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ON SOME RADICALS IN NEAR-RINGS WITH A DEFECT OF DISTRIBUTIVITY

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Abstract. We consider some properties of the radical $J_2(R)$ and the Levitzki radical $L(R)$ in a near-ring R with a defect of distributivity. With and additional assumption that the defect D of R is nilpotent or D is contained in the commutator subgroup of $(R,+)$ we generalize some results of Freidman [**6**, Theorems 1, 2], and of Beidleman [**1**, Th. 16]. Also, we give a slight version of the Theorem 2.5 of [**3**]. By using the notation of a relative defect, we consider some properties of minimal nonnilpotent R-subgroups and we generalize some results of Beidleman [**2**, Theorems 2.4, 2.6, 2.7, 3.1].

1. Preliminaries

A left zero-symmetric near-ring R is a set with two binary operations $+$ and $\cdot$ such that

(1°) $(R,+)$ is a group (not necessarily abelian)

(2°) $(R,\cdot)$ is a semigroup

(3°) The left distributivity law holds, i.e.

$$x(y+z)=xy+xz, \quad \text{for all } \ x,y,z\in R.$$

Also we suppose $0x=0$ for all $x\in R$.

Let R be a near-ring and let $(S,\cdot)$ be a multiplicative subsemigroup of $(R,\cdot)$ whose elements generate $(R,+)$. We say that S is a set of generators of the near-ring R. Thus, every element $r\in R$ can be represented as a finite sum $\sum_i(\pm s_i)$, $(s_i\in S)$. Denote by $D=D(S)$ the normal subgroup of the group $(R,+)$ generated by the set $\{d:\ d=-(xs+yx)+(x+y)s,\ \ x,y\in R, s\in S\}$. It was proved in [**4**] that D is an ideal of R. If $S\subset R$ is a proper subset of R, then we say that R is a near-ring with the defect of distributivity D. If we wish to stress the set of generators, then we write (R,S). Thus, in the near-ring (R,S) with the defect D, for all $x,y\in R$ and $s\in S$ there exists $d\in D$ such that $(x+y)s=xs+ys+d$. Specially, if $D=\{0\}$ then R is a distributively generated (briefly d.g.) near-ring. If

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$S = R$, then we say that R is a D-distributive near-ring and then for all $x, y, z \in R$ there exists $d \in D$ such that $(x + y)z = xz + yz + d$. Specially, if $D = \{0\}$ then R is a distributive near-ring.

A right ideal K of R is a normal subgroup of $(R, +)$ such that $(x+a)y-xy \in K$ for all $a \in K$, $x, y \in R$. An ideal H of R such that $ra \in H$ for all $a \in H$, $r \in R$. An R-subgroup B of R is subgroup $(R, +)$ such that $br \in B$ for all $b \in B$, $r \in R$.

Let A be a nonempty subset of R and let A' be the normal subgroup of $(R, +)$ generated by A. The normal subgroup $D_r(A)$ of $(R, +)$ generated by the elements of the form

$$d = -(xs + a's) + (x + a')s, \quad (x \in R, a' \in A', s \in S)$$

is called the relative defect of the subset A with respect to R. In [**5**] it was proved that the relative defect $D_r(A)$ of some ideal A of R is an ideal of R too.

2. Some properties of the radical $J_2(R)$ and the Levitzki radical

A right ideal B of a near-ring R is called modular (strictly maximal) if B is maximal as an R-subgroup. Let I denote the colletion of all modular right ideals of R. We define the radical $J_2(R)$ of R by radical $J_2(R)$ of R by $J_2(R) = \bigcap_{B \in I} B$. The radical subgroup $R_s(R)$ of R is the intersection of all maximal R-subgroup of R.

In a zeror-symmetric near-ring R every right ideal is an R-subgroup, hence a subnear-ring. We recall that a near-ring R is locally nilpotent if for every finite subset H of R there exists a positive integer $n = n(H)$ such that the product of every n elements from H is zero. An ideal of R is locally nilpotent if it is locally nilpotent as a near-ring ideals of R is called the Levitzki radical $L(R)$ of R.

The following two results generalize respectively theorems 1 and 2 of [**6**]. Namely, we extend these results of Friedman about distributive near-rings to a wider class of D-distributive near-rings. In addition we only require that the defect D is nilpotent (Th. 2.1) and that the defect D is contained in the commutator subgroup of R (Th. 2.2).

THEOREM 2.1. *Let A be an ideal of a D-distributive near-ring R with a nilpotent defect D: Then the Levitzki radical $L(A)$ of A is a locally nilpotent ideal of R and $L(A) = L(R) \cap A$.*

Proof. The relative defect $D_r(A)$ of an ideal A is an ideal of R too, and it is contained in the defect D [**5**, Th. 4]. Thus, $D_r(A)$ is nilpotent. Hence, $D_r(A) \subseteq L(A)$, i.e. $D(A) \subseteq D_r(A) \subseteq L(A)$, where $D(A)$ is the defect of the ideal A, considering A as a near-ring. By Proposition 5 of [**5**], $A/L(A)$ is a distributive near-ring and has no non-zero locally nilpotent ideals. Using Lemma 3 in [**6**], it follows that $L(A)$ is a locally nilpotent ideal of R. Thus, $L(A) \subseteq L(R) \cap A$. Also, $L(R) \cap A$ is a locally nilpotent ideal of A. Consequently $L(R) \cap A \subseteq L(A)$ i.e. $L(R) \cap A = L(A)$.

THEOREM 2.2. *Let R be a D-dislributive near-ring with a nilpotent defect D which is contained in the commutator subgroup R' of $(R, +)$. Then the factor near-ring $R/L(R)$ is a ring, where $L(R)$ is the Levitzki radical of R.*

Proof. Since D is a nilpotent ideal of R we have $D \subseteq L(R)$. Thus, by Proposition 5 ot [**5**], $R/L(R)$ is distributive. On the other hand, by Theorem 2 of [**5**], R' is a nilpotent ideal of R, i.e. $R' \subseteq L(R)$. Therefore, $R/L(R)$ is an abelian group with respect to addition; thus $R/L(R)$ is a ring.

An ideal B of a near-ring R is called strictly small if and only if $R = C$ for each R-subgroup C such that $R = B + C$. The following result generalizes Theorem 16 of [**1**]. Namely, we extend this result of Beidleman about d.g. near-rings to a wider class of near-rings with a defect of distributivity. In this goal we only require that the defect D is contained in the commutator subgroup of R.

THEOREM 2.3. *Let R be a near-ring whose defect D is contained in the commutator subgroup R' of $(R,+)$. If $(R,+)$ is a finitely generated nilpotent group, then the radical $J_2(R)$ is a strictly small ideal.*

Proof. In view of Theorem 6 [**1**], we need to show that $J_2(R) = \bigcap_{B \in I'} B$, where I' is the set of maximal R-subgroups. Thus it suffices to show that every maximal R-subgroup of R is right ideal of R too. Let B be a maximal R-subgroup of R. Since $(R,+)$ is finitely generated, it follows that there exists a maximal sugroup B_1 of $(R,+)$, $B \subseteq_1$B,. By Corollary 10. 3.2. of [**7**] it follows that B_1 is a proper normal subgroup of $(R,+)$ and $R' \subseteq B$,and so $D \subseteq B_1$. If B_2 the normal subgroup of $(R,+)$ generated by the set B, then it is easy to show that $B_2 + D$ is a right ideal of R and $B \subseteq B_2 + D \neq R$. Since B is maximal R-subgroup of R, it follows that $B = B_2 + D$, whence B is a right ideal of R.

We now give a slightly modified version of some earlier results [**3**, Corollary to Th. 2.5]. We say that B is a small normal subgroup of $(R,+)$ if and only if $R = C$ for each normal subgroup C of $(R,+)$ such that $(R,+) = (B,+) + (C,+)$.

THEOREM 2.4. *Let R be a near-ring whose defect D is a small normal subgroup of $(R,+)$. If $(R,+)$ is a nilpotent group and R has the identity, then $J_2(R) = R_s(R)$.*

Proof. Obviously $R_s(R) \subseteq J_2(R)$. We need to show that $J_2(R) \subseteq R_s(R)$. Let B be a maximal R-subgroup. It is well known [**9**, Th. 6.4.10.] that B is a term of a normal series for $(R,+)$. Thus, B is contained in a proper normal subgroup C of $(R,+)$. But, $C + D$ is a normal subgroup of $(R,+)$ and $C + D \neq R$, because $C \neq R$ and D is a small normal subgroup of $(R,+)$. Thus, there exists a proper normal subgroup $C + D$ of $(R,+)$ containing B and D. Therefore, the normal subgroup B_1, of $(R,+)$ generated by the set B is contained in $C + D$, so $B_1 + D \neq R$. It is easy to see that $B_1 + D$ is a right ideal of R which contains the R-subgroup B. Since B is a maximal R-subgroup, it follows that $B = B_1 + D$, i.e. B is a right ideal of R. Thus, every R-subgroup is a right ideal of R.

COROLLARY. *Let R be a near-ring whose defect D is a small normal subgroup of $(R,+)$. If $(R,+)$ is a nilpotent group and R has the identity, then the radical $J_2(R)$ is a quasi-regular ideal of R.*

Throughout this section we shall assume that R satisfies the descending chain condition on R-subgroups, R has the identity and the radical $J_2(R)$ is a nilpotent ideal.

An R-subgroup B of a near-ring R is called minimal nonnilpotent if B is nonnilpotent and every proper R-subgroup in B is nilpotent. A proper right ideal B of R is said to be complemented in R if there exists an R-subgroup H of R such that $R = B + H$ and $B \cap H = \{0\}$.

We first need the following

PROPOSITION 3.1. *Let B be an R-subgroup and let A be a right ideal of R. If the relative defect of the subset B is contained in B, then $B \cap A$ is a right ideal of R.*

Proof. We have only to show that the relative defect of the subset $b \cap A$ is contained in $B \cap A$. By definition of the relative defect, $D_r(B \cap A)$ is generated by all elements d in R for which there exist $x \in R$, $s \in S$ and $b \in B \cap A$ such that $d = -bs - xs + (x + b)s$. Since $D_r(B) \subseteq B$, it follows that $d \in B$. By Lemma 2.3. of [**4**], we have $d \in A$, because A is a right ideal of R. Thus, for all $d \in D_r(B \cap A)$, it follows that $d \in B \cap A$, i.e. $D_r(B \cap A) \subseteq B \cap A$. Therefore by Lemma 3.2. of [**4**], we have that $B \cap A$ is a right ideal of R.

The following results are generalizations of some results of Beidleman [**2**, Theorems 2.4, 2.6, 3.1]. The results of Beidleman refer to a class of d.g. near-rings. We transmit these results over a wider class of near-rings with a defect of distributivity. Here we only impose an additional condition of the form: every minimal nonnilpotent R-subgroup contains the relative defect of its own (Theorems 3.2, 3.3.), or every nonnilpotent R-subgroup contains the relative defect of its own (Theorems 3.4, 3.5).

THEOREM 3.2. *Let R be a near-ring with a defect of distributivity and let every minimal nonnilpotent R-subgroup of a near-ring R contain the relative defect of its own. If B is a minimal nonnilpotent R-subgroup of R, then $B \cap J_2(R)$ is the unique strictly maximal right ideal of B.*

Proof. From Proposition 3.1. it follows that $B \cap J_2(R)$ is an ideal of B. The proof of the remaining part is the same as that of Theorem 2.4. of [**2**].

THEOREM 3.3. *Let R be a near-ring with a defect of distributivity and let every minimal nonnilpotent R-subgroup of R contain the relative defect of its own. Further, let $f : R \to R_2$ denote the natural near-ring homomorphism of the near-ring R onto the near-ring $R_2 = R/J_2(R)$. If B is a minimal nonnilpotent R-subgroup of R, then $(B)f$ is a minimal R_2 subgroup of R_2.*

Proof. By the First isomorphism theorem we have

$$(B + J_2(R))/J_2(R) \simeq B/B \cap J_2(R).$$

On the other hand $(B)f = (B + J_2(R))f = (B + J_2(R))/J_2(R)$.

From Theorem 3.2. it follows that $B \cap J_2(R)$ is a strictly maximal ideal of B. Thus, $(B)f$ is a minimal R-subgroup of the R-group R_2, i.e. $(B)f$ is a minimal R_2-subgroup of R_2.

THEOREM 3.4. *Let R be a near-ring with a defect of distributivity and let every nonnilpotent R-subgroup of R contains the relative defect of its own. A nonnilpotent R-subgroup B of R is minimal nonnilpotent if and only if B contains no proper nonzero normal R-subgroups which are complemented in B.*

Proof. Assume that B is a minimal nonnilpotent R-subgrup. Let B_1 be a proper nonzero normal R-subgroup of B that is complemented in B by an R-subgroup $B_2 \subseteq B$. From definition of a minimal nonnilpotent R-subgroup, it follows that the R-subgroups B_1 and B_2 are nilpotent. The radical $J_2(R)$ contains all nilpotent R-subgroups [**8**, Corollary 5.45]. Since $J_2(R)$ nilpotent, we have that $B = B_1 + B_2$ is nilpotent whih contradicts the assumption above. Conversely, let a nonnilpotent R-subgroup B contains no proper nonzero normal R-subgroups that are complemented in B. We assume that there exists a minimal nonnilpotent R-subgroup C which is contained in B and we seek a contradiction to this assumption. Namely, by using Theorem 3.51 of [**8**], C contains an idempotent c such that $cR = C$ and $R = cR + A(c)$, $A(c) = \{r : cr = 0, r \in R\}$. Hence, $B = cR + A(c) \cap B$. Since $A(c)$ is a right ideal of R, it follows that $A(c) \cap B$ is a normal R-subgroup of the R-group B. From Proposition 3.1. we have that $A(c) \cap B$ is a right ideal of R, i.e. $A(c) \cap B$ is a right ideal of B. The proof of the remaining part is the same as that of the Theorem 2.7 in [**2**].

THEOREM 3.5. *Let R be a near-ring with a defect of distributivity and let every nonnilpotent R-subgroup of R contains the relative defect of its own. Further, let $R_2 = R/J_2(R)$ be a ring and let b be an idempotent of R. Then $B = bR = cR + A(c) \cap B$, where c is an idempotent element of R contained in B. Moreover, the R-subgroup $A(c) \cap B$ is nilpotent if and only if, $b_2R_2b_2$ is a division ring, where $b_2 = (b)f$ (f is the natural near-ring homomorphism of R onto R_2).*

Proof. The proof is the same as that of Theorem 3 of [**2**], whereby we use as jet the result of the Proposition 3.1 and the result of the Theorem 3.4.

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REFERENCES

[1] J.C. Beidleman, *A radical for near-ring modules*, Michigan Math. J. **12**(1965), 377–383.

[2] J.C. Beidleman, *Nonsemisimple distributively generated near-rings with minimum condition*, Math. Annalen **170**(1967), 206–213.

[3] V. Dašić, *On the radicals of near-rings with defect of distributivity*, Publ. Inst. Math. (Beograd) **28(43)**(1980), 51–59.

[4] V. Dašić, *A defect of distributivity of the near-ring*, Math. Balkanica **8: 8**(1978), 63–75.

[5] V. Dašić, *Some properties of the defect of distributivity of a near-rings*, Third Algebraic Conference Beograd, 1982, 67–71.

[6] P.A. Freidman, *Distributively solvable near-rings*, Proc. Riga Seminar on Algebra (Russian), Riga, 1969, 279–309.

[7] M. Hall, *The Theory of Groups*, Macmillan, New York, 1969.

[8] G.G. Pilz, *Near-rings*, Horth-Holland, Amsterdam, 1977.

[9] W.R. Scott, *Group Theory*, Prentice Hall, New Jersey, 1964.

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