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CAUSALITY AND STOCHASTIC REALIZATION PROBLEM

Ljiljana Petrović

Abstract. The basic idea is to relate some concepts of causality to the stohastic realization problem. Especially, a new definition of causality "$\mathbf{F}^1$ is a cause of $\mathbf{F}^2$ within $\mathbf{F}^3$", which is a generalization of a corresponding definition from [**3**], is given. The problem of determining possible states of the stohastic dynamic system S_1 with known outputs, having a certain causality relationship with another stohastic dynamic system S_2 is considered. More precisely, problems formulated in [**1**] are investigated in the sense of a new definition of cuasality.

1. Preliminary Notions and Definitions. We forst give precise definitions of all the terms used. These definitions are given in terms of Hilbert space.

Let $\mathbf{F} = (F_t)$, $t \in R$, be a family of Hilbert spaces. We think about F_t as about the information aviable at time t. Total information $F_{<\infty}$ carried by $\mathbf{F}$ wll be defined by $F_{<\infty} = \vee_{t\in R} F_t$, while past and future information of $\mathbf{F}$ at t will be defined as $F_{\leq t} = \vee_{s\leq t} F_s$ and $F_{\geq t} = \vee_{s\geq t} F_s$, respectively. It should be clear that $F_{<t} = \vee_{s<t} F_s$ and $F > t = \vee_{s>t} F_s$ need not coincide with $F_{\leq t}$ and $F_{\geq t}$ respectively; $F_{\leq t}$ and $F_{>t}$ are sometimes called the real past and the real future of $\mathbf{F}$ at t.

If F_1 and F_2 are arbitrary subspaces of a Hilbert space $\mathcal{H}$, then $P(F_1 \mid F_2)$ will denote the orthogonal projection of F_1 onto F_2 and $F_1 \ominus F_2$ will denote a Hilbert space generated by all elements $x - P(x \mid F_2)$, where $x \in F_1$.

Definition 1.1. We say that $\mathbf{F}^1$ is *submitted* to $\mathbf{F}^2$ (and write $\mathbf{F}^1 \subseteq \mathbf{F}^2$) if and only if $F^1_{\leq t} \subseteq F^2_{\leq t}$ for each t.

We say that families $\mathbf{F}^1$ and $\mathbf{F}^2$ are *equivalent* (and write $\mathbf{F}^1 = \mathbf{F}^2$) if and only if $\mathbf{F}^1 \subseteq \mathbf{F}^2$ and $\mathbf{F}^2 \subseteq \mathbf{F}^1$.

Definition 1.2. We say that $\mathbf{F}^1$ is *strictly submitted* to $\mathbf{F}^2$ (and write $\mathbf{F}^1 \leq \mathbf{F}^2$) if and only if $F^1_t \subseteq F^2_t$ for each t.

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It is easy to see that submission implies strict submission and that the converse does not hold.

The notion of minimality of families of Hilbert spaces is specified in the following definition.

Definition 1.3. We say that $\mathbf{F}$ is a *minimal* (respectively, *strictly minimal*) family having a certain property if and only if there is no family $\mathbf{F}^*$ with the same property submitted (respectively, strictly submitted) to $\mathbf{F}$.

It should be clear that a minimal (respectively, strictly minimal) family having a certain property is not necessarily unique.

Definition 1.4. (cf. [**4**] and [**6**] If F_1, F_2 and F are arbitrary Hilbert spaces, then we say that F is a *splitting* for F_1 and F_2 (and write $F_1 \perp F_2 \mid F$) if and only if $F_1 \ominus F \perp F_2 \ominus F$

Definition. 1.5. (cf. [**6**]). The family $\mathbf{F}$ is called *markovian* if and only if $P(F_{\geq t} \mid F_{\leq t}) = F_t$ for each t.

It is easy to see that the family F is markovian if and only if $F_{\geq t} \perp F_{\leq t} \mid F_t$, for each t

Definition 1.6. (cf. [**4**]) A stohastic dynamic system (s.d.s) is a set of two families $\mathbf{F}^1$ (outputs) and $\mathbf{F}^2$ (states) which satisfy the condition

$$F^1_{<t} \vee F^2_{<t} \perp F^1_{>t} \vee F^2_{>t} \mid F^2_t \tag{1.1}$$

For a given family of outputs $\mathbf{F}^1$, any family $\mathbf{F}^2$ satisfying (1.1) is called a realization of an s.d.s. with those outputs.

The following results concerning splitting and s.d.s. will be used later (for the proofs of these results see the given references).

Lemma 1.1. (*cf.* [**1**] *and* [**4**]) *$F_1 \perp F_2 \mid F$ if and only if $P(F_i \mid F_j \vee F) \subseteq F$, for $i, j = 1, 2,\ i \neq j$.*

Proof. Let, for example, $F_1 \perp F_2 \mid F$. Then, obviously, $F_1 \ominus F \perp F_2$ (which follows from $F_1 \ominus F \perp F_2 \ominus F$ and $F_1 \ominus F \perp F$), which, together with the obvious equality $F_1 \ominus F = (F_1 \vee F) \ominus F$, implies $P(F_2 \mid F_1 \vee F) \subseteq F$. The other half of the statement is obvious

Corollary 1.1.1. *$F_1 \perp F_2 \mid F$ if and only if $F_1' \perp F_2' \mid F$ for all $F_i' \subseteq F_i \vee F,\ i = 1, 2$.*

Theorem 1.2. [**2**] *The space F is a minimal one such that $F_1 \perp F_2 \mid F$ if and only if $F = P(F_1 \mid S)$ for some space S such that $F_2 \subseteq S \subseteq (F_2 \vee P(F_2 \mid F1)) \oplus (F_1 \vee F_2)^{\perp}$.*

Corollary 1.2.1. [**2**] *The space $F \subseteq F_1 \vee F_2$ is a minimal one such that $F_1 \perp F_2 \mid F$ if and only if $F = P(F_1 \mid S)$ for some space S such that $F_2 \subseteq S \subseteq F_2 \vee P(F_2 \mid F_1)$.*

THEOREM 1.3. [2] *A family* $\mathbf{F}^2 = (F_t^2)$, $t \in R$, *is a strictly minimal realization of a s.d.s. with outputs* $\mathbf{F}_1 = (F_t^1)$, $t \in R$, *if only and if:*

(1) F_t^2 *is a minimal space such that* $F_{<t}^1 \perp F_{>t}^1 \mid F_t^2$ *for each* t;

(2) *there exists a family* $\mathbf{F}^3 = (F_t^3)$, $t \in R$, *with the property* $F_t^3 \subseteq (F_{<\infty}^1 \vee F_{>t}^2)^\perp$ *for each* t *such that the family* $\mathbf{S} = (S_t)$, $S_t = (F_t^1 \vee F_t^2) \oplus F_t^3$, $t \in R$, *is nondecreasing, i. e.* $S_{t_1} \subseteq S_{t_2}$ *whenever* $t_1 \leq t_2$.

This theorem has a simpler version if $F_t^2 \subseteq F_{<\infty}^1$.

COROLLARY 1.3.1. [2] *A family* $\mathbf{F}^2 = (F_t^2)$, $t \in R$, *of subspaces from* $F_{<\infty}^1$ *is a strictly minimal realization of an s. d. s. with outputs* $\mathbf{F}^1 = (F_t^1)$, $t \in R$, *if and only if condition* (1) *of Theorem* 1.3 *holds and the family* $\mathbf{S} = (S_t)$, $S_t = F_{<t}^1 \vee F_t^2$, $t \in R$, *is nondecreasing.*

Now we shall give an intuitively plausible notion of causality. Let $\mathbf{F}^1, \mathbf{F}^2$ and $\mathbf{F}^3$ be arbitrary families of Hilbert spaces. We may say that "$\mathbf{F}^2$ is a cause of $\mathbf{F}^1$ within $\mathbf{F}^3$" if

$$F_{<\infty}^1 \perp F_{\leq t}^3 \mid F_{\leq t}^2 \tag{1.2}$$

because the essense of (1.2) is that all information about $F_{<\infty}^1$ given by $F_{\leq t}^3$ comes via $F_{\leq t}^2$ for arbitrary t; equivalently, for arbitrary t, the information about $F_{<\infty}^1$ proveded by $F_{\leq t}^3$ is not "bigger" than that provided by$F_{\leq t}^2$, or,$F_{\leq t}^2$ contains all the informations from $F_{\leq t}^3$ needed for predicting $F_{<\infty}^1$. According to Corollary 1/1/1, (1.2) is equivalent to $F_{<\infty}^1 \perp F_{\leq t}^3 \vee F_{\leq t}^2 \mid F_t^2$. The last relation, means that condition $\mathbf{F}^2 \subseteq F^3$ does not represent an essential restriction. Thus, it is natural to introduce the following definition of causality between familes of Hilbert spaces:

Definition 1.7. We say that $\mathbf{F}^2$ is a *cause* of $\mathbf{F}^1$ within $\mathbf{F}^3$ (and write $\mathbf{F}^1 \mid< \mathbf{F}^2; \mathbf{F}^3$) if and only if $\mathbf{F}^2 \subseteq \mathbf{F}^3$ and $F_{<\infty}^1 \perp F_{\leq t}^3 \mid F_{\leq t}^2$ for each t.

A definition analogous to Definition 1.7, formulated in terms of σ-algebras, was first given in [**3**]; however, a strict Hilbert space version of the definition from [**3**] contains also the condition $\mathbf{F}^1 \subseteq \mathbf{F}^3$ which does not have an intuitive justification. Since Definition 1.7 is more general than the Hilbert space version of the definition in [**3**], all results related to causality in the sence of Definition 1.7 will be also true in the sense of the Hilbert space version of the definition in [**3**], when we add condition $\mathbf{F}^1 \subseteq \mathbf{F}^3$ to them.

If $\mathbf{F}^1$ and $\mathbf{F}^2$ are such that $\mathbf{F}^1 \mid< \mathbf{F}^1; \mathbf{F}^2$ we say that $\mathbf{F}^1$ is its own cause within $\mathbf{F}^2$ (cf. [**3**]). It should be mentioned that the notion of subordination (as indroduced in [**5**]) **is equivalent to the notion of being one's own cause as defined here.**

If $\mathbf{F}^1$ and $\mathbf{F}^2$ are such that $\mathbf{F}^1 \mid< \mathbf{F}^1; \mathbf{F}^1 \vee \mathbf{F}^2$ (where $\mathbf{F}^1 \vee \mathbf{F}^2$ is a family determined by $(F^1 \vee F^2)_t = F_t^1 \vee F_t^2$), we say that $\mathbf{F}^2$ does not cause $\mathbf{F}^1$. It can be shown without difficulties that this is identical to "$\mathbf{F}^2$ does not anticipate $\mathbf{F}^1$" (as introduced in [6]).

We shall now prove some results which will be needed later.

LEMMA 1.4. *From* $F^1_{<\infty} \subseteq F^2_{<\infty}$ *and* $\mathbf{F}^2 \mid< \mathbf{F}^2; F^3$ *it follows that* $\mathbf{F}^1 \mid< F^2; \mathbf{F}^3$. *The proof is quite similar.*

LEMMA 1.5. c.f. [1]) $\mathbf{F}^1 \mid< \mathbf{F}^2; \mathbf{F}^3$ *if and only if* $\mathbf{F}^2 \subseteq \mathbf{F}^3$ *and* $P(F^1_{<\infty} \mid F^3_{\leq t}) = P(F^1_{<\infty} \mid F^2_{\leq t})$ *for each* t.

Proof. **According to Lemma 1.1,** $\mathbf{F}^1 \mid \mathbf{F}^2; \mathbf{F}^3$ **is equivalent to** $P(F^1_{<\infty} \mid F^3_{\leq t} \vee F^2_{\leq t}) \subseteq F^2_{\leq t}$ **and** $\mathbf{F}^2 \subseteq \mathbf{F}^3$ **which implies** $P(F^1_{<\infty} \mid F^3_{\leq t}) = P(P(F^1_{<\infty} \mid F^3_{\leq t}) \mid F^2_{\leq t}) = P(F^1_{<\infty} \mid F^2_{\leq t})$, **so that one half of the statement is proved. The other half is obvious.**

LEMMA 1.6. **(cf.** [1]) *From* $\mathbf{F}^1 \mid< \mathbf{F}^2; \mathbf{F}^3$ *and* $\mathbf{F}^1 \subseteq \mathbf{F}^3$ *it follows that* $\mathbf{F}^1 \subseteq \mathbf{F}^2$.

Proof. **If, for some** t, x_t **is an arbitrary element from** $F^1_{\leq t}$, **then**

$$x_t - P(x_t \mid F^2_{\leq t}) \perp F^3_{\leq t} \ominus F^2_{\leq t},$$

which (because $\mathbf{F}^1 \subseteq \mathbf{F}^3$**) implies** $x_t - P(x_t \mid F^2_{\leq t}) \perp x_t - P(x_t \mid F^2_{\leq t})$ **that is,** $x_t = P(x_t \mid F^2_{\leq t})$. **The proof is completed.**

2. Main Results. The results of this section will tell us under which conditions, concerning the relationship between (known) information E about an s.d.s. S^2 **and (known) outputs H of an s.d.s.** S_1, **it is possible to find a realizations of an s.d.s.** S_1 **which are in a certain causality relationship with E and H. More precisely, we shall consider the following two cases (see [1]): (1) the available information E about** S_2 **is a cause of states of** S_1 **within outputs H of** S_2**; (2) states of** S_1 **are a cause of the available information E about** S_2 **within outputs H of** S_1. **The problem of minimality and strict minimality is disscused and partially solved.**

The next two theorems deal with case (1), while the other results deal with case (2):

THEOREM 2.1. [1] *If* $\mathbf{G}$ *is its own cause within* $\mathbf{H}$, *then* $\mathbf{G}$ *is a realization of an s.d.s. with outputs* $\mathbf{H}$ *if and only if* $\mathbf{G}$ *is markovian and* $H_{<t} \perp_{>t}\mid G_t$ *for each* t.

For the proof of this theorem see [1].

The previous theorem gives conditions under which G is a realization of an s.d.s. with known outputs H, while Lemma 1.4 and Lemma 1.6 (for $\mathbf{G} = \mathbf{F}^1$, $\mathbf{E} = \mathbf{F}^2$, $\mathbf{H} = \mathbf{F}^3$**) complete the solution of the problem (1); i.e. Lemma 1.4 gives conditions under which** $\mathbf{G} \mid< \mathbf{R}; \mathbf{H}$ **holds and conversely, Lemma 1.6 explains connections between families** $\mathbf{G}, \mathbf{H}, \mathbf{E}$ **if** $\mathbf{G} \mid< \mathbf{E}; \mathbf{H}$ **holds.**

An example which illustrates the results above and an example which shows that a given realization is not a minimal realization of S_1 **are given in [1]. The problem of determining the minimal realization G**

of an s.d.s. with outputs H, which will be caused by a given E within H, is still open.

The next result gives conditions under which G is stricly minimal realization of an s.d.s. with outputs H so that G $|<$ E; H holds.

THEOREM 2.2. *The family* $\mathbf{G}$ *is a unique strictly minimal realizatin (of an s.d.s. with outputs* $\mathbf{H}$*), such that* $G_{\leq t} \subseteq H_{<t}$*, for each* t*, if and only if it is defined by*

$$G_t = P(H_{>t} \mid H_{<t}),\ \ t \in R. \tag{2.1}$$

Every family $\mathbf{E}$*, such that* $\mathbf{E} \mid< \mathbf{E}; \mathbf{H}$ *and* $P(H_{>t} \mid H_{<t}) \subseteq E_{<\infty}$ *for each* t*, is a cause of the realization* $\mathbf{G}$*, defined by* (2.1)*, within* $\mathbf{H}$*.*

Proof. **From Theorem 1.3 and Corollary 1.3.1 it follows that G is a strictly minimal realization (of an s.d.s. with outputs H) such that** $G_{\leq t} \subseteq H_{<t}$ **for each** t**, if and only if G is defined by (2.1). To prove that G is unique like this realization, let us suppose that** $\mathbf{G}^*$ **is an other strictly minimal realization (of an s.d.s. with outputs H such that** $G^*_{<t} \subseteq H_{<t}$. **According to Lemma 1.1,** $P(H> t \vee G^*_{\geq t} \mid H_{<t}) \subseteq G^*_t$ **that is,**

$$PH_{>t} \mid H_{<t}) \vee P(G^*_{\geq t} \mid H_{<t}) \subseteq G^*_t,$$

and, thus $G_t \subseteq G^*_t$ **as we wanted to prove. The assumption** $P(H_{>t} \mid H_{<t}) \subseteq E_{<\infty}$ **implies** $G_{<\infty} \subseteq E_{<\infty}$**, so that, according to Lemma 1.4 (for** $\mathbf{G} = \mathbf{F}^1$, $\mathbf{E} = \mathbf{F}^2$, $\mathbf{H} = \mathbf{F}^3$**), it follows that** $\mathbf{G} \mid< \mathbf{E}; \mathbf{H}$**. The proof is completed.**

If families $\mathbf{G}, \mathbf{H}$ **and E are such that E** $|<$ **E; H and** $\mathbf{G} \subseteq \mathbf{E}$**, then it can be shown (Corollary 1.1.1 and Lemma 1.1) that** $P(G_{\geq t} \mid H_{\leq t} = P(G_{\geq t} \mid E_{\leq t})$ **for each** t**. Thus, in this case, the problem of predicting the future behaviour of G has the same solution, no matter which one of the families E or H is used.**

In the remaining part of the paper we consider the problem of determining possible realizations G (of an s.d.s. S_1 **with outputs H) which are a cause of the information E (about the s.d.s.** S_2**) within the family** $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$**; especially, we consider the problem (2) above. More precisely, solutions of the problem (2) are obtained as consequences of the following more general result which gives conditions under which G is a minimal realization such that** $\mathbf{E} \mid< \mathbf{G}; \mathbf{H} \vee \mathbf{E}$ **holds.**

THEOREM 2.3. *Let* $\mathbf{H}$ *and* $\mathbf{E}$ *be such that* $P(H_t \mid E_{<\infty}) \subseteq H_{\leq t}$ *and*

$$H_{<t} \perp H_{>t} \mid P(H_t \vee E_t \mid E_{<\infty})$$

for each t*. If* $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ *is markovian, then the family* $\mathbf{G}$*, defined by*

$$G_t = P(H_t \vee E_t \mid E_{<\infty}),\ \ t \in R, \tag{2.2}$$

is minimal realization (of an s.d.s. with outputs $\mathbf{H}$*) which is a cause of* $\mathbf{E}$ *within* $\mathbf{E}^1$*.*

Proof. From $G_{\leq t} = P(E^1_{\leq t} \mid E_{<\infty})$ and Lemma 1.1 it follows that $E_{<\infty} \perp E^1_{\leq t} \mid G_{\leq t}$. Also, the definition of **G** and the assumption $P(H_t \mid E_{<\infty}) \subseteq H_{\leq}$ imply $G_{\leq t} \subseteq E^1_{\leq t}$ which, together with the previous conclusion, means that **E** $|<$ **G**; $\mathbf{E}^1$. The minimality of **G** follows from Corollary 1.2.1.

From **E** $|<$ **G**; $\mathbf{E}^1$ and the obvious equality $G_{<\infty} = E_{<\infty}$ it follows that **G** $|<$ **G**; $\mathbf{E}^1$ (and, in particular, **G** $|<$ **G**; **H**). From $G_{\leq t} \subseteq E^1_{\leq t}$, the fact that

$$P(G_{\geq t} \mid G_{\leq t}) = P(E^1_{\geq} t \mid G_{\leq t})$$

(which follows from $G_{<\infty} = E_{<\infty}$), and the assumption that $\mathbf{E}^1$ is markovian, we obtain:

$$P(G_{\geq t} \mid G_{\leq t}) = P(P(E^1_{\geq t} \mid E^1_{\leq t}) \mid G_{\leq t}) = P(E^1_t \mid G_{\leq t}). \tag{2.3}$$

However, **G** $|<$ **G**; $\mathbf{E}^1$ means in particular that $E^1_t \perp G_{<\infty} \ominus G_{\leq t}$, so that (2.3) becomes

$$P(G_{\geq t} \mid G_{\leq t}) = P(E^1_t \mid G_{<\infty}) = P(E^1_t \mid E_{<\infty}) = G_t$$

which means that **G** is markovian. Now, Theorem 2.1 completes the proof.

The next example shows that the family **G**, defined by (2.2), is not a strictly minimal realization (of an s.d.s. S_1) such that **E** $|<$ **G**; $\mathbf{H} \vee \mathbf{E}$.

Example 2.1. Let A and B be arbitrary Hilbert spaces and let $\mathbf{H} = (H_t)$ and $\mathbf{E} = (E_t)$, $t \in \{1,2,3\}$ be defined by

$$H_1 = A,\ \ H_2 = B,\ \ H_3 = A,\ \ E_1 = A,\ \ E_2 = A,\ \ E_3 = B.$$

Family $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ is then given by $E^1_1 = A$, $E^1_2 = A \vee B$, $E^1_3 = A \vee B$. It is easy to see that $\mathbf{E}^1$ is markovian and $P(H_t \mid E_{,\infty}) \subseteq H_{\leq t}$, $H_{<t} \perp H_{>t} \mid P(H_t \vee E_t \mid E_{<\infty})$ for each t. If family **G** is drfined by (2.2), then

$$G_1 = A,\ \ G_2 = A \vee B,\ \ G_3 = A \vee B.$$

According to Theorem 2.3, **G** is a realization (of an s.d.s. with outputs **H**) and **E** $|<$ **G**; $\mathbf{E}^1$. However, the family $\mathbf{G}^* = (G^*_t)$, $t \in \{1,2,3\}$ defined by $G^*_1 = A$, $G^*_2 = A \vee B$, $G^*_3 = \{0\}$ is another realization of the same s.d.s., and **E** $|<$ $\mathbf{G}^*$; $\mathbf{E}^1$. Obviously, $\mathbf{G}^* \leq \mathbf{G}$.

The problem of determining the realization **G** (of an s.d.s. with outputs **H**) which is a cause of $\mathbf{E}^1$ within **H**, in the case when family $\mathbf{E}^1 = \mathbf{H} \vee \mathbf{E}$ is not markovian, is still open. One way to try to solve this problem is to find families $\mathbf{E}^* \subseteq \mathbf{E}$ such that $\mathbf{H} \vee \mathbf{E}^*$ is markovian, and then, analogously to what we had in Theorem 2.3, find a realization **G** (of an s.d.s. with outputs **H**) such that **E** $|<$ **G**; $\mathbf{H} \vee \mathbf{E}^*$. Thus, with a "sacrifice" of a part of an inforfation **E** it would be possible to find a realization (of an s.d.s. with outputs **H**) which is in a certain causal relationship with information thus "made smaller"

The next corollary of Theorem 2.3 gives a partial solution (under the condition that H is markovian) of problem (2).

COROLLARY 2.3.1. *Let* $\mathbf{H}$ *and* $\mathbf{E}$ *be such that* $E_{<\infty} \subseteq H_{<\infty}$, $P(H_t \mid E_{<\infty}) \subseteq H_{\leq t}$ *and* $H_{<t} \perp H_{>t} \mid P(H_t \mid E_{<\infty})$ *for each t. If* $\mathbf{H}$ *is markovian, then the family* $\mathbf{G}$ *defined by* $G_t = P(H_t \mid E_{<\infty})$, $t \in R$, *is a minimal realization (of an s.d.s. with outputs* $\mathbf{H}$*) which is a cause of* $\mathbf{E}$ *within* $\mathbf{H}$.

We obtain a simpler version of the above result if E is its own cause within H.

COROLLARY 2.3.2. *Let* $\mathbf{E}$ *be its own cause within* $\mathbf{H}$ *and* $H_{<t} \perp H_{>t} \mid P(H_t \mid E_{\leq t})$ *for each t. If* $\mathbf{H}$ *is markovian, then the family* $\mathbf{G}$*, defined by* $G_t = P(H_t \mid E_{\leq t})$, $t \in R$, *is a minimal realization (of an s.d.s. with outputs* $\mathbf{H}$*) which is a cause of* $\mathbf{E}$ *within* $\mathbf{H}$.

The assumption (in Corollary 2.3.1 and Corollary 2.3.2) that H itself is markovian is rather strong, because H represent the outputs of an s.d.s., and thus the properties of H could hardly be controlled. The following result does not require H to be markovian, but provides a realization whose present information at t **is equal to its total information accumulated up to** t**.**

THEOREM 2.4. *Let* $\mathbf{H}$ *and* $\mathbf{E}$ *be such that* $E_{<\infty} \subseteq H_{<\infty}$, $P(H_t \mid E_{<\infty}) \subseteq H_{\leq t}$ *and* $H_{<t} \perp H_{>t} \mid P(H_{\leq t} \mid E_{<\infty})$ *for each t. The family* $\mathbf{G}$*, defined by*

$$G_t = P(H_{\leq t} \mid E_{<\infty}), \quad t \in R, \tag{2.4}$$

is a minimal realization (of an s.d.s. with outputs $\mathbf{H}$*) which is a cause of* $\mathbf{E}$ *within* $\mathbf{H}$.

Proof. **Since** $G_t = G_{\leq t}$ **for all** t**, it is immediately clear that G is markovian. From Lemma 1.1 it follows** $E_{<\infty} \perp H_{\leq t} \mid G_t$**; that is,** $E_{<\infty} \perp H_{\leq t} \mid G_{\leq t}$**, which, together with** $\mathbf{G} \subseteq \mathbf{H}$**, means that** $\mathbf{E} \mid< \mathbf{G}; \mathbf{H}$**. From the last relation and** $G_{<\infty} = E_{<\infty}$ **it follows that** $\mathbf{G} \mid< \mathbf{G}; \mathbf{H}$**. Now, Theorem 2.1 implies that G is a realization (of an s.d.s. with outputs H). The minimality of G follows from Corollary 1.2.1.**

COROLLARY 2.4.1. *Under the conditions of Theorem* 2.4*, the family* $\mathbf{G}$ *defined by* (2.4) *is such that* $G_t = E_{\leq t}$ *if and only if* $\mathbf{E} \mid< \mathbf{E}; \mathbf{H}$.

It would certainly be intresting to find conditions for the existence of a realization with certain properties less restrictive than those obtained in this paper.

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Odsek za matematiku
Prirodno-matematički fakultet
34000 Kragujevac
Jugoslavija

(Received 23 03 1988)