
Studentski trg 16
11000 Beograd
Jugoslavija

(Received 15 05 1990)
(Revised 28 08 1990)