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BIANCHI IDENTITIES IN RECURRENT FINSLER SPACES

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Abstract. We give Bianchi identities for Finsler space with recurrent metric tensor, which was defined by (1.8)–(1.13).

1. Some definitions and notations. The recurrent Finsler space is a generalization of non-recurrent one (when in (1.6 c) and (1.6 d) $\lambda_\delta = 0$, $\mu_\delta = 0$).Moór's generalization in [**1**] is such that the usual conditions for Finsler space

$$A^\alpha_{\delta 0} = 0, \qquad T|_\alpha l^\alpha = 0$$

(where T is any tensor) remains unchanged. Here the generalization is going in the other direction. The above conditions here are not satisfied, but the condition of Varga [**2**] $C^i_{j0} = 0$ is satisfied. The difference occurs because in [**1**] $A^\alpha_{\beta\gamma}$ from (1.1) is symmetric in the first two indices, but here it is symmetric in the first and the last one. The two generalizations coincide in the case when $\mu_\gamma = 0$. As the connection coefficients $\Gamma^{*\beta}_{\alpha\ \gamma}$ from (1.1) are the same in both cases, so the curvature tensor R is also the same, but tensors P and S are here defined in a different manner ((1.11), (1.13)).

If $\xi^\alpha(\chi, \dot\chi)$ are coordinates of a vector field homogeneous of degree zero in $\dot\chi$, then

$$D\xi^\alpha = d\xi^\alpha + \Gamma^{*\alpha}_{\beta\ \gamma}\xi^\beta d\xi^\gamma + A^\alpha_{\beta\gamma}\xi^\beta Dl^\gamma, \tag{1.1}$$

$$D\xi^\alpha = \xi^\alpha|_\beta d\chi^\beta + \xi^\alpha|_\beta Dl^\beta, \tag{1.2}$$

where

$$\xi^\alpha|_\beta = \partial_\beta\xi^\alpha - \dot\partial_\chi\xi^\alpha\Gamma^{*\chi}_\beta + \Gamma^{*\alpha}_{\chi\beta}\xi^\chi \qquad (\Gamma^{*\chi}_\beta = \Gamma^{*\chi}_{\alpha\beta}\dot\chi^\alpha = L\Gamma^{*\chi}_{0\beta}) \tag{1.3}$$

$$\xi^\alpha|_\beta = L\dot\partial_\chi\xi^\alpha(\delta^\chi_\beta - A^\chi_{0\beta}) + A^\alpha_{\chi\beta}\xi^\beta. \tag{1.4}$$

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The connection coefficients are given in [**3**]. They satisfy the conditions:

$$\text{(1.5)}\qquad \text{a)}\ \Gamma^{*\alpha}_{\beta\gamma}=\Gamma^{*\alpha}_{\gamma\beta},\qquad \text{b)}\ A^{\alpha}_{\beta\gamma}=A^{\alpha}_{\gamma\beta},\qquad \text{c)}\ g_{\alpha\beta|\delta}=\lambda_\delta g_{\alpha\beta},\qquad \text{d)}\ g_{\alpha\beta|\delta}=\mu_\delta g_{\alpha\beta}$$

and have the form

$$\Gamma^{*\alpha}_{\beta\gamma}=\tilde{\Gamma}^{*\alpha}_{\beta\gamma}T_\beta\gamma^\alpha(g,\lambda)$$
$$A^{\alpha}_{\beta\gamma}=\tilde{A}^{\alpha}_{\beta\gamma}+Q^{\alpha}_{\beta\gamma}(g,\mu),$$

where $\tilde{\Gamma}^{*\alpha}_{\beta\gamma}$ and $\tilde{A}^{\alpha}_{\beta\gamma}$ are connection coefficients of the ordinary Finsler space ($\lambda_\delta=0$, $\mu_\delta=0$) and $T^{\alpha}_{\beta\gamma}(g,\lambda)$, $Q^{\alpha}_{\beta\gamma}(g,\mu)$ are tensors which are equal to zero for $\lambda_\delta=0$ and $\lambda_\delta=0$ respectively. We shall use the relation

$$\text{(1.6)}\qquad \text{a)}\ 2A_{00\gamma}=-\mu_\gamma,\qquad \text{b)}\ L|_\gamma=2^{-1}L\lambda_\gamma,\qquad \text{c)}\ L|_\gamma=L(l_\gamma+2^{-1}\mu_\gamma)\qquad \text{d)}\ \chi^{\alpha}_{|\gamma}=0.$$

In the recurrent Finsler space, from $g_{\alpha\beta}l^\alpha l^\beta=1$, in view of (1.5c) and (1.5d) we have

$$\text{(1.7)}\qquad \text{a)}\ \lambda_\gamma[d\chi^\gamma\Delta l^\delta]+(2l_\gamma+\mu_\gamma)[Dl^\gamma\Delta l^\delta]=0$$
$$\text{b)}\ \lambda_\delta[d\chi^\gamma\delta\chi^\delta]+(2l_\delta+\mu_\delta)[d\chi^\gamma\Delta l^\delta]=0.$$

The curvature tensors in the recurrent Finsler space are defined by

$$\text{(1.8)}\qquad 2^{-1}K^{\beta}_{\alpha\gamma\delta}=\partial_{[\delta}\Gamma^{*\beta}_{|\alpha|\gamma]}-\dot{\partial}\Gamma^{*\beta}_{\alpha[\gamma}\Gamma^{*\iota}_{\delta]}+\Gamma^{*\chi}_{\alpha[\gamma}\Gamma^{*\beta}_{|\chi|\delta]}$$

$$\text{(1.9)}\qquad 2^{-1}LK^{\beta}_{0\gamma\delta}=\partial_{[\delta}\Gamma^{*\beta}_{\gamma]}-\dot{\partial}\Gamma^{*\beta}_{[\gamma}\Gamma^{*\iota}_{\delta]}$$

$$\text{(1.10)}\qquad R^{\beta}_{\alpha\gamma\delta}=K^{\beta}_{\alpha\gamma\delta}+A^{\beta}_{\alpha\iota}K^{\iota}_{0\gamma\delta}$$

$$\text{(1.11)}\qquad P^{\beta}_{\alpha\gamma\delta}=L\dot{\partial}_\chi\Gamma^{*\beta}_{\alpha\gamma}(\delta^\chi_\delta-A^\chi_{0\delta})-A^{\beta}_{alpha\delta|\gamma}+A^{\beta}_{\alpha_\iota}\dot{\chi}^\chi\dot{\partial}_\gamma\Gamma^{*\iota}_{\chi\delta}$$

$$\text{(1.12)}\qquad LP^{\chi}_{0\gamma\delta}=L(\dot{\partial}_\chi\Gamma^{*\beta}_{\alpha\gamma})\dot{\chi}^\alpha(\delta^\chi_\delta-A^\chi_{0\delta})-L_{|\gamma}A^{\beta}_{0\delta}-LA^{\beta}_{0\delta|\gamma}+LA^{\beta}_{0\iota}\dot{\chi}^\chi\dot{\partial}_\gamma\Gamma^{*\iota}_{\chi\delta}$$

$$\text{(1.13)}\qquad 2^{-1}S^{\beta}_{\alpha\gamma\delta}=L\dot{\partial}_\chi A^{\beta}_{\alpha[\gamma}(\delta^\chi_{\delta]}-A^\chi_{|0|\delta]})+A^\chi_{\alpha[\gamma}A^\varepsilon_{|\chi|\delta}$$

These tensors are formed with connection coefficients of the recurrent Finsler space. In the case of the ordinary Finsler space ($\lambda_\delta=0$; $\mu_\delta=0$) these connection coefficients reduce to the correspondent connection coefficients of the ordinary Finsler space and the above defined curvature tensors become the well known curvature tensors in the non-recurrent Finsler space (where $A_0{}^\chi_\delta=0$).

The tensors defined by (1.8)–(1.13) satisfy the following relations [**4**]

$$\text{(1.14)}\qquad -K_{\alpha\beta\gamma\delta}-K_{\beta\alpha\gamma\delta}-L\dot{\partial}_\chi g_{\alpha\beta}K^\chi_{0\gamma\delta}=2\lambda_{[\gamma|\delta]}g_{\alpha\beta}$$

$$\text{(1.15)}\qquad -P_{\alpha\beta\gamma\delta}-P_{\beta\alpha\gamma\delta}-g_{\alpha\beta}\lambda_\chi A^\chi_{\gamma\delta}+L\dot{\partial}_\chi g_{\alpha\beta}[A^\chi_{0\delta|\gamma}-(\dot{\chi}^\theta\dot{\partial}_\iota\Gamma\theta\gamma^{*\chi}+$$
$$+2^{-1}\lambda_\gamma\delta^\iota_\chi)(\delta^\delta_\iota-A^\iota_{0\delta})]=(\lambda_{\gamma|\delta}-\mu_{\delta|\gamma})g_{\alpha\beta}$$

$$\text{(1.16)}\qquad -S_{\alpha\beta\gamma\delta}-S_{\beta\alpha\gamma\delta}-2L^2\dot{\partial}_\chi g_{\alpha\beta}(\dot{\partial}_{[\delta}A^\chi_{|0|\gamma]}-\dot{\partial}_\iota A^\chi_{0[\gamma}A^\iota_{|0|\delta]})=2\mu_{[\gamma|\delta]}g_{\alpha\beta}$$

The Bianchi identities for a contravariant vector field

Starting from (1.3) and (1.4), by direct calculation we obtain:

$$\xi^{\alpha}_{|\gamma|\delta} - \xi^{\alpha}_{|\delta|\gamma} = K^{\alpha}_{\chi\gamma\delta}\xi^{\chi} - LK^{\chi}_{0\gamma\delta}\dot{\partial}_{\chi}\xi^{\alpha} \tag{2.1}$$

$$\xi^{\alpha}_{|\gamma|\delta} - \xi^{\alpha}_{|\delta|\gamma} = P^{\alpha}_{\chi\gamma\delta}\xi^{\chi} - LP^{\chi}_{0\gamma\delta}\dot{\partial}_{\chi}\xi^{\alpha} - A^{\chi}_{\gamma\delta}\xi^{\alpha}_{|\chi} - \dot{\partial}_{\delta}\xi^{\alpha}L_{|\gamma} \tag{2.2}$$

$$\begin{aligned}\xi^{\alpha}_{|\gamma|\delta} - \xi^{\alpha}_{|\delta|\gamma} &= S^{\alpha}_{\chi\gamma\delta}\xi^{\chi} - 2L^{2}\dot{\partial}_{\chi}\xi^{\alpha}\dot{\partial}_{\iota}A^{\chi}_{0[\gamma}(\delta^{\iota}_{\delta]} - A^{\iota}_{0\delta]})+ \\ &+2L\dot{\partial}_{\chi}\xi^{\alpha}(l_{[\delta} + 2^{-1}\mu_{[\delta})(\delta^{\iota}_{\gamma]} - A^{\iota}_{|0|\gamma]}\end{aligned} \tag{2.3}$$

where we have used (1.6). The above formulae cannot be obtained using the differential forms. We are going to show why it is not possible. If D and Δ are absolute differentials which correspond to the change of the line element $(\chi, \dot{\chi})$ to $(\chi + d\chi, \dot{\chi} = d\dot{\chi})$ and $(\chi + \delta\chi, \dot{\chi} + \delta\dot{\chi})$ respectively, then from (1.1) we obtain

$$\begin{aligned}(\Delta D - D\Delta)\xi^{\alpha} &= 2^{-1}K^{\alpha}_{\chi\gamma\delta}\xi^{\chi}[d\chi^{\gamma}\delta\chi^{\delta}]+ \\ &+(P^{\alpha}_{\chi\gamma\delta} - A^{\alpha}_{\chi\iota}\dot{\partial}_{\gamma}\Gamma^{*\iota}_{\delta})\xi^{\chi}[d\chi^{\gamma}\Delta l^{\delta}] + 2^{-1}S^{\alpha}_{\chi\gamma\delta}\xi^{\chi}[Dl^{\gamma}\Delta l^{\delta}] + A\end{aligned} \tag{2.4}$$

where

$$A = (\delta d - d\delta)\xi^{\alpha} + \Gamma^{*\alpha}_{\beta\gamma}(\delta d - d\delta)\chi^{\gamma} + A^{\alpha}_{\beta\gamma}\xi^{\beta}(\delta D - d\Delta)l^{\gamma}. \tag{2.5}$$

From

$$Dl^{\gamma} = dl^{\gamma} + \Gamma^{*\gamma}_{0\chi}d\chi^{\chi} + A^{\gamma}_{0\chi}Dl^{\chi} \tag{2.6}$$

$$dl^{\gamma} = L^{-1}d\dot{\chi}^{\gamma} + \dot{\chi}^{\gamma}dL^{-1}, \qquad dL^{-1} = -L^{-2}dL \tag{2.7}$$

we obtain

$$d\dot{\chi}^{\gamma} = L(\delta^{\gamma}_{\chi} - A^{\gamma}_{0\chi})Dl^{\chi} - \Gamma^{*\gamma}_{\chi}d\chi^{\chi} + L^{-1}\dot{\chi}^{\gamma}dL. \tag{2.8}$$

Using the homogenity of degree zero of $A^{\gamma}_{0\chi}$ and degree one of $\Gamma^{*\gamma}_{\chi}$ in $\dot{\chi}$, using (2.6), (2.8) and after that (1.9) and (1.12) we obtain

$$\begin{aligned}A &= \xi^{\alpha}_{|\gamma}(\delta d - d\delta)\chi^{\gamma} + \xi^{\alpha}_{|\gamma}(\delta D - d\Delta)l^{\gamma} - 2^{-1}L\dot{\partial}_{\chi}\xi^{\alpha}K^{\chi}_{0\gamma\delta}[d\chi^{\gamma}\delta\chi^{\delta}]- \\ &L\dot{\partial}_{\chi}\xi^{\alpha}[P^{\chi}_{0\gamma\delta} + A0\iota^{\chi}\dot{\partial}_{\gamma}\Gamma^{*\iota}_{\delta} - \Gamma^{*\chi}_{\gamma\delta} + L^{-1}L_{1\gamma}A^{\chi}_{0\delta}][d\chi^{\gamma}\Delta l^{\delta}]- \\ &-L^{2}\dot{\partial}_{\chi}\xi^{\alpha}[\dot{\partial}_{\iota}A^{\chi}_{0[\gamma}(\delta^{\iota}_{\delta]} - A^{\iota}_{|0|\delta]})[Dl^{\gamma}\Delta l^{\delta}]\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.4) we obtain

$$(\Delta D - D\Delta)\xi^{\alpha} = B - \dot{\partial}_{\iota}\xi^{\alpha}L_{|\gamma}A^{\iota}_{0\delta}[d\chi^{\gamma}\Delta l^{\delta}] \tag{2.10}$$

where

$$\begin{aligned}B &= 2^{-1}(K^{\alpha}_{\chi\gamma\delta}\xi^{\chi} - L\dot{\partial}_{\chi}\xi^{\alpha}K^{\chi}_{0\gamma\delta})[d\chi^{\gamma}\delta\chi^{\delta}]+ \\ &+[P^{\alpha}_{\chi\gamma\delta}\xi^{\chi} - L\dot{\partial}_{\chi}\xi^{\alpha}P^{\chi}_{0\gamma\delta} - \dot{\chi}^{\chi}\dot{\partial}_{\gamma}\Gamma^{*\iota}_{\chi\delta}(\xi^{\alpha}_{|\iota} - \dot{\partial}_{\iota}\xi^{\alpha}) - \Gamma^{*\iota}_{\gamma\delta}\xi^{\alpha}_{|\iota}][d\chi^{\gamma}\Delta l^{\delta}]+ \\ &+2^{-1}[S^{\alpha}_{\chi\gamma\delta}\xi^{\chi} - L^{2}\dot{\partial}_{\chi}\xi^{\alpha}\dot{\partial}_{\iota}A^{\chi}_{0[\gamma}(\delta^{\iota}_{\delta]} - A^{\iota}_{|0|\delta]})][Dl^{\gamma}\Delta l^{\delta}]+ \\ &+\xi^{\alpha}_{|\gamma}(\delta d - d\delta)\chi^{\gamma} + \xi^{\alpha}_{|\gamma}(\delta D - d\Delta)l^{\gamma}.\end{aligned} \tag{2.11}$$

Starting from (1.2) and using

$$\begin{aligned}&\Delta Dl^\chi = \delta Dl^\chi + \Gamma^{*\chi}_{\gamma\delta} Dl^\gamma \delta\chi^\delta + A^\chi_{\gamma\delta} Dl^\gamma \Delta l^\delta \Rightarrow\\ &\Rightarrow (\Delta D - D\Delta) l^\chi = (\delta D - d\Delta) l^\chi - \Gamma^{*\chi}_{\gamma\delta}[d\chi^\gamma \Delta l^\delta],\\ &\Delta d\chi^\chi = \delta d\chi^\chi + \Gamma^{*\chi}_{\gamma\delta} d\chi^\gamma \delta\chi^\delta + A^\chi_{\gamma\delta} d\chi^\gamma \Delta l^\delta \Rightarrow\\ &\Rightarrow (\Delta d - D\delta)\chi^\chi = (\delta d - d\delta)\chi^\chi + A^\chi_{\gamma\delta}[d\chi^\gamma \Delta l^\delta]\end{aligned}$$

we obtain

$$\begin{aligned}(\Delta D - D\Delta)\xi^\alpha = \xi^\alpha_{[|\gamma|\delta]}[d\chi^\gamma \delta\chi^\delta] +\\ (\xi^\alpha_{|\gamma|\delta} - \xi^\alpha_{|\delta|\gamma} - \xi^\alpha_{|\chi}\Gamma^{*\chi}_{\gamma\delta} + \xi^\alpha_{|\chi} A^\chi_{\gamma\delta})[d\chi^\gamma \Delta l^\delta] +\\ \xi^\alpha_{|[\gamma|\delta}[Dl^\gamma \Delta l^\delta] + \xi^\alpha_{|\gamma}(\delta d - d\delta)\chi^\gamma + \xi^\alpha_{|\gamma}(\delta D - d\Delta) l^\gamma.\end{aligned} \tag{2.12}$$

It is obvious that comparing coefficients beside $[d\chi^\gamma \delta\chi^\delta]$ in (2.10) and (2.12) we obiain (2.1) but comparing coefficients beside $[d\chi^\gamma \Delta l^\delta]$ and $[Dl^\gamma \Delta l^\delta]$ we do not get (2.2) and (2.3). We are going to show that the sum of terms remain is zero. Substituting (2.1), (2.2) and (2.3) isto (2.12) we obtain

$$\begin{aligned}(\Delta D - D\Delta)\xi^\alpha = B - \dot\partial_\delta \xi^\alpha L_{|\gamma}[d\chi^\gamma \Delta l^\delta] +\\ 2^{-1} L\dot\partial_\iota \xi^\alpha[(l_\delta + 2^{-1}\mu_\delta)(\delta^\iota_\gamma - A^\iota_{0\gamma}) - (l_\gamma + 2^{-1}\mu_\gamma)(\delta^\iota_\delta - A^\iota_{0\delta})][Dl^\gamma \Delta l^\delta]\end{aligned} \tag{2.13}$$

where B is determined by (2.11). Equating the right hand side of (2.10) and (2.13), using the relation $L_{|\gamma} = 2^{-1} L\lambda_\gamma$ we obtain:

$$\begin{aligned}-2^{-1} L\dot\partial_\iota \xi^\alpha(\delta^\iota_\delta - A^\iota_{0\delta})[d\chi^\gamma \Delta l^\delta] +\\ 2^{-1} L\dot\partial_\iota \xi^\alpha[(l_\gamma + 2^{-1}\mu_\gamma)(\delta^\iota_\delta - A^\iota_{0\delta}) - (l_\delta + 2^{-1}\mu_\delta)(\delta^\iota_\gamma - A^\iota_{0\gamma})][Dl^\gamma \Delta l^\delta] = 0.\end{aligned} \tag{2.13}$$

From this relation we obtain

$$\dot\partial_\iota \xi^\alpha(\delta^\iota_\delta - A^\iota_{0\delta})\{\lambda_\gamma[d\chi^\gamma \Delta l^\delta] + (2l_\gamma + \mu_\gamma)[Dl^\gamma \Delta l^\delta]\} = 0$$

which is true in view of (1.7b).

3. Bianchi identities for the curvature tensors. As in the recurrent Finsler spaces the relations (1.7a) and (1.7b) are valid, so there is no use of forming the expression $[D\Delta\mathfrak{D}]\xi^\alpha$, and so the Bianchi identities can be found only by direct calculation. Tensors R and P are connected by the relation

$$\begin{aligned}\sigma_{\gamma\delta\theta}\{R^\beta_{\alpha\gamma\delta|\theta} + P^\beta_{\alpha\gamma\iota} K^\iota_{0\delta\theta} - P^\beta_{\alpha\delta\iota} K^\iota_{0\gamma\theta}\} =\\ \sigma_{\gamma\delta\theta}\{A^\beta_{\alpha\iota} K^\chi_{0\gamma\delta}(\dot\partial_\theta \Gamma^{\cdot\iota}_{\chi} - \dot\partial_\chi \Gamma^{\cdot\iota}_{\theta}) - L\dot\partial_\chi \Gamma^{\cdot\beta}_{\alpha\theta} A^\chi_{0\iota} K^\iota_{0\gamma\delta} - 2^{-1} A^\beta_{\alpha\iota} K^\iota_{0\gamma\delta}\lambda_\theta\}\end{aligned} \tag{3.1}$$

where

$$\sigma_{\gamma\delta\theta} T_{\gamma\delta\theta} = T_{\gamma\delta\theta} + T_{\delta\theta\gamma} + T_{\theta\gamma\delta}.$$

Since

$$\dot\partial_\theta \Gamma^{*\iota}_{\chi} = \dot\chi^\varepsilon \dot\partial_\theta \Gamma^{*\iota}_{\varepsilon\chi} + \Gamma^{*\iota}_{\theta\chi}$$

$\dot\partial_\theta \Gamma^{\cdot\iota}_{\chi}$ is not a tensor, but $(\dot\partial_\theta \Gamma^{\cdot\iota}_{\chi} - \dot\partial_\chi \Gamma^{\cdot\iota}_{\theta})$ is a tensor because of the symmetry of $\Gamma^{\cdot\iota}_{\theta\chi}$ in θ and χ. It is evident that (3.1) has a different form from the analogous one in the non-recurrent Finsler space. In the case of the ordinary Finsler space the right hand side of (3.1) vanishes because there

$$\lambda_\theta = 0, \qquad \dot\partial_\theta \Gamma^{*\iota}_{\chi} - \dot\partial_\chi \Gamma^{*\iota}_{\theta} = 0, \qquad A^{\chi}_{0\iota} = 0.$$

The other formula which connects tensors R, P and S is

$$\begin{aligned}
&\sigma_{\gamma\delta\theta}\{R^{\beta}_{\alpha\gamma\delta|\theta} + (P^{\beta}_{\alpha\gamma\delta|\theta} - P^{\beta}_{\alpha\delta\gamma|\theta}) - S^{\beta}_{\alpha\chi\theta}K^{\chi}_{0\gamma\delta} - P^{\beta}_{\alpha\theta\chi}(\dot\partial_\delta \Gamma^{*\chi}_{\gamma} - \dot\partial_\gamma \Gamma^{*\chi}_{\delta})\} = \\
&\sigma_{\gamma\delta\theta}\{-2L\dot\partial_\chi \Gamma^{*\beta}_{\alpha\theta}(\dot\partial_\delta \Gamma^{*\chi}_{\gamma} - \dot\partial_\gamma \Gamma^{*\chi}_{\delta}) - L^{-1}L_{|\theta}A^{\beta}_{\alpha\iota}K^{\iota}_{0\gamma\delta} + \\
&+ 2A^{\beta}_{\alpha\iota}\dot\partial_\theta(LK^{\iota}_{0\gamma\delta}) - L\dot\partial_\iota A^{\beta}_{\alpha\theta}A^{\iota}_{0\chi}K^{\chi}_{0\gamma\delta} - A^{\beta}_{\alpha\iota}\dot\partial_\varepsilon(LK^{\iota}_{0\gamma\delta})A^{\varepsilon}_{0\theta} + \\
&+ L_{|\theta}(\dot\partial_\delta \Gamma^{*\beta}_{\alpha\gamma} - \dot\partial_\gamma \Gamma^{*\beta}_{\alpha\delta}) + A^{\beta}_{\alpha\iota}(\dot\partial_\chi \Gamma^{*\iota}_{\theta} - \dot\partial_\theta \Gamma^{*\iota}_{\chi})(\dot\partial_\delta \Gamma^{*\chi}_{\gamma} - \dot\partial_\gamma \Gamma^{*\chi}_{\delta}) + \\
&+ L\dot\partial_\chi \Gamma^{*\beta}_{\alpha\theta}(\dot\chi^{\varepsilon}\dot\partial_\gamma \Gamma^{*\chi}_{\varepsilon\theta} - P^{\chi}_{0\theta\gamma}) - L\dot\partial_\chi \Gamma^{*\beta}_{\alpha\gamma}(\dot\partial_\delta \Gamma^{*\chi}_{\varepsilon\theta}\dot\chi^{\varepsilon} - P^{\chi}_{0\theta\delta})\}.
\end{aligned} \tag{3.2}$$

In the above formula the right hand side is a function of λ_γ and μ_γ, in view of (1.5).

In the case of non-recurrent Finsler space we have:

$$\begin{gathered}
\dot\chi^{\varepsilon}\dot\partial_\gamma \Gamma^{*\chi}_{\varepsilon\theta} - P^{\chi}_{0\theta\gamma} = 0, \qquad L_{|\theta} = 0, \qquad L_{|\theta} = Ll_\theta, \qquad A^{\varepsilon}_{0\theta} = 0, \\
\dot\partial_\gamma \Gamma^{*\chi}_{\theta} - \dot\partial_\theta \Gamma^{*\chi}_{\gamma} = 0, \qquad \sigma_{\gamma\delta\theta}\{\dot\partial_\theta(LK^{\iota}_{0\gamma\delta})\} = 0
\end{gathered}$$

and in this case (3.2) becomes

$$\sigma_{\gamma\delta\theta}\{R^{\beta}_{\alpha\gamma\delta|\theta} + 2P^{\beta}_{\alpha[\gamma\delta]|\theta} - S^{\beta}_{\alpha\chi\theta}K^{\chi}_{0\gamma\delta} = \sigma_{\gamma\delta\theta}\{-l_\theta A^{\beta}_{\alpha\iota}K^{\iota}_{0\gamma\delta}\}. \tag{3.2a}$$

Tensors P and S are connected by the formula

$$\begin{gathered}
\{P^{\beta}_{\alpha\gamma\delta|\theta} + A^{\iota}_{\gamma\theta}P^{\beta}_{\alpha\chi\delta} + S^{\beta}_{\alpha\iota\delta}\dot\chi^{\chi}\dot\partial_\theta \Gamma^{*\iota}_{\chi\gamma}\} - \{\delta/\theta\} + S^{\beta}_{\alpha\delta\theta|\gamma} = \\
\{L_{|\theta}(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})\dot\partial_\iota \Gamma^{*\beta}_{\alpha\gamma} + L^2(\delta^{\chi}_{\delta} - A^{\chi}_{0\theta})\dot\partial_\chi[(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})\dot\partial_\iota \Gamma^{*\beta}_{\alpha\gamma}] + \\
+(\dot\partial_\gamma \Gamma^{*\iota}_{\delta} - \dot\partial_\delta \Gamma^{*\iota}_{\gamma})S^{\beta}_{\alpha\iota\theta} + \dot\partial_\theta A^{\beta}_{\alpha\delta}L_{|\gamma} + A^{\beta}_{\alpha\iota}\dot\partial_\chi\dot\partial_\gamma \Gamma^{*\iota}_{\delta}(\delta^{\chi}_{\delta} - A^{\chi}_{0\theta}) - \\
-L\dot\partial_\iota A^{\beta}_{\alpha\delta}[(\dot\chi^{\varepsilon}\dot\partial_\theta \Gamma^{*\iota}_{\varepsilon\gamma} - P^{\iota}_{0\gamma\theta}) + (\dot\partial_\gamma \Gamma^{*\iota}_{\theta} - \dot\partial_\theta \Gamma^{*\iota}_{\gamma})]\} - \{\delta/\theta\}
\end{gathered} \tag{3.3}$$

where

$$\{Q^{\beta}_{\alpha\gamma\delta\theta}\} - \{\delta/\theta\} = Q^{\beta}_{\alpha\gamma\delta\theta} - Q^{\beta}_{\alpha\gamma\theta\delta}.$$

The right hand side of (3.3) is a function of vector fields μ and λ. In the case of no non-recurrent Finsler space the right hand side of (3.3) reduces to $2Ll_{[\theta}\dot\partial_{\delta]}\Gamma^{*\beta}_{\alpha\gamma}$.

By a cyclic permutation of indexes $\gamma\delta\theta$ in (3.3) we obtain

$$\begin{gathered}
\sigma_{\gamma\delta\theta}(2P^{\beta}_{\alpha\gamma\delta|\theta} - 2P^{\beta}_{\alpha\gamma\theta|\delta} + S^{\beta}_{\alpha\delta\theta|\gamma}) = \\
\sigma_{\gamma\delta\theta}\{\{L_{|\theta}(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})\dot\partial_\iota \Gamma^{*\beta}_{\alpha\gamma} + L^2(\delta^{\chi}_{\delta} - A^{\chi}_{0\theta})\dot\partial_\chi[(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})\dot\partial_\iota \Gamma^{*\beta}_{\alpha\gamma}] + \\
+\dot\partial_\theta A^{\beta}_{\alpha\delta}L_{|\gamma} + A^{\beta}_{\alpha\iota}\dot\partial_\chi\dot\partial_\gamma \Gamma^{*\iota}_{\delta}A^{\chi}_{0\theta} - \\
-L\dot\partial_\iota A^{\beta}_{\alpha\delta}[(\dot\chi^{\varepsilon}\dot\partial_\theta \Gamma^{*\iota}_{\varepsilon\gamma} - P^{\iota}_{0\gamma\theta}) + (\dot\partial_\gamma \Gamma^{*\iota}_{\theta} - \dot\partial_\theta \Gamma^{*\iota}_{\gamma})]\} - \{\delta/\theta\}\}
\end{gathered} \tag{3.4}$$

Because

$$\sigma_{\gamma\delta\theta}(A^{\chi}_{\gamma[\theta}P^{\beta}_{|\alpha\chi|\delta]}) = 0\sigma_{\gamma\delta\theta}(\dot{\chi}^{\chi}S^{\beta}_{\alpha\iota[\delta}\dot{\partial}_{\theta]}\Gamma^{*\iota}_{\chi\gamma}) = \sigma_{\gamma\delta\theta}(S^{\beta}_{\alpha\iota\gamma}\dot{\partial}_{\theta]}\Gamma^{*\iota}_{\theta]}).$$

In the case of non-recurrent Finsler space (3.4) becomes

$$\sigma_{\gamma\delta\theta}\{2P^{\beta}_{\alpha\gamma\delta|\theta} + S^{\beta}_{\alpha\delta\theta|\gamma}\} = \sigma_{\gamma\delta\theta}\{2Ll_{[\theta}\dot{\partial}_{\delta]}\Gamma^{*\beta}_{\alpha\gamma}\} \tag{3.4a}$$

where P and S reduce to the tensors P and S in the ordinary Finsler space.

In the formulae above the tensor P is defined by (1.11). If we define a new tensor P' in such a way that the last term in (1.11) is replaced by $A^{\chi}_{\alpha\iota}\dot{\chi}^{\varepsilon}\dot{\partial}_{\delta}\Gamma^{*\iota}_{\varepsilon\gamma}$ then we have

$$P'^{\beta}_{\alpha\gamma\delta} = L\dot{\partial}_{\chi}\Gamma^{*\beta}_{\chi\gamma}(\delta^{\chi}_{\delta} - A^{\chi}_{0\delta}) - A^{\beta}_{\alpha\delta|\gamma} + A^{\chi}_{\alpha\iota}\dot{\chi}^{\varepsilon}\dot{\partial}_{\delta}\Gamma^{*\iota}_{\varepsilon\gamma}.$$

In the non-recurrent Finsler space both definitions of P are the same because there

$$\dot{\chi}^{\varepsilon}\dot{\partial}_{\delta}\Gamma^{*\iota}_{\varepsilon\gamma} = \dot{\chi}^{\varepsilon}\dot{\partial}_{\gamma}\Gamma^{*\iota}_{\varepsilon\delta} = A^{\iota}_{\gamma\delta|o}, \quad \text{where } A^{\iota}_{\gamma\delta} = LC^{\iota}_{\gamma\delta}.$$

With this P' the above formulae have a less complicated form:

$$\begin{aligned}(3.1)'\quad &\sigma_{\gamma\delta\theta}\{R^{\beta}_{\alpha\gamma\delta|\theta} + P'^{\beta}_{\alpha\gamma\iota}K^{\iota}_{0\delta\theta} - P'^{\beta}_{\alpha\delta\iota}K^{\iota}_{0\gamma\theta}\} = \\ &\sigma_{\gamma\delta\theta}\{-L\dot{\partial}_{\chi}\Gamma^{*\beta}_{\alpha\theta}A^{\chi}_{0\iota}K^{\iota}_{0\gamma\delta} - 2^{-1}A^{\beta}_{\alpha\iota}K^{\iota}_{0\gamma\delta}\lambda_{\theta}\},\end{aligned}$$

$$\begin{aligned}&\sigma_{\gamma\delta\theta}\{R^{\beta}_{\alpha\gamma\delta|\theta} + (P'^{\beta}_{\alpha\gamma\delta|\theta} - P'^{\beta}_{\alpha\delta\gamma|\theta}) - S^{\beta}_{\alpha\chi\theta}K^{\chi}_{0\gamma\delta} + P'^{\beta}_{\alpha\theta\chi}(\dot{\partial}_{\delta}\Gamma^{*\chi}_{\gamma} - \dot{\partial}_{\gamma}\Gamma^{*\chi}_{\delta})\} = \\ (3.2)'\quad &\sigma_{\gamma\delta\theta}\{-L^{-1}L_{|\theta}A^{\beta}_{\alpha\iota}K^{\iota}_{0\gamma\delta} - L\dot{\partial}_{\iota}A^{\beta}_{\alpha\theta}A^{\iota}_{0\chi}K^{\chi}_{0\gamma\delta} \\ &-A^{\iota}_{\alpha\beta}\dot{\partial}_{\varepsilon}(LK^{\iota}_{0\gamma\delta})A^{\varepsilon}_{0\theta} + L_{|\theta}(\dot{\partial}_{\delta}\Gamma^{*\beta}_{\alpha\gamma} - \dot{\partial}_{\gamma}\Gamma^{*\beta}_{\alpha\delta}) \\ &+L\dot{\partial}_{\chi}\Gamma^{*\beta}_{\alpha\delta}(\dot{\chi}^{\varepsilon}\dot{\partial}_{\gamma}\Gamma^{*\chi}_{\varepsilon\theta} - P'^{\chi}_{0\theta\gamma}) - L\dot{\partial}_{\chi}\Gamma^{*\beta}_{\alpha\delta}(\dot{\chi}^{\varepsilon}\dot{\partial}_{\delta}\Gamma^{*\chi}_{\varepsilon\theta} - LP'^{\chi}_{0\theta\delta})\},\end{aligned}$$

$$\begin{aligned}(3.3)'\quad &2(P'^{\beta}_{\alpha\gamma\delta|\theta} + A^{\chi}_{\gamma\theta}P'^{\beta}_{|\alpha\chi|\delta]} + S^{\beta}_{\alpha\iota[\delta}\dot{\chi}^{\chi}\dot{\partial}_{\theta]}\Gamma^{*\iota}_{\varepsilon\gamma}) + S^{\beta}_{\alpha\delta\theta|\gamma} = \\ &\{L_{|\theta}(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})\dot{\partial}_{\iota}\Gamma^{*\beta}_{\alpha\gamma} + L^{2}(\delta^{\chi}_{\delta} - A^{\chi}_{0\theta})\dot{\partial}_{\chi}[\dot{\partial}_{\iota}\Gamma^{*\beta}_{\alpha\gamma}(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})] \\ &+\dot{\partial}_{\theta}A^{\beta}_{0\delta}L_{|\gamma} - A^{\beta}_{\alpha\iota}\dot{\partial}_{\chi}\dot{\partial}_{\delta}\Gamma^{*\iota}_{\gamma}A^{\chi}_{0\theta} - L\dot{\partial}_{\iota}A^{\beta}_{\alpha\delta}(\dot{\chi}^{\varepsilon}\dot{\partial}_{\theta}\Gamma^{*\iota}_{\varepsilon\gamma} - P'^{\iota}_{0\gamma\theta})\} - \{\delta/\theta\},\end{aligned}$$

$$\begin{aligned}(3.4)'\quad &\sigma_{\gamma\delta\theta}(2P'^{\beta}_{\alpha[\delta|\theta} + 2S^{\beta}_{\alpha\iota\gamma}\dot{\partial}_{[\delta}\Gamma^{*\iota}_{\theta]} + S^{\beta}_{\alpha\delta\theta|\gamma}) = \\ &\sigma_{\gamma\delta\theta}\{\{L_{|\theta}(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})\dot{\partial}_{\iota}\Gamma^{*\beta}_{\alpha\gamma} + L^{2}(\delta^{\chi}_{\theta} - A^{\chi}_{0\theta})\dot{\partial}_{\chi}[(\delta^{\iota}_{\delta}\Gamma^{*\beta}_{\alpha\gamma}(\delta^{\iota}_{\delta} - A^{\iota}_{0\delta})] \\ &+\dot{\partial}_{\theta}A^{\beta}_{0\delta}L_{|\gamma} - A^{\beta}_{\alpha\iota}\dot{\partial}_{\chi}\dot{\partial}_{\delta}\Gamma^{*\iota}_{\gamma}A^{\chi}_{0\theta} \\ &-L\dot{\partial}_{\iota}A^{\beta}_{\alpha\delta}[(\dot{\chi}^{\varepsilon}\dot{\partial}_{\theta}\Gamma^{*\iota}_{\varepsilon\gamma} - P'^{\iota}_{0\gamma\theta})\} - \{\delta/\theta\}\}.\end{aligned}$$

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