

ON A RUSCHEWEYH TYPE GENERALIZATION OF THE PASCU CLASS OF ANALYTIC FUNCTIONS

R. Parvatham and S. Radha

Abstract. New classes $M_a(\alpha; h)$, $R_a(\alpha, h)$ and $I_a(\alpha; h)$ of analytic functions are defined and studied. results of this paper generalize mainly results of Padmanabham and Manjini [5], Padmanabhan and Parvatham [4] and Pascu [5].

Let $E = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disc in \mathbf{C} and $H(E)$ be the class of functions $f(z)$ holomorphic in E . Let $A = \{f \in H(E); f(0) = 0 = f'(0) - 1\}$. By $f * g$ we denote the Hadamard product or convolution of $f, g \in H(E)$; that is if $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $g(z) = \sum_{j=0}^{\infty} b_j z^j$ then $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$.

Let g and G be two functions in $H(E)$. Then $g(z)$ is said to be subordinate to $G(z)$ (written $g(z) \prec G(z)$) if G is univalent, $g(0) = G(0)$ and $g(E) \subset G(E)$. Let $k_a(z) = z(1-z)^{-a}$, where a is any real number. In the sequel $h \in H(E)$ is a convex univalent function in E with $h(0) = 1$ and $\operatorname{Re} h(z) > 0$ in E .

In this paper we define certain new classes of functions holomorphic in E with Montel's normalizations and study these classes in detail. To establish results of this paper connected with these new classes, we require the following two theorems. Theorem A is due to Eenigenburg, Miller, Mocanu and Reade [2] and Theorem B may be found in [4].

THEOREM A. Let $\beta, \gamma \in \mathbf{C}$ and $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \gamma) > 0$ in E , and let $p(z) = 1 + p_1 z + \dots \in H(E)$. Then $p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec h(z)$ implies $p(z) \prec h(z)$.

THEOREM B. Let $\beta, \gamma \in \mathbf{C}$, $k \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}(\beta h(z) + \gamma) > 0$ in E . Let $q \in H(E)$ with $q(0) = 1$ and $q(z) \prec h(z)$ in E . If $p(z) = 1 + p_1 z + \dots \in H(E)$ then $p(z) + \frac{z p'(z)}{\beta q(z) + \gamma} \prec h(z)$ implies $p(z) \prec h(z)$.

First let us define a new class $M_a(\alpha; h)$ of holomorphic functions in E and study its properties.

Definition 1. Let $M_a(\alpha; h)$ denote the class of functions f with

$$(k_a * f)'(z)(k_a * f)(z) \neq 0 \text{ in } E - \{0\} \text{ satisfying}$$

$$\frac{\alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z)}{\alpha z(k_a * f)'(z) + (1 - \alpha)(k_a * f)(z)} \prec h(z) \text{ for } \alpha \geq 0.$$

Note 1. When $\alpha = 0$ this class coincides with the class $S_a(h)$ studied in [4] and when $\alpha = 1$ this is the same class as $K_a(h)$ in [3]. Also $M_1(\alpha; (1 - z)(1 + z)^{-1})$ is the class introduced by Pascu and Podaru [6].

THEOREM 1. For $0 < \alpha \leq 1$ we have $M_a(\alpha; h) \subset M_a(0; h) = S_a(h)$.

Proof. Let $f \in M_a(\alpha; h)$ and $p(z) = \frac{z(k_a * f)'(z)}{(k_a * f)(z)}$. Then

$$\begin{aligned} & \alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z) \\ &= \alpha z(k_a * f)(z)p'(z) + \alpha z p(z)(k_a * f)'(z) + (1 - \alpha)p(z)(k_a * f)(z) \\ &= (\alpha z p'(z) + p(z)(\alpha p(z) + (1 - \alpha)))(k_a * f)(z); \\ & \alpha z(k_a * f)'(z) + (1 - \alpha)(k_a * f)(z) = (\alpha p(z) + (1 - \alpha))(k_a * f)(z) \end{aligned}$$

Hence

$$\begin{aligned} \frac{\alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z)}{\alpha z(k_a * f)'(z) + (1 - \alpha)(k_a * f)(z)} &= \frac{\alpha z p'(z) + p(z)(\alpha p(z) + (1 - \alpha))}{\alpha p(z) + (1 - \alpha)} \\ &= \frac{z p'(z)}{p(z) + (\alpha^{-1} - 1)} + p(z) \prec h(z) \end{aligned}$$

because $f \in M_a(\alpha; h)$. Since $0 < \alpha \leq 1$, an application of Theorem A gives $p(z) \prec h(z)$ in E which implies $f \in S_a(h)$.

THEOREM 2. Let $f \in M_a(\alpha; h)$. Then for $0 < \alpha \leq 1$ we have

$$F(z) = \alpha^{-1} z^{1-1/\alpha} \int_0^z t^{1/\alpha-2} f(t) dt \in M_a(\alpha; h).$$

Proof. Differentiating $F(z) = \alpha^{-1} z^{1-1/\alpha} \int_0^z t^{1/\alpha-2} f(t) dt$ with respect to z and simplifying we get $\alpha z F'(z) + (1 - \alpha)F(z) = f(z)$. This, by convolution with $k_a(z)$, gives

$$\alpha z(k_a * F)'(z) + (1 - \alpha)(k_a * F)(z) = (k_a * f)(z)$$

where we used the fact that $k_a * zF'(z) = z(k_a * F)'(z)$. Taking logarithmic derivative with respect to z and multiplying by z we get

$$\frac{\alpha z(z(k_a * F)'(z))' + (1 - \alpha)z(k_a * F)'(z)}{\alpha z(k_a * F)'(z) + (1 - \alpha)(k_a * F)(z)} = \frac{z(k_a * f)'(z)}{(k_a * f)(z)}.$$

The member on the right hand side is subordinate to $h(z)$ since $f \in M_a(\alpha; h) \subset S_a(h)$ by the previous theorem. Also $F(z) = \gamma(z) * f(z)$ where $\gamma_\alpha(z) = \sum_{n=1}^{\infty} \frac{1/\alpha}{1/\alpha+n-1} z^n$. Since $(k_a * f)(z) \neq 0$, $(k_a * f)'(z) \neq 0$ in $E - \{0\}$ and $\alpha > 0$ we have $(k_a * F)(z) = \gamma_\alpha(z)(k_a * f)(z) \neq 0$, hence $(k_a F)'(z) \neq 0$ in $E - \{0\}$. Thus $F \in M_a(\alpha; h)$.

We now obtain an estimate for the moduli of the coefficients, $|a_n|$ where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_a(\alpha; h)$

THEOREM 3. *Let $f(z) = z + \sum_{i=2}^{\infty} a_i z^i$ be in $M_a(\alpha; h)$ Then*

$$(1) \quad |a_i| \leq \frac{|h_1| (1 + |h_1|) \cdots (i - 2 + |h_1|)}{((i - 1)\alpha + 1)a(a + 1) \cdots (a + i - 2)}, \quad i \geq 2,$$

where $h(z) = 1 + h_1 z + \cdots$.

Proof. Let
$$\frac{\alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z)}{\alpha z(k_a * f)'(z) + (1 - \alpha)(k_a * f)(z)} = p(z) = 1 + p_1 z + \cdots$$

Since $f \in M_a(\alpha; h)$, $p(z) \prec h(z) = 1 + h_1 z + h_2 z^2 + \cdots$. It is well known that $|p_i| \leq |h_1|$ for all $i \geq 2$. Now,

$$(2) \quad \alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z) = p(z)(\alpha z(k_a * f)'(z) + (-\alpha)(k_a * f)(z))$$

and $k_\alpha(z) = z(1 - z)^{-\alpha} = z + \sum_{i=2}^{\infty} b_i z^i$ where $b_i = a(a + 1) \cdots (a + i - 2)/(i - 1)!$.

By actual computation we have

$$\begin{aligned} \alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z) &= z + \sum_{i=2}^{\infty} i((i - 1)\alpha + 1)a_i b_i z^i; \\ p(z)(\alpha z k_a * f)'(z) + (1 - \alpha)(k_a * f)(z) &= \\ &= (1 + p_1 z + \cdots) \left(z + \sum_{i=2}^{\infty} ((i - 1)\alpha + 1)a_i b_i z^i \right). \end{aligned}$$

Now comparing the coefficients on either side of (2) we get

$$\begin{aligned} i((i - 1)\alpha + 1)a_i b_i &= p_{i=1} + p_{i-2} a_2 b_2 (\alpha + 1) + p_{i-3} a_3 b_3 (2\alpha + 1) \\ &+ \cdots + p_1 a_{i-1} b_{i-1} ((i - 2)\alpha + 1) + a_i b_i ((i - 1)\alpha + 1). \end{aligned}$$

Let (1) be true for all $i = 2, \dots, n - 1$. In other words

$$(3) \quad |a_i b_i| \leq \frac{|h_1| (1 + |h_1|) \cdots (1 + |h_1| (i - 2)^{-1})}{(i - 1)((i - 1)\alpha + 1)} \quad \text{for } i = 2, 3, \dots, n - 1.$$

Now for $i = n$ we have

$$n(n-1)\alpha + 1)a_n b_n = p_{n-1} + p_{n-2}a_2 b_2(\alpha + 1) + p_{n-3}a_3 b_3(2\alpha + 1) = \cdots \\ \cdots + p_1 a_{n-1} b_{n-1}((n-2)\alpha + 1) + a_n b_n((n-1)\alpha + 1);$$

$$(n-1)((n-1)\alpha + 1)a_n b_n = p_{n-1} + p_{n-2}a_2 b_2(\alpha + 1) + \cdots \\ \cdots + p_1 a_{n-1} b_{n-1}((n-2)\alpha + 1);$$

$$(n-1)((n-1)\alpha + 1) |a_n b_n| \leq |h_1| + |h_1|(\alpha + 1) |a_2 b_2| + \cdots \\ \cdots + |h_1|((n-2)\alpha + 1) |a_{n-1} b_{n-1}| \leq |h_1| + |h_1|^2/2 \cdot (1 + |h_1|) + \cdots \\ \cdots + |h_1|^2/(n-2) \cdot (1 + |h_1|) \cdots (1 + |h_1|/(n-3)) \\ = |h_1|(1 + |h_1|)(1 + |h_1|/2) \cdots (1 + |h_1|/(n-2)).$$

Hence

$$|a_n b_n| \leq \frac{|h_1|(1 + |h_1|)(1 + |h_1|/2) \cdots (1 + |h_1|/(n-2))}{(n-1)((n-1)\alpha + 1)}$$

which means (3) is true for $i = n$ provided it is true for $i = 2, 3, \dots, n-1$. It is easy to see that (1) is true for $i = 2$ and hence it is true for all $i \geq 2$.

Remark. The results above generalize many results found in [3, 4] and [6].

Now we define another new class of functions $R_a(\alpha; h)$ which generalizes both the class $C(\alpha; h)$ of Pascu [5] and the class $C_a(h)$ in [4].

Definition 2. Let $R_a(\alpha; h)$ denote the class of functions $f \in A$ such that $\frac{\alpha z(z(k_a * f)'(z))' + (1-\alpha)z(k_a * f)'(z)}{\alpha z(k_a * \varphi)'(z) + (1-\alpha)(k_a * \varphi)(z)} \prec h(z)$ for some $\varphi \in M_a(\alpha; h)$ and $\alpha \geq 0$.

Here we prove an inclusion relation and also the fact that this class is closed under a certain integral operator.

THEOREM 4. $R_a(\alpha; h) \subset R_a(0; h) = C_a(h)$ for $0 < \alpha < 1$.

Proof. Let $f \in M_a(\alpha; h)$. Setting

$$p(z) = \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \quad \text{and} \quad q(z) = \frac{z(k_a * \varphi)'(z)}{(k_a * \varphi)(z)},$$

we have

$$\frac{\alpha z(z(k_a * f)'(z))' + (1-\alpha)z(k_a * f)'(z)}{\alpha z(k_a * \varphi)'(z) + (1-\alpha)(k_a * \varphi)(z)} \\ = \frac{azp'(z) + p(z) \left(\frac{\alpha z(k_a * \varphi)'(z)}{(k_a * \varphi)'(z)} + 1 - \alpha \right)}{\frac{\alpha z(k_a * \varphi)'(z)}{(k_a * \varphi)'(z)} + (1-\alpha)} \\ = p(z) + \frac{zp'(z)}{\frac{z(k_a * \varphi)'(z)}{(k_a * \varphi)(z)} + \left(\frac{1}{\alpha} - 1 \right)} \prec h(z)$$

since $f \in R_a(\alpha; h)$. Here $q(z) \prec h(z)$ by Theorem 1. Since $\alpha \leq 1$ an application of Theorem B yields $p(z) \prec h(z)$ there by establishing the theorem.

THEOREM 5. $F(z) = \alpha^{-1} z^{1-1/\alpha} \int_0^z t^{1/\alpha-2} f(t) dt \in R_a(\alpha; h)$ whenever $f \in R_a(\alpha; h)$, for $0 < \alpha \leq 1$.

Proof. Differentiating F with respect to z we have

$$\alpha z F'(z) + (1 - \alpha)F(z) = f(z).$$

This on convolution with $k_a(z)$ yields

$$\alpha z(k_a * F)'(z) + (1 - \alpha)(k_a * F)(z) = (k_a * f)(z),$$

where we used the identity $(k_a * zF')(z) = z(k_a * F)'(z)$. Again differentiating with respect to z we get

$$(4) \quad \alpha z(z(k_a * F)'(z))' + (1 - \alpha)z(k_a * F)'(z) = z(k_a * f)'(z), \quad f \in R_a(\alpha; h).$$

Hence there exist a $\varphi \in M_a(\alpha; h)$ such that

$$\frac{\alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z)}{\alpha z(k_a * \varphi)'(z) + (1 - \alpha)z(k_a * \varphi)(z)} \prec h(z) \text{ in } E.$$

Then by Theorem 2, Φ defined by

$$(5) \quad \Phi(z) = \alpha^{-1} z^{1-1/\alpha} \int_0^z t^{1/\alpha-2} \varphi(t) dt$$

is in $M_a(\alpha; h)$ for $1 \geq \alpha > 0$. Differentiating (5) with respect to z and convoluting the result with $k_a(z)$ yields after simplification

$$(6) \quad \alpha z(k_a * \Phi)'(z) + (1 - \alpha)(k_a * \Phi)(z) = (k_a * \varphi)(z).$$

Finally (4) and (6) together yield,

$$\frac{\alpha z(z(k_a * F)'(z))' + (1 - \alpha)z(k_a * F)'(z)}{\alpha z(k_a * \varphi)'(z) + (1 - \alpha)(k_a * \varphi)(z)} = \frac{z(k_a * f)(z)}{(k_a * \varphi)(z)}.$$

Since $f \in R_a(\alpha; h)$ by Theorem 4 we get $\frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \prec h(z)$ for $z \in E$, $0 < \alpha \leq 1$.

In the same way as in Theorem 2 we can show that $(k_a * F)'(z) \neq 0$, $(k_a * F)(z) \neq 0$ in $E - \{0\}$ from the fact that $(k_a * f)'(z) \neq 0$, $(k_a * f)(z) \neq 0$ in $E - \{0\}$ for $\alpha > 0$. Thus we get $F \in R_a(\alpha; h)$.

Remark. Theorem 4 and Theorem 5 generalize results in [4] and [5].

Now let us establish a representation theorem for function belonging to the class $R_a(\alpha; h)$.

THEOREM 6. A functions f belongs to $R_a(\alpha; h)$ if and only there exists a function $G \in H(E)$ with $G(0) = 0$ such that $zG'(z)/G(z) \prec h(z)$ in E and an analytic function $p(z)$ with $p(0) = 1$ and $p(z) \prec h(z)$ in E such that

$$(k_a * f)'(z) = \alpha^{-1} z^{-1/\alpha} \int_0^z p(t)G(t)t^{1/\alpha-2} dt, \text{ if } \alpha \neq 0;$$

$$(k_a * f)'(z) = p(z)G(z)/z, \text{ if } \alpha = 0.$$

Proof. $f(z) \in R_a(\alpha; h)$ means that there exists, a $\varphi \in M_a(\alpha; h)$ such that

$$\frac{\alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z)}{\alpha z(k_a * \varphi)'(z) + (1 - \alpha)(k_a * \varphi)(z)} \prec h(z).$$

Since $\varphi(z) \in M_a(\alpha; h)$, there exists a $G(z)$ such that

$$\frac{zG'(z)}{G(z)} = \frac{\alpha z(z(k_a * \varphi)'(z))' + (1 - \alpha)z(k_a * \varphi)'(z)}{\alpha z(k_a * \varphi)'(z) + (1 - \alpha)(k_a * \varphi)(z)} \prec h(z).$$

This on integration gives $G(z) = \alpha z(k_a * \varphi)'(z) + (1 - \alpha)(k_a * \varphi)(z)$ and so $\alpha z(z(k_a * f)'(z))' + (1 - \alpha)z(k_a * f)'(z) = p(z)G(z)$ where $p(z) \prec h(z)$. In $\alpha \neq 0$, multiplying by $\alpha^{-1}z^{1/\alpha-2}$ and integrating we get

$$(k_a * f)'(z) = \alpha^{-1}z^{-1/\alpha} \int_0^s p(t)G(t)t^{1/\alpha-2} dt.$$

Conversely, it is easy to see that if $f(z)$ has the above integral representation then $f(z) \in R_a(\alpha; h)$. For $\alpha = 0$ let $G(z) = (k_a * \varphi)(z)$. Then $(k_a * f)'(z) = p(z)G(z)/z$, where $p(z) \prec h(z)$ and the converse is trivially true.

Finally we define a new class $I_a(\alpha; h)$ which coincides with $C_a(h)$ of [4] when $\alpha = 0$. In particular if $\varphi(z)$ coincides with $f(z)$ then $I_a(\alpha; h)$ is nothing but $K_a^\alpha(h)$ studied in [3].

Definition 3. Let $I_a(\alpha; h)$ denote the class of functions $f \in A$ such that

$$\frac{\alpha(z(k_a * f)'(z))'}{(k_a * \varphi)'(z)} + (1 - \alpha)\frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \prec h(z)$$

for some $\varphi \in S_a(h)$ and $\alpha \geq 0$ in E .

Remark. Though for $a = 1$ the class $I_a(\alpha; h)$ coincides with $C_a^\alpha(h)$ studied in [4] for other values of a , $I_a(\alpha; h)$ is certainly different from $C_a^\alpha(h)$.

THEOREM 7. We have the following inclusion relation $I_a(\alpha; h) \subset I_a(0; h) = C_a(h)$.

Proof. Let $f \in I_a(\alpha; h)$. Setting

$$p(z) = \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \quad \text{and} \quad q(z) = \frac{z(k_a * \varphi)'(z)}{(k_a * \varphi)(z)}$$

we have

$$\begin{aligned} \frac{\alpha(z(k_a * f)'(z))'}{(k_a * \varphi)'(z)} + (1 - \alpha)\frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} &= \\ &= \alpha(zp'(z)/q(z) + p(z)) + (1 - \alpha)p(z) = p(z) + \alpha zp'(z)/q(z). \end{aligned}$$

Since $f(z) \in I_a(\alpha; h)$ we have $p(z) + \alpha zp'(z)/q(z) \prec h(z)$ and $q(z) \prec h(z)$. Now an application of Theorem B yields that $p(z) \prec h(z)$ which implies $f \in I_a(\alpha; h)$.

THEOREM 8. For $\alpha > \beta \geq 0$ we have $I_a(\alpha; h) \subset I_a(\beta; h)$.

Proof. The case $\alpha = 0$ has already been proved in Theorem 7. Hence we can assume $\beta \neq 0$. Suppose that $f(z) \in I_a(\alpha; h)$. Then,

$$\frac{\alpha(z(k_a * f)'(z))'}{(k_a * \varphi)(z)} + (1 - \alpha) \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \prec h(z).$$

Let z_1 be an arbitrary point in E . Then,

$$(7) \quad \frac{\alpha(z_1(k_a * f)'(z_1))'}{(k_a * \varphi)'(z_1)} + (1 - \alpha) \frac{z_1(k_a * f)'(z_1)}{(k_a * \varphi)(z_1)} \in h(E).$$

Because of Theorem 7 we have $\frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \prec h(z)$. Hence,

$$(8) \quad \frac{z_1(k_a * f)'(z_1)}{(k_a * \varphi)(z_1)} \in h(E).$$

Also

$$\begin{aligned} & \frac{\beta(z(k_a * f)'(z))'}{(k_a * \varphi)'(z)} + (1 - \beta) \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} = \\ & = \left(1 - \frac{\beta}{\alpha}\right) \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} + \frac{\beta}{\alpha} \frac{\alpha(z(k_a * f)'(z))'}{(k_a * \varphi)'(z)} + (1 - \alpha) \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)}, \end{aligned}$$

Since $\beta/\alpha < 1$ and $h(E)$ is convex we have

$$\frac{\beta(z_1(k_a * f)'(z_1))'}{(k_a * \varphi)'(z_1)} + (1 - \beta) \frac{z_1(k_a * f)'(z_1)}{(k_a * \varphi)(z_1)} \in h(E) \text{ by (7) and (8).}$$

Thus it follows

$$\frac{\beta(z(k_a * f)'(z))'}{(k_a * \varphi)'(z)} + (1 - \beta) \frac{z(k_a * f)'(z)}{(k_a * \varphi)(z)} \prec h(z)$$

which means $f(z) \in I_a(\beta; h)$.

Now, let us state a representation Theorem for functions belonging to the class $I_a(\alpha; h)$ without a proof since it follows the idea of the proof of Theorem 6.

THEOREM 9. A function $f(z)$ belongs to $I_a(\alpha; h)$ if and only if there exists a function $G(z) \in H(E)$ with $G(0) = 0$ such that $zG'(z)/G(z) \prec h(z)$ in E and an analytic function $p(z)$ with $p(0) = 1$ and $p(z) \prec h(z)$ in E such that

$$\begin{aligned} (k_a * f)'(z) &= \alpha^{-1} z^{-1/\alpha+1} \int_0^z p(t) G^{1/\alpha}(t) G'(t) dt \text{ if } \alpha \neq 0 \text{ and} \\ (k_a * f)'(z) &= p(z) G(z)/z, \text{ if } \alpha = 0, \end{aligned}$$

Remark. If $a = 1$ and $h(z) = (1 - z)/(1 + z)$. Theorem 7, Theorem 8 and Theorem 9 reduce to Theorem 1, Theorem 2 and Theorem 3 respectively of Chichra [1].

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Ramanujan Institute
University of Madras
Madras, 6000 005
India

J. B.A. S. Women's College
(Affiliated to Univ. of Madras)
Madras, 600 018
India

(Received 26 03 1987)