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CHARACTERIZATIONS OF SOME DISTRIBUTIONS CONNECTED WITH EXTREMAL-TYPE DISTRIBUTIONS

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Abstract. It is shown that the class of distribution functions, which preserve the type when submitted to the operation of screening the maxima coincides with the three types of distributions obtained as limiting in a special case of the transfer theorem for maxima.

In [**1**], a limit theorem for the sequence of maxima of a random number of independent, identically distributed random variables is proved, which runs as follows:

Let us suppose that the following sequences are given:

1) sequence of integres k_n, $k_n \to \infty$, $n \to +\infty$;

2) ν_n-nonnegative, integer valued random variables;

3) ξ_{nk}-independent random variables for each n and independent of the variables ν_n, such that $P\{\xi_{nk} < x\} = F_n(x)$.

THEOREM [**1**]. *If, when $n \to +\infty$, the following conditions*

A) $P\{\max\{\xi_{n1};\xi_{n2};\dots;\xi_{nk_n}\} < x\} \to \Phi(x)$,

B) $P\{\nu_n/k_n < x\} \to A(x)$

are satisfied, then

$$P\{\max\{\xi_{n1};\xi_{n2};\dots;\xi_{n\nu_n}\} < x\} \to \Psi(x),$$

where $\Psi(x) = \int_0^{+\infty}[\Phi(x)]^z dA(z)$.

In [**1**] it was pointed out that as a special case of the theorem, when $k_n = n$, the random index ν_n has a geometric distribution function with parameter $a_n = 1/n$ and $\xi_{nk} = (\xi_k - A_n)/B_n$, where ξ_k, $k = 1,2,\dots$ are independent, identically distributed random variables with the distribution function $F(x)$, A_n and B_n are appropriately chosen constants (it is well known that in this case the distribution

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$\Phi(x)$ is of the extremal type) – then the three following distribution functions are limiting

$$\begin{aligned} &1. \quad \Psi(x) = 1/(1+e^{-x}), && -\infty < x < +\infty; \\ &2. \quad \Psi(x) = 1/(1+x^{-\alpha}), && \alpha > 0, 0 \le x < +\infty; \\ &3. \quad \Psi(x) = 1/(1+(-x)^{\alpha}), && \alpha > 0, -\infty < x \le 0, \end{aligned} \tag{1}$$

the first of them being the well known logistic distribution.

Our intention is to characterize the three types of distributions mentioned above.

Let us consider the following operation: $X_1, X_2, \ldots, X_k, \ldots$ be a sequence of independent, identically distributed random variables with the distribution function $F(x)$. Let us form the variables $Y_1 = X_1$ $Y_2 = \max\{X_1; X_2\}, \ldots, Y_k = \max\{X_1; X_2; \ldots; X_k\}, \ldots$. For given $p(0 < p < 1)$, we eliminate (independently of the others) the variable Y_k with probability p. Let us denote by $F_p(x)$ the distribution function of the first retained variable. Then we have:

$$F_p(x) = q \sum_{k=1}^{+\infty} p^{k-1} (D(x))^k = \frac{qF(x)}{1-pF(x)} \tag{2}$$

where $q = 1-p$, $p+q = 1$.

We are interested in the class of random variables for which the distribution function $F_p(x)$ (2) of the first retained maximum has, for every p, the same type as the distribution $F(x)$. In other words, let us suppose that, for each $p, q(p, q > 0, p+q = 1)$ there exist real numbers $a(q) > 0$ and $b(q)$ such that the following equation holds:

$$\frac{qF(x)}{1-pF(x)} = F(a(q)x + b(q)), \quad -\infty < x < +\infty. \tag{3}$$

We want to find the class of random variables whose distribution functions satisfy the equation (3) for some $a(q) > 0$ and $b(q)$, $0 < q < 1$. The following two theorems will give all solutions $(F(x), a(q), b(q))$, $-\infty < x < +\infty$, $0 < q < 1$, of the functional equation (3). The case of degenerated $F(x)$ is not considered since it is trivial.

THEOREM 1. *Let us suppose that in the equation (3), the function $a(q)$ is identically equal to one. Then the pair $(F(x), b(q))$ is the solution of the equation (3) if nnd only if*

$$F(x) = 1/(1+ce^{-\alpha x}), \quad -\infty < x < +\infty, \ \alpha > 0, \ c > 0 \tag{4}$$

and

$$b(q) = (\ln q)/\alpha, \quad 0 < q < 1. \tag{5}$$

Proof. The direct part of the theorem is proved immediately by inspection. Now, let us prove the converse: if the equation

$$\frac{qF(x)}{1-(1-q)F(x)} = F(x + b(q)), \quad -\infty < x < +\infty \tag{6}$$

holds, then (4) and (5) follow.

From (6) it follows that the function $b(q)$, $0 < q < 1$; has no zero, because if there exists a q_0 such that $b(q_0) = 0$, then (6) becomes

$$\frac{q_0 F(x)}{1-(1-q_0)F(x)} = F(x),$$

whence $F(x)$ takes only values 0 or 1, contrary to our assumption that $F(x)$ is nondegenerated.

Let us show that $F(x) > 0$, for every x. If there existed an x_0 such that $F(x_0) = 0$, then from (6) we would have $F(x_0 + nb(q)) = 0$ and $F(x_0 - nb(q)) = 0$, $n = 1, 2, \ldots$, which for $b(q) > 0$, as well as for $b(q) < 0$, implies the impossible $F(+\infty) = 0$. Since $F(x) > 0$, for every x, we can write $F(x) = 1/(1 + v(x))$, $-\infty < x < +\infty$, where $v(x) \geq 0$, $v(+\infty) = 0$, $v(-\infty) = +\infty$. It is easy to check that in terms of the function $v(x)$, the equation (6) reduces to the following simpler form

$$v(x) = qv(x + b(q)). \tag{7}$$

We shall show that $b(q)$ is increasing and $b(q) < 0$ for every $0 < q < 1$. If, for some q_0, we had $b(q_0) > 0$, then from (7) we should have for every x and $n = 1, 2, \ldots$ that $v(x) = q_0^n v(x + nb(q_0))$, which means that $v(x) \equiv 0$, contrary to our assumption. So, we must have $b(q) < 0$.

Let us consider $q = q_1 q_2$. According to (7),

$$v(x + b(q)) = v(x)/q = v(x)/(q_1 q_2) = v(x + b(q_1) + b(q_2));$$

hence, we conclude that

$$b(q_1 q_2) = b(q_1) + b(q_2) \tag{8}$$

Since $b(q) < 0$, $0 < q < 1$, it follows from (8) that $b(q)$ is strictly increasing. From (7) and from the monotonicity of $F(x)$ and $b(q)$, the continuity of $F(x)$ and $b(q)$ follows.

If we make a substitution $f(x) = b(e^{-x})$ then (8) reduces to $f(x+y) = f(x) + f(y)$, $x, y > 0$. The general solution of the preceding equation is [**2**]: $f(x) = -ax$, $x > 0$, $a > 0$. We have here $a > 0$ because $b(q) < 0$, $0 < q < 1$. Comming back to the function $b(q)$, we have $b(q) = f(-\ln q) = a \ln q$, $a > 0$.

One solution of the functional equation (7) is obvious, namely $v_1(x) = e^{-\alpha x}$, $-\infty < x < +\infty$, $\alpha > 0$. Let us now suppose that the equation (7) has also another solution $v_2(x)$. In order to determine $v_2(x)$, let us consider the ratio $q(x) = v_2(x)/v_1(x)$. From (7) we have

$$g(x + b(q)) = \frac{v_2(x + b(q))}{v_1(x + b(q))} = \frac{v_2(x)}{v_1(x)} = g(x), \qquad -\infty < x < +\infty;$$

whence the function $g(x)$ is constant $g(x) = c$, $-\infty < x < +\infty$. Therefore, every solution of (7) is of the form $v(x) = ce^{-\alpha x}$, $-\infty < x < +\infty$, $\alpha > 0$, $c > 0$. So we have proved that every solution of (6) is the distribution function $F(x) =$

$1/(1+ce^{-\alpha x})$, $-\infty < x < +\infty$, $\alpha > 0$, $c > 0$, which belongs to the logistic distribution type. From (6) it follows that $b(q) = \ln q/\alpha$, $0 < q < 1$, $\alpha > 0$. □

THEOREM 2. *Let us suppose that the function $a(q)$ in the functional equation* (3) *is not identically equal to one. Then the triple $(F(x), a(q), b(q))$ is the solution of the equation* (3) *if and only if*

$$\text{a)}\quad F(x) = \begin{cases} 0 & x \leq \beta \\ 1/(1+\gamma(x-\beta)^{-\alpha}), & x > \beta,\ \alpha > 0,\ \gamma >, 0 \end{cases}$$
$$a(q) = q^{1/\alpha},\ b(q) = \beta(1-q^{1/\alpha}),\ 0 < q < 1,\ \alpha > 0, \tag{9}$$

or

$$\text{b)}\quad F(x) = \begin{cases} 1/(1+\gamma(-x+\beta)^{\alpha}), & x > \beta,\ \alpha > 0,\ \gamma > 0, \\ 1 & x \geq \beta \end{cases}$$
$$a(q) = q^{-1/\alpha},\ b(q) = \beta(1-q^{-1/\alpha}),\ 0 < q < 1,\ \alpha > 0, \tag{10}$$

Proof In one direction the theorem is proved immediately by inspection. Now let us prove the converse, i.e. that if the equation

$$\frac{qF(x)}{1-(1-q)F(x)} = F(a(q)x + b(q)),\ a(q) > 0,\ a(q) \not\equiv 1,\ 0 < q < 1 \tag{11}$$

holds, then all the solutions are of the form (9) or (10).

From (11) it follows that for an arbitrary number β,

$$\frac{qF(x+\beta)}{1-(1-q)F(x+\beta)} = F(a(q)x + a(q)\beta + b(q)) \tag{12}$$

is valid. Let us fix a number q such that $a(q) \neq 1$, $0 < q < 1$. For such a q, let us denote by β_q a number which satisfies the following equation: $\beta_q = a(q)\beta_q + b(q)$, wherefrom we have

$$\beta_q = b(q)/(1-a(q)). \tag{13}$$

Let us denote

$$\Phi_q(x) = F(x + \beta q). \tag{14}$$

For such a q, the equality (12) for the function $\Phi_q(x)$ reduces to

$$\frac{q\Phi_q(x)}{1-(1-q)\Phi_q(x)} = \Phi_q(a(q)x). \tag{15}$$

It follows from (15) that either $\Phi_q(0) = 0$ or $\Phi_q(0) = 1$. Let us prove that. Indeed for $x = 0$, it follows from (15) that

$$\frac{q\Phi_q(0)}{1-(1-q)\Phi_q(0)} = \Phi_q(0);$$

hence

$$0 = \frac{q\Phi_q(0)}{1-(1-q)\Phi_q(0)} - \Phi_q(0) = \frac{(1-q)((\Phi_q(0)))^2 - \Phi_q(0)}{1-(1-q)\Phi_q(0)},$$

and so, since $q < 1$, $\Phi_q(0)$ could take only two values: 0 and 1.

First we shall consider the following case:

a) $\Phi_q(0) = 0$. We shall show that $\Phi_q(0) > 0$ for every $x > 0$. Let us suppose the contrary, i.e. that for some $x_0 > 0$ we have $\Phi_q(x_0) = 0$. For our fixed q, the function $a(q)$ could be either strictly less ot strictly greater than 1 (because we chose q so that $a(q) \neq 1$ is valid). Let us suppose that $a(q) > 1$. Then it follows from (15) that for each $n \in N$, $0 = \Phi_q((a(q)^n x) \to \Phi_q(-\infty)$ is valid, contrary to our assumption that we deal with nondegenerated distribution functions.

If $a(q) < 1$, then we have from (15): $\Phi_q(x_0/(a(q))^n) \to \Phi_q(+\infty) = 0$, $n \to +\infty$, which is impossible. So, we proved that if $\Phi_q(0) = 0$, then the function $\Phi_q(x)$ is never zero for $x > 0$.

Now, let us consider the case:

b) $\Phi_q(0) = 1$. Let us show that then $0 < \Phi_q(x) < 1$, for every $x < 0$. Let us suppose that for some $x_0 < 0$ we have $\Phi_q(x_0) = 0$. If $a(q) < 1$, then it follows from (15) that $0 = \Phi_q((a(q))^n x) \to \Phi_q(0_-)$, $n \to +\infty$; hence $\Phi_q(x)$ is degenerated at zero. If $a(q) > 1$, we have as $n \to +\infty$, $0 = \Phi_q(x/(a(q))^n) \to \Phi_q(0_-)$, and hence again we have that $\Phi_q(x)$ is degenerated at zero, which is impossible. So, we must have $\Phi_q(x) > 0$ for every $x < O$. Let us show that $\Phi_q(x) < 1$, for every $x < O$. Let us suppose that $\Phi_q(x_0) = 1$ for some $x_0 < 0$. lf $a(q) < 1$, then, when $n \to +\infty$ we have $\Phi_q((a(q))^n x) \to \Phi_q(-\infty) = 1$ which is impossible; if $a(q) > 1$, it follows from (15), that, when $n \to +\infty$, we have $\Phi_q(x_0/(a(q))^n) \to \Phi_q(-\infty) = 1$, which is again impossible. So we have proved that $0 < \Phi_q(x) < 1$, for every $x < 0$.

Now we shall prove that there is no $q (0 < q < 1)$ such that $a(q) = 1$. If for some q_0, we had $a(q_0) = 1$, then (11) would become

$$\frac{q_0 F(x)}{1 - (1 - q_0) F(x)} = F(x + b(q_0)). \tag{16}$$

In Theorem 1 we proved that it follows from (16) that $F(x) > 0$ for every x. Let us show that from (16) it follows that $F(x) < 1$, for every x. If $F(x_0) = 1$ for some x_0, then from (16) it follows that if $b(q_0) < 0$, we have that, as $n \to +\infty$, $1 = F(x_0 + nb(q_0)) \to F(-\infty)$, and if $b(q_0) > 0$ we have that, as $n \to +\infty$, $1 = F(x_0 - nb(q_0)) \to F(-\infty)$, both cases both impossible. If $b(q_0) = 0$, it follows that $F(x)$ takes only the values 0 or 1, which is contrary to our assumptions. So we have that if $a(q_0) = 1$ for some q_0, then $0 < F(x) < 1$, $-\infty < x < +\infty$. But, by assumption $a(q)$ is not identically equal to 1, which means that there exists a q such that $a(q) \neq 1$. We already showed that for such a q the function $\Phi_q(0) = F(\beta_q)$ takes only the values 0 or 1, contrary to the condition $O < F(x) < 1$, $-\infty < x < +\infty$, obtained from the assumption that there exists a q_0 such that $a(q_0) = 1$. Hence, from the supposition that there exists at least one q such that $a(q) \neq 1$, it follows that $a(q) \neq 1$ for every $0 < q < 1$.

Let us consider two numbers q_1 and q_2, $q_1 \neq q_2$ such that the corresponding β_{q_1}, β_{q_2}, (13) are not equal, $\beta_{q_1} \neq \beta_{q_2}$. Let us suppose that $\beta_{q_1} < \beta_{q_2}$. If $\Phi_{q_1}(0) = 0$, then we have that $\Phi_{q_1}(x) > 0$, for $x > 0$, or, equivalently

$$F(\beta_{q_1}) = 0, \quad F(x + \beta_{q_1}) > 0, \quad x > 0. \tag{17}$$

We know that $\Phi_{q_2}(0)$ could be 0 or 1. Let $\Phi_{q_2}(0) = 0$ and consequently $\Phi_{q_2}(x) > 0$ for $x > 0$. Hence $F(\beta_{q_2}) = 0$, contrary to (17). Let $\Phi_{q_2}(0) = 1$. But then we have $\Phi_{q_2}(x) > 0$, for every $x < 0$, contrary to (17).

Let us suppose that $\Phi_{q_1}(0) = 1$, wherefrom we have

$$F(\beta_{q_1}) = 1, \quad 0 < F(x + \beta_{q_1}) < 1, \quad x < O. \tag{18}$$

If $\Phi_{q_2}(0) = 0$, we have $F(x + \beta_{q_2}) = 0$ for $x < 0$, contrary to (18). If $\Phi_{q_2}(0) = 1$, we have

$$F(\beta_{q_2}) = 1, \quad 0 < F(x + \beta_{q_2}) < 1, \quad x < 0. \tag{19}$$

But, from (18) we have $F(\beta_{q_1}) = 1$, contrary to (19) which states that there must be $F(\beta_{q_1}) = 1$, since $\beta_{q_1} < \beta_{q_2}$.

From the assumption that there are numbers q_1, q_2 such that $\beta_{q_1} \neq \beta_{q_2}$ we obtained a contradiction, which proves that all β_q, $0 < q < 1$, are equal. Let us denote by β the following expression, which, as it is shown, does not depend on q: $\beta = b(q)/(1 - a(q))$, $0 < q < 1$. Next, let $\Phi(x) = F(x + \beta)$. In terms of the function $\Phi(x)$, the equation (11) reduces to

$$\frac{q\Phi(x)}{1 - (1 - q)\Phi(x)} = \Phi(a(q)x), \quad 0 < q < 1. \tag{20}$$

It is already shown that either $\Phi(0) = 0$ or $\Phi(0) = 1$. We shall consider each of these cases separately.

a) case $\Phi(0) = 0$. Since in this case $\Phi(x) > 0$ for every $x > 0$, we can write $\Phi(x) = 1/(1 + \nu(x))$, $x > 0$, where $\nu(x) \geq 0$, $\nu(+\infty) = 0$. Equation (20) becomes $1/(1 + q^{-1}\nu(x)) = 1/(1 + \nu(a(q)x))$ or

$$q^{-1}\nu(x) = \nu(a(q)x). \tag{21}$$

Let us show that $a(q)$ is increasing and $a(q) < 1$ for every $0 < q < 1$. If $a(q) > 1$ for some q then from (21) we have: $\nu(x) = q^n \nu((a(q))^n x)$, $x > 0$, $n = 1, 2, \ldots$, which means that $\nu(x) \equiv 0$. So it must be that $a(q) < 1$ for $0 < q < 1$. Let us consider $q = q_1 q_2$ where $0 < q_1 < 1$, $0 < q_2 < 1$. From (21) we have $\nu(x) = q\nu(a(q)x) = q_1 q_2 \nu(a(q_1)a(q_2)x)$, wherefrom

$$a(q_1 q_2) = a(q_1)a(q_2). \tag{22}$$

Since $a(q) < 1$ for $0 < q < 1$ it follows that $a(q)$ is strictly increasing. The continuity of the functions $\Phi(x)$ and $a(q)$ follows from (20). Let us find the solution of (20). Let $f(x) = \ln a(e^{-x})$, $x > 0$. The continuity of $f(x)$ follows from the continuity of $a(q)$; from $a(q) < 1$ it follows that $f(x) < 0$. In terms of $f(x)$, (22) becomes $f(x + y) = f(x) + f(y)$, $x, y > 0$. The general solution of the preceding equation is (21): $f(x) = -x/\alpha$, $\alpha > 0$, wherefrom $a(q) = e^{f(-\ln q)} = q^{1/\alpha}$, $\alpha > 0$, $0 < q < 1$. So, equation (21) reduces to

$$q^{-1}\nu(x) = \nu(q^{1/\alpha}x), \;\; x > 0, \;\; O < q < 1, \;\; \alpha > 0. \tag{23}$$

One solution of (23) is obvious, namely $\nu_1(x) = x^{-\alpha}$, $\alpha > 0$. Let us suppose that this is not the only solution of (23), and let $\nu_2(x)$ be the other solution of (23). Let us consider the ratio $g(x) = \nu_2(x)/\nu_1(x)$, $x > 0$.

By virtue of (23) we have

$$g(q^{1/\alpha}x) = \frac{\nu_2(q^{1/\alpha}x)}{\nu_1(q^{1/\alpha}x)} = \frac{\nu_2(x)}{\nu_1(x)} = g(x) \quad x > 0,\ 0 < q < 1.$$

Hence the function $g(x)$ is constant $g(x) = \gamma > 0$ and the general solution of (21) is $\nu(x) = \gamma x^{-\alpha}$, $x > 0$, $\alpha > 0$, $\gamma > 0$, so that in case $\Phi(0) = 0$ we have

$$F(x) = \Phi(x - \beta) = \begin{cases} 0 & x \le \beta \\ 1/(1 + \gamma(x-\beta)^{-\alpha}), & x > \beta,\ \alpha > 0,\ \gamma > 0, \end{cases}$$

$$a(q) = q^{1/\alpha},\ 0 < q < 1,\ \alpha > 0,$$

and from the equation $\beta = b(q)/(1 - a(q))$ we obtain $b(q) = \beta(1 - q^{1/\alpha})$, $0 < q < 1$, $\alpha > 0$.

b) Case $\Phi(0) = 1$. We already know that $\Phi(x) > 0$ for $x < 0$, and hence we can write $\Phi(x) = 1/(1 + \nu(x))$, $x < 0$, where $\nu(x) \ge 0$, $\nu(-\infty) = +\infty$. From (20) we have $1/(1 + q^{-1}\nu(x)) = 1/(1 + \nu(a(q)x))$, $x < 0$, or

$$q^{-1}\nu(x) = \nu(a(q)x). \tag{24}$$

In this case we have $a(q) > 1$ for $0 < q < 1$. If, for some q, we had $a(q) < 1$, then it would follow from (24) that for $x < 0$ and $n = 1, 2, \ldots$ $\nu(x) = q^n\nu((a(q))^n x)$, wherefrom $\nu(x) \equiv 0$, which is impossible. Now we shall prove that $a(q)$ is decreasing. Let us consider $q = q_1q_2$, $0 < q_1 < 1$, $0 < q_2 < 1$; it follows from (24) that $\nu(x) = q\nu(a(q)x) = q_1q_2\nu(a(q_1)a(q_2)$, or, equivalently

$$a(q_1q_2) = a(q_1)a(q_2). \tag{25}$$

Since $a(q) > 1$ for every $0 < q < 1$, it follows that $a(q)$ is decreasing. The continuity of $\Phi(x)$ and $a(q)$ is the consequence of equation (20). We solve the equation (25) by the substitution $f(x) = \ln a(e^x)$, $x < 0$. Since $a(q) > 1$, we have $f(x) > 0$. The continuity of $f(x)$ follows from the continuity of $a(q)$. From (25) we get $f(x + y) = f(x) + f(y)$, $x, y < 0$, and the general solution of this equation is $f(x) = -x/\alpha$, $x < 0$, $\alpha > 0$. Hence we have $a(q)\ a(q) = e^{f(\ln q)} = q^{-1/\alpha}$, $0 < q < 1$, $\alpha > 0$ and we see that (24) reduces to

$$q^{-1}\nu(x) = \nu(q^{-1/\alpha}x),\ x < 0,\ 0 < q < 1,\ \alpha > 0. \tag{26}$$

One solution of (26) is obvious $\nu_1(x) = (-x)^\alpha$, $\alpha > 0$, $x < 0$. Let us denote by $\nu_2(x)$ another solution of (26); we have

$$g(q^{-1/\alpha}x) = \frac{\nu_2(q^{-1/\alpha}x)}{\nu_1(q^{-1/\alpha}x)} = \frac{\nu_2(x)}{\nu_1(x)} = g(x), \quad x < 0,\ 0 < q < 1.$$

It follows that $g(x)$ is a constant function, $g(x) = \gamma > 0$, and so the general solution of (24) is $\nu(x) = \gamma(-x)^\alpha$, $x < 0$, $\alpha > 0$, $\gamma > 0$. So, in the case b), we have

$$F(x) = \begin{cases} 1/(1+\gamma(-x+\beta)^\alpha), & x > \beta, \ \alpha > 0, \ \gamma > 0, \\ 1 & x \geq \beta, \end{cases}$$
$$a(q) = q^{-1/\alpha}, \ \ b(q) = \beta(1-q^{-1/\alpha}), \ \ 0 < q < 1. \ \square$$

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