PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 47 (61), 1990, 68–70

ENLARGEMENTS OF MACKEY'S TOPOLOGIES

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Abstract. A new proof is given of the fact that finite-condimensional subspace of a quasibarrelled space is the space of the same sort. This generalizes a result from [**1**], but the main contribution is in the simplicity of the proof.

Dieudonné showed in [**1**] that, in barrelled (resp. bornological) space, any hyperplane, and so any vector subspace of finite condimension, is barrelled (resp. bornological). The same result for quasibarrelled spaces was established by Valdivia in [**2**], using the mentioned Dieudonné's result for the bornological spaces. This paper is concerned with the same problem. It is inspired by the Dieudonné's method for the proof that the property of being barrelled (resp. bornological) is stable under finite-condimensional subspace.

Throughout the paper, E is a locally convex Hausdorff space with the topological dual E' and the algebraic dual E^*. We suppose that E is with the Mackey topology $\tau(E,E')$. If E^b denotes the set of bounded linear functionals on E, then $E' \subset E^b$ and $\tau(E,E^b)$ is the bornological topology associated with E. A subspace F of E^* which contains E', and in which E' is finite-condimensional is said to be a finite-dimensional enlargement of E' in E^*. The corresponding Mackey topology $\tau(E,F)$ will be called an enlargement of $\tau(E,E')$. It should be mentioned that Dieudonné also showed in [**1**] that E remains barrelled (resp. bornological) under the new Mackey topology $\tau(E,F)$. It is clear that H is a $\tau(E,E')$-dense hyperplane in E iff E' is a $\sigma(F,E)$-dense hyperplane in F, where $F = E + \operatorname{span}\{u\}$ and $H = u^{-1}(0)$. The subspace F is then one-dimensional enlargement of the dual E', which is associated to the hyperplane H. In this case the above mentioned Dieudonné's result shows that the Mackey's topologies $\tau(E,E')$ and $\tau(E,F)$ coincide on H. Therefore the question arises naturally whether the corresponding result for a quasibarrelled (resp. ultra-bornological) space is true? The next theorem generalizes Theorem 3 in [**1**] and uses essentially the same arguments.

Theorem 1. *Let E be a quasibarrelled space and H a finite-codimensional subspace of E. Then H is quasibarrelled with respect to the relative topology.*

AMS Subject Classification (1985): Primary 46 A 05

Proof. If E is a barrelled or bornological space, the proof follows from [**1**]. If E is a quasibarrelled non-bornological barrelled space, it is with Mackey topology $\tau(E,E')$ and $E' \subsetneq E^b$. It suffices to prove the result when H is a $\tau(E,E')$-dense hyperplane in E. In this case, by the above, we have on E the new Mackey topology $\tau(E,F)$, where $F \subset E^b$ is one-dimenzional enlargement of the dual E' in $E^b \subset E^*$.

Claim 1. The topologies $\beta(E',E)$ and $\beta(F,E)$ coincide on E'.

Proof. The topologies $\tau(E,E') < \tau(E,F) < \tau(E,E^b)$ have the same bounded sets in E.

Claim 2. For every $\beta(F,E)$-bounded set A in F, there exist a $\beta(E',E)$-bounded set B in E' and bounded finite-dimensional set C in $\operatorname{span}\{u\}$ so that $A \subset B + C$.

Proof. Indeed, if $x \in A$, then there exist $y \in E'$ and $\lambda \in K$ such that $x = y + \lambda u$ (unique decomposition) where $|\lambda| \leq \alpha$ for some α. If the set of these λ is not bounded, there exist $x_n \in A$, $y_n \in E'$ and $\lambda_n \in K$ such that $x_n = y_n + \lambda_n u$ and $|\lambda_n| \to +\infty$. Now, we have that $\lambda_n^{-1} x_n = \lambda_n^{-1} y_n + u \to 0$ in the topology $\beta(F,E)$ i.e. the sequence $\lambda_n^{-1} y_n = \lambda_n^{-1} x_n - u$ converges to $-u$ in the topology $\beta(E',E)$ (Claim 1). Since the space $(E', \beta(E',E))$ is sequentially complete it follows that $u \in E'$. But this is a contradiction. Hence, the set $\{x - \lambda u,\, x \in A, |\lambda| \leq \alpha\} \subset E'$ is $\beta(E',E)$-bounded, completing the proof of the Claim 2.

Claim 3. The space E with the topology $\tau(E,F)$ is quasibarrelled.

Proof. By the Claim 2 and the fact that E with the topology $\tau(E,E')$ is quasibarrelled, it follows that every convex $\beta(F,E)$-bounded set in F lies in the sum of two compact sets, i.e. it is $\tau(E,F)$-equicontinuous. This shows that $(E, \tau(E,F))$ is a quasibarrelled space.

Claim 4. The topologies $\tau(E,E')$ and $\tau(E,F)$ coincide on H.

Proof. It is obvious that $\tau(E,E')|H \leq \tau(E,F)|H$. On the other hand, if A is a convex $\sigma(F,E)$-compact set in F, then there exist sets B and C as in Claim 2 such that $A \subset B + C$. From that it follows that $A^\circ \cap H \supset (B+C)^\circ \cap H = B^\circ \cap H$, i.e. $\tau(E,E')|H \geq \tau(E,F)|H$.

Claim 5. The subspace H is quasibarrelled with respect to the relative topology.

Proof. H is a $\tau(E,F)$-closed hyperplane in E and then the proof follows from the Claim 3.

Remark 1. Claim 1 is true for the bornological space too, but the proof is not the same, because in this case the topologies $\tau(E,E')$ and $\tau(E,F)$ need not have the same bounded sets in E. This shows that in bornological case too the proof that the topologies $\tau(E,E')$ and $\tau(E,F)$ coincide on H can be as in our Theorem 1, i.e. without using convergence of ultrafilters (see [**1**]). On the other hand, our Theorem 1 is an easy generalization of the Dieudonné's method, only we use topologies $\beta(F,E)$ and $\beta(E',E)$ instead of $\sigma(F,E)$ and $\sigma(E',E)$ in [**1**].

Remark 2. Claim 2 remains also true for every locally convex space whose strong dual is sequentially complete.

Remark 3. We know from [**3**] that the property of being ultrabornological is not stable under one-condimensional subspace. From this it follows that this property is not stable under one-dimensional enlargement of the dual E' in E^* either. Indeed, let E be an ultra-bornological space with the Mackey topology $\tau(E,E')$ and H a $\tau(E,E')$-dense hyperplane which is not ultra-bornological [**3**]. By the above notation, we have the new Mackey topology $\tau(E,F)$. If the space $(E,\tau(E,F))$ is ultra-bornological, then $(H,\tau(E,E')|H)$ is of the same type, since by the Theorems 2 and 3 of [**1**] the topologies $\tau(E,E')$ and $\tau(E,F)$ coincide on H and H is $\tau(E,F)$-closed. This is not true by [**3**].

We conclude this paper with the following result.

THEOREM 2. *Let $(E,\tau(E,E'))$ be a barrelled (resp. bornological; ultra-bornological; quasibarrelled) locally convex space, H a $\tau(E,E')$-dense hyperplane and F the corresponding enlargement of the dual E' in E^*. Then $(H,\tau(E,E')|H)$ is barrelled (resp. bornological; ultra-bornological; quasibarrelled) iff $(E,\tau(E,F))$ is of the same sort.*

Proof. $(E,\tau(E,F)) = (H,\tau(E,F)|H) \oplus L = (H,\tau(E,E')|H) \oplus L$, where L is a one-dimensional vector space.

Remark 4. We conclude that the property of being barrelled (resp. bornological; ultra-bornological; quasibarrelled) is stable under taking finite codimensional subspaces iff it is stable under finite-dimensional enlargment of the Mackey topology.

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(Received 21 05 1989)