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ON SOME CURVATURE TENSORS OF COMPLEX ANALYTIC AND LOCALLY DECOMPOSABLE RIEMANNIAN SPACES WITH $({}^1F, {}^2F)$-CONNECTION

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1. Introduction. Let $\mathfrak{M}^n$ be any differentiable n-dimensional manifold and let $T_p(\mathfrak{M})$ be the tangent space of the manifold $\mathfrak{M}$ at any point p. The elements of $T_p(\mathfrak{M})$ are called vectors and are denoted by X_p, Y_p, $Z_p, \dots$. The corresponding C^∞ vector fields are denoted by X, Y, $Z, \dots$. These vector fields compose a real vector space $\mathfrak{X}(\mathfrak{M})$. We denote by $C^\infty(\mathfrak{M})$ the collection of all real valued C^∞ functions defined on $\mathfrak{M}$. More details about this and the notions mentioned below can be found in [**3**], [**5**].

C^∞ connection ∇ on a manifold $\mathfrak{M}$ is a mapping

$$\nabla : \mathfrak{X}(\mathfrak{M}) \times \mathfrak{X}(\mathfrak{M}) \to \mathfrak{X}(\mathfrak{M})$$

(notation: $\nabla : (X, Y) \to \nabla_X Y$) such that for every $f \in C^\infty(\mathfrak{M})$; $X, Y, Z \in \mathfrak{X}(\mathfrak{M})$ the following equalities are valid

$$\begin{aligned} &\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z \\ &\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z \\ &\nabla_{fX} Z = f\nabla_X Z \\ &\nabla_X(fY) = (Xf)Y = f\nabla_X Y. \end{aligned} \tag{1.1}$$

If $\mathcal{A}$ is any field of 1-forms, then we have

$$(\nabla_X \mathcal{A})(Y) = X\mathcal{A}(Y) - \mathcal{A}(\nabla_X Y). \tag{1.2}$$

Let ∇ be any connection with torsion defined on a manifold $\mathfrak{M}^n$ and $X, Y \in \mathfrak{X}(\mathfrak{M})$. It is possible to define two connections ${}^1\nabla$ and ${}^2\nabla$ so that we have

$${}^1\nabla_X Y = \nabla_X Y \tag{1.3}$$

and

$${}^2\nabla_X Y = \nabla_X Y + [X, Y], \tag{1.4}$$

where $[X,Y]$ is the commutator of vector fields X, Y.

For connections ${}^1\nabla$ and ${}^2\nabla$ there exist four fields of curvature tensors ${}_1R(X,Y)Z$, ${}_2R(X,Y)Z$, ${}_3R(X,Y)Z$ and ${}_4R(X,Y)Z$ [**9**], defined by the following relations

$$
\begin{aligned}
&{}_1R(X,Y)Z ={}^1\nabla_X{}^1\nabla_Y Z -{}^1\nabla_Y{}^1\nabla_X Z -{}^1\nabla_{[X,Y]}Z\\
&{}_2R(X,Y)Z ={}^2\nabla_X{}^2\nabla_Y Z -{}^1\nabla_Y{}^2\nabla_X Z -{}^2\nabla_{[X,Y]}Z\\
(1.5)\qquad &{}_3R(X,Y)Z ={}^2\nabla_X{}^1\nabla_Y Z -{}^1\nabla_Y{}^2\nabla_X Z +\nabla^2_{{}^1\nabla_Y X}Z -{}^1\nabla_{{}^2\nabla_X Y}Z,\\
(1.6)\qquad &{}_4R(X,Y)Z ={}^2\nabla_X{}^1\nabla_Y Z -{}^1\nabla_Y{}^2\nabla_X Z +\nabla^2_{{}^2\nabla_Y X}Z -{}^1\nabla_{{}^1\nabla_X Y}Z,
\end{aligned}
$$

The first Ricci transformation at any point $p\in\mathfrak{M}$ with respect to any pair of vectors $Y_p,\ Z_p\in T_p(\mathfrak{M})$ is the linear transformation

$$ {}_1^{\prime}R_{Y_p,Z_p}: T_p(\mathfrak{M})\to T_p(\mathfrak{M}) $$

defined by the formula

$$ {}_1^{\prime}R_{Y_p,Z_p}(X_p)={}_1R(X_p,Y_p)Z_p. $$

Ricci tensor ${}_1^{\prime}\mathcal{R}(Y_p,Z_p)$ of a manifold $\mathfrak{M}$ at a point p is the bilinear mapping

$$ T_p(\mathfrak{M})\times T_p(\mathfrak{M})\to\mathbf{R} $$

given by the relation

$$ (1.7)\qquad {}_1^{\prime}\mathcal{R}(Y_p,Z_p)=\operatorname{tr}{}_1^{\prime}R_{Y_p,Z_p}, $$

where by tr we denote the trace of a linear mapping.

The second Ricci transformation at an arbitrary point $p\in\mathfrak{M}$ with respect to any pair of vectors $Y_p, Z_p\in T_p(\mathfrak{M})$ is the linear transformation

$$ {}_1^{\prime\prime}R_{Y_p,Z_p}: T_p(\mathfrak{M})\to T_p(\mathfrak{M}) $$

defined by the formula

$$ {}_1^{\prime\prime}R_{Y_p,Z_p}(X_p)={}_1R(Y_p,X_p)Z_p. $$

The Ricci tensor $\mathcal{R}_1^{\prime\prime}(Y_p,Z_p)$ of a manifold $\mathfrak{M}$ at point p is the bilinear mapping

$$ T_p(\mathfrak{M})\times T_p(\mathfrak{M})\to\mathbf{R} $$

given by the relation

$$ (1.8)\qquad {}_1^{\prime\prime}\mathcal{R}(Y_p,Z_p)=\operatorname{tr}{}_1^{\prime\prime}R_{Y_p,Z_p}. $$

The third Ricci transformation at a point $p\in\mathfrak{M}$ with respect to pair of vectors $Y_p, Z_p\in T_p(\mathfrak{M})$ is the linear transformation

$$ {}_1^{\prime\prime\prime}R_{Y_p,Z_p}: T_p(\mathfrak{M})\to T_p(\mathfrak{M}) $$

defined by the formula

$$ {}_1^{\prime\prime\prime}R_{X_p,Z_p}(X_p) =_1 R(Y_p,Z_p)X_p. $$

The Ricci tensor ${}_1^{\prime\prime\prime}\mathcal{R}(Y_p,Z_p)$ of a manifold $\mathfrak{M}$ at a point p is the bilinear mapping

$$ T_p(\mathfrak{M}) \times T_p(\mathfrak{M}) \to \mathbf{R} $$

given by the relation

$$ {}_1^{\prime\prime\prime}\mathcal{R}(Y_p,Z_p) = \mathrm{tr}_1^{\prime\prime\prime}\mathcal{R}_{Y_p,Z_p}. \tag{1.9} $$

Ricci tensors corresponding to the other curvature tensors are defined analogously.

For brevity, when it is clear from the context what we mean, we will often call "fields of tensors" simply "tensors" and vestors will be denoted like fields of vectors without lower index p.

An almost complex structure on an evendimensional differentiable manifold $\mathfrak{M}^{2n}$ [**1, 4**] is a field of endomorphisms of the tangent spaces such that

$$ F^2 = -I, \tag{1.10} $$

where I denotes the identity endomorphism. Such a manifold is an almost complex space.

A connection ${}^0\nabla$ is a symmetric affine F-connection if the following conditions are fulfilled

$$ \begin{aligned} {}^0\nabla_X Y &={}^0 \nabla_Y X + [X,Y], \\ ({}^0\nabla_X F)(Y) &= 0. \end{aligned} \tag{1.11} $$

An almost complex manifold $\mathfrak{M}$ is complex analytic if and only if its Nijenhuis tensor vanishes [**6**], or if and only if there is a symmetric F-connection [**7, 8**].

A Kähler space is an evendimensional manifold with almost complex structure F and Riemannian metric g, which satisfies the following conditions

$$ \begin{aligned} g(FX,FY) = g(X,Y) \quad & X,Y \in \mathfrak{X}(\mathfrak{M}) \\ ({}^0\nabla_X F)(Y) = 0&, \end{aligned} \tag{1.12} $$

${}^0\nabla$ being the Riemann-Christoffel connection formed with g.

An almost product structure on a manifold $\mathfrak{M}^n$ [**12**] is a field F of endomorphisms of the tangent spaces such that

$$ F^2 = I. \tag{1.13} $$

A manifold with this structure is called an almost product space.

A locally product space [**12**] is an almost product space with symmetric F-connection ${}^0\nabla$, so that we have

$$ ({}^0\nabla_X F)(Y) = 0. \tag{1.14} $$

An almost product manifold $\mathfrak{M}^n$ is called a locally decomposable Riemannian space [**12**] if in $\mathfrak{M}^n$ a positive definite Riemannian metric tensor field g is given, satisfying the conditions

$$F^2 = I, \quad g(FX, FY) = g(X, Y)$$
$$({}^0\nabla_X F)(Y) = 0, \qquad X, Y \in \mathfrak{X}(\mathfrak{M}), \tag{1.15}$$

where ${}^0\nabla$ is the Riemann-Christofell connection formed with g.

A locally decomposable space can be covered by a separating coordinate system [**12**], i.e. by a system of coordinate neighbourhoods (x^i) such that in any intersection of two coordinate neighbourhods (x^i) and $(x^{i'})$ we have

$$x^{a'} = x^{a'}(x^a), \ \ x^{y'} = x^{y'}(x^y),$$

where the indices $a, b, c, d, \dots$ range over $1, 2, \dots, p$ and the indices $x, y, z, t, \dots$ range over $p+1, \ p+2, \dots, p+q = n$. By $\mathfrak{M}^p$ we denote the system of subspaces defined by $x^y = 0$, and by $\mathfrak{M}^q$ the system of subspaces defined by $x^a = 0$. Then our space $\mathfrak{M}^n$ is locally the product $\mathfrak{M}^p \times \mathfrak{M}^q$ of two spaces. With respect to a separating coordinate system we also have

$$(F^i_j) = \begin{pmatrix} \delta^a_b & 0 \\ 0 & -\delta^x_y \end{pmatrix}.$$

Therefore

$$\varphi = F^i_i = p - q. \tag{1/16}$$

M. Prvanović [**10**] has investigated covariant differentiation with respect to the connections ${}^1\nabla$ and ${}^2\nabla$ on almost complex and almost product spaces and has found that in a locally coordinate system

$$\Gamma^i_{jk} = {}^\circ\Gamma^i_{jk} + A^i_{jk} + \omega A^b_{ak} F^a_j F^i_b + \omega A^b_{ja} F^a_k F^i_b + \omega A^i_{ab} F^a_j F^b_k \tag{1.17}$$

(where A^i_{jk} is any tensor of covariant type 2 and contravariant type 1, F^i_j is either an almost complex structure and $\omega = -1$ or an almost product structure and $\omega = +1$) is the most general form of connection such that

$$({}^1\nabla_X F)(Y) = ({}^2\nabla_X F)(Y) = ({}^0\nabla_X F)(Y).$$

This connection is called $({}^1F, {}^2F)$-connection. We have considered connections induced by special $({}^1F, {}^2F)$-connections on an almost complex spaces [**2**].

In this paper we suppose that $2A = \mathcal{A} \otimes F$, where $\mathcal{A}$ is a field of 1-forms and find the invariants for transformations defined by relations (2.1) and (2.2) for complex analytic space and (3.2) and (3.3) for locally decomposable Riemannian space, i.e. we obtain the tensors independent on the field $\mathcal{A}$ satisfying some conditions. Further, in the sence of Klein's "Erlangen's" programm of geometry as the theory of the invariants of certain group of transformations we investigate curvature tensors ${}_3R(X, Y)Z$ and ${}_4R(X, Y)Z$ of this spaces. Especially, we find the conditions of flatness of the spaces mentioned above.

2. Complex analytic space. Let $\mathfrak{M}^{2n}(n>1)$ be a complex analytic space. This means, by [**6, 7, 8**], that on this space there exists an almost complex structure F and a symmetric affine F-connection ${}^0\nabla$. We consider on this space $\mathfrak{M}^{2n}$ the $({}^1F,{}^2F)$-connection under the assumption

$$2A^k_{ji} = F^k_j \mathcal{A}_i,$$

where $\mathcal{A}_i$ is a field of 1-forms and F^k_j is a field of tensors of almost complex structure. In that case the $({}^1F,{}^2F)$-connection given by the formula (1.17) has the form

$$\Gamma^k_{ji} ={}^0 \Gamma^k_{ji} + F^k_j \mathcal{A}_i + \mathcal{A}_a F^a_i \delta^k_j.$$

Using the notion of a tensor as a multilinear transformation and the definition of a connection given in section 1., we can write this relation in the following way

$$\nabla_X Y ={}^0 \nabla_X Y + \mathcal{A}(X)F(Y) + \mathcal{A}F(X)\mathcal{A}(Y)$$

and bearing in mind (1.3) and (1.4), we thus obtain the connections ${}^1\nabla,{}^2\nabla$ in the following form

$$(2.1)\qquad {}^1\nabla_X Y ={}^0 \nabla_X Y + \mathcal{A}(X)F(Y) + \mathcal{A}F(X)Y,$$

$$(2.2)\qquad {}^2\nabla_X Y ={}^0 \nabla_X Y + \mathcal{A}(Y)F(X) + \mathcal{A}F(Y)X,$$

Substituting (2.1) and (2.2) into (1.5), using (1.1) and (1.11), after some calculation we get

$$(2.3)\qquad \begin{aligned} {}_3R(X,Y)Z = {} & K(X,Y)Z + [\mathcal{A}({}^0\nabla_Y FZ) - Y\mathcal{A}F(Z) + \mathcal{A}F(Y)\mathcal{A}F(Z)]X + \\ & + \mathcal{A}(Y)\mathcal{A}F(Z)F(X) + [\mathcal{A}({}^0\nabla_Y Z) - Y\mathcal{A}(Z) + \mathcal{A}F(Y)\mathcal{A}(Z)]F(X) - \\ & - \mathcal{A}(Y)\mathcal{A}(Z)X - [\mathcal{A}({}^0\nabla_X FY) - X\mathcal{A}F(Y) + \mathcal{A}F(Y)\mathcal{A}F(X)]Z - \\ & - \mathcal{A}(Y)\mathcal{A}F(X)F(Z) - [\mathcal{A}({}^0\nabla_X Y) - X\mathcal{A}(Y) + \mathcal{A}F(Y)\mathcal{A}(X)]F(Z) \\ & + \mathcal{A}(X)\mathcal{A}(Y)Z, \end{aligned}$$

where by $K(X<Y)Z$ we denote the curvature tensor corresponding to the symmetric connection ${}^0\nabla$:

$$(2.4)\qquad K(X,Y)Z ={}^0 \nabla^0_X \nabla_Y Z -{}^0 \nabla^0_Y \nabla_X Z +{}^0 \nabla_{{}^\circ\nabla_Y X} Z -{}^0 \nabla_{{}^\circ\nabla_X Y} Z.$$

Let us now prove the following theorem:

THEOREM 2.1. *The tensors*

$${}_3R(X,Y)Z - \frac{1}{2(n-1)}[{}'_3\mathcal{R}(Y,Z)X -{}'_3\mathcal{R}(FY,Z)FX -{}'_3\mathcal{R}(X,Y)Z +{}'_3\mathcal{R}(FX,Y)FZ]$$

is independent on 1-form $\mathcal{A}$, which is either:

a) *covariantly constant with respect to the connection ${}^0\nabla$ in any direction Y:*

$$({}^0\nabla_Y \mathcal{A})(X) = Y\mathcal{A}(X) - \mathcal{A}({}^0\nabla_Y X) = 0$$

or

b) *the covariant derivative of* $\mathcal{A}$ *with respect to the connection* ${}^0\nabla$ *in any direction* Y *is equal to* ${}'\mathcal{K}(FX,Y)$

$$({}^0\nabla_Y\mathcal{A})(X) = Y\mathcal{A}(X) - \mathcal{A}({}^0\nabla_Y X) = {}'\mathcal{K}(FX,Y).$$

Moreover, this tensor is equal to the tensor

$$K(X,Y)Z - \frac{1}{2(n-1)}[{}'\mathcal{K}(Y,Z)X - {}'\mathcal{K}(FY,Z)FX - {}'\mathcal{K}(X,Y)Z + {}'\mathcal{K}(FX,Y)FZ].$$

Proof. Let us prove the theorem under the assumption a). Due to the assumption, the 1-form $\mathcal{A}$ is covariantly constant with respect to the affine symmetric F-connection ${}^0\nabla$; using (1.2) we have

$$X\mathcal{A}(Y) - \mathcal{A}({}^0\nabla_X Y) = 0$$

and because of this relation the formula (2.3) has the form

$$\begin{aligned} {}_3R(X,Y)Z = K(X,Y)Z &+ [\mathcal{A}F(Y)\mathcal{A}F(Z) - \mathcal{A}(Y)\mathcal{A}(Z)]X + \\ &+ [\mathcal{A}(Y)\mathcal{A}F(Z) + \mathcal{A}F(Y)\mathcal{A}(Z)]F(X) - [\mathcal{A}F(Y)\mathcal{A}F(X) - \\ &- \mathcal{A}(X)\mathcal{A}(Y)]Z - [\mathcal{A}F(Y)\mathcal{A}(X) + \mathcal{A}(Y)\mathcal{A}F(X)]F(Z). \end{aligned} \tag{2.6}$$

Now we define the linear map

$$f : \mathfrak{X}(\mathfrak{M}^{2n}) \to \mathfrak{X}(\mathfrak{M}^{2n})$$

by the following relation

$$\begin{gathered} f(X) = {}_3R(X,Y)Z - K(X,Y)Z - \\ -[\mathcal{A}F(Y)\mathcal{A}F(Z) - \mathcal{A}(Y)\mathcal{A}(Z)]X - [\mathcal{A}(Y)\mathcal{A}F(Z) + \mathcal{A}F(Z)\mathcal{A}(Z)]F(X) \\ +[\mathcal{A}F(Y)\mathcal{A}F(X) - \mathcal{A}(X)\mathcal{A}(Y)]Z + [\mathcal{A}F(Y)\mathcal{A}(X) + \mathcal{A}(Y)\mathcal{A}F(X)]F(Z), \end{gathered}$$

for any $X, Y, Z \in \mathfrak{X}(\mathfrak{M}^{2n})$.

Using the notion of first Ricci tensor, given by the relation (1.7), and relations (1.10), (2.6) and

$$\operatorname{tr} I = 2n, \quad \operatorname{tr} F = 0, \tag{2.7}$$

where I is the identity map, we can find the trace of the previously defined linear map f, and from that we obtain

$${}_3'\mathcal{R}(Y,Z) - {}'\mathcal{K}(Y,Z) - 2(n-1)[\mathcal{A}F(Y)\mathcal{A}F(Z) - \mathcal{A}(Y)\mathcal{A}(Z)] = 0,$$

or

$$\mathcal{A}F(Y)\mathcal{A}F(Z) - \mathcal{A}(Y)\mathcal{A}(Z) = \frac{1}{2(n-1)}[{}_3'\mathcal{R}(Y,Z) - {}'\mathcal{K}(Y,Z)],$$

since by assumption we have $n > 1$. Let us substitute FY for Y in the last relation. Using (1.10), we obtain

$$\mathcal{A}(Y)\mathcal{A}F(Z) - \mathcal{A}F(Y)\mathcal{A}(Z) = \frac{1}{2(n-1)}[{}_3'\mathcal{R}(FY,Z) -' \mathcal{K}(FY,Z)],$$

Substituting the corresponding expressions given by the last two relations into (2.6), after some calculation we obtain the equality

$$_3R(X,Y)Z - \frac{1}{2(n-1)}[{}_3'\mathcal{R}(Y,Z)X - {}_3'\mathcal{R}(FY,Z)FX - {}_3'\mathcal{R}(X,Y)Z + {}_3'\mathcal{R}(FX,Y)FZ]$$
$$= K(X,Y)Z - \frac{1}{2(n-1)}['\mathcal{K}(Y,Z)X - '\mathcal{K}(FY,Z)FX - '\mathcal{K}(X,Y)Z + \mathcal{K}'(FX,Y)FZ].$$

Hence, we have proved the theorem in case a). In case b), the theorem is proved similarly.

For Kähler spaces the following theorem holds:

THEOREM 2.2. *If $\mathfrak{M}^{2n}(n > 1)$ is a Kähler space with Riemann-Christoffel connection ${}^0\nabla$ and 1-form $\mathcal{A}$ satisfying one of the following conditions:*

a) $({}^0\nabla_Y\mathcal{A})(X) = 0$, *or* b) $({}^0\nabla_Y\mathcal{A})(X) =' \mathcal{K}(FX,Y)$

and if the third Ricci ${}_3'''\mathcal{R}(X,Y)$ is equal to zero, then the curvature tensor ${}_3R(X,Y)Z$ is equal to the curvature tensor $K(X,Y)Z$.

Proof. Let us define the linear map

$$g : \mathfrak{X}(\mathfrak{M}^{2n}) \to \mathfrak{X}(\mathfrak{M}^{2n})$$

by the following relation

$$g(X) =_3 R(Y,Z)X - K(Y,Z)X -$$
$$- [\mathcal{A}F(Z)\mathcal{A}F(X) - \mathcal{A}(Z)\mathcal{A}(X)]Y - [\mathcal{A}(Z)\mathcal{A}F(X) + \mathcal{A}F(Z)\mathcal{A}(X)]F(Y)$$
$$= [\mathcal{A}F(Z)\mathcal{A}F(Y) - \mathcal{A}(Y)\mathcal{A}(Z)]X + [\mathcal{A}F(Z)\mathcal{A}(Y) + \mathcal{A}(Z)\mathcal{A}F(Y)]F(X),$$

for any $X, Y, Z \in \mathfrak{X}(\mathfrak{M}^{2n})$. Since ${}^0\nabla$ is a Riemann-Christoffel connection, bearing in mind the notation (2.4) and the definition of the third Ricci tensor given by the formula (1.9), we get

$$'''\mathcal{K}(Y,Z) = 0.$$

Using the notion of the third Ricci tensor given by the relation (1.9), and the relations (1.10), (2.6), (2.7), we can compute the trace of the linear map g defined above; it follows that

$${}_3'''\mathcal{R}(Y,Z) + 2(n-1)[\mathcal{A}F(Z)F(Y) - \mathcal{A}(Z)\mathcal{A}(Y)] = 0,$$

and due to $n > 1$, we get

$$\mathcal{A}(Y)\mathcal{A}(Z) - \mathcal{A}F(Y)\mathcal{A}F(Z) = \frac{1}{2(n-1)}\,{}_3'''\mathcal{R}(FY,Z).$$

If we substitute FY for Y in the last relation, keeping in mind (1.10) we get

$$\mathcal{A}F(Y)\mathcal{A}(Z) + \mathcal{A}(Y)\mathcal{A}F(Z) = \frac{1}{2(n-1)}\, {}_3^{'''}\mathcal{R}(FY, Z).$$

Using the last two relations, the expression (2.6) can be written in the following form

$$\begin{aligned}{}_3R(X,Y)Z = K(X,Y)Z + \frac{1}{2(n-1)}[-{}_3^{'''}\mathcal{R}(Y,Z)X +{}_3^{'''} \mathcal{R}(FY,Z)FX+ \\ +{}_3^{'''}\mathcal{R}(X,Y)Z -{}_3^{'''} \mathcal{R}(FY,X)FZ].\end{aligned}$$

The conclusion follows from this formula and from the assumption that the third Ricci tensor ${}_3^{'''}\mathcal{R}(X,Y)$ vanishes.

Theorem 2.2 in case b) can be proved similarly.

Let us now consider the curvature tensor ${}_4R(X,Y)Z$ given by the formula (1.6). Similarly as in the case curvature tensor ${}_3R(X,Y)Z$ we prove the following two theorems.

THEOREM 2.3. *The tensor*

$$\begin{aligned}{}_4R(X,Y)Z - \frac{1}{2(n-1)}[{}_4^{''}\mathcal{R}(X,Z)Y -{}_4^{''} \mathcal{R}(FX,Z)FY -{}_4^{''} \mathcal{R}(X,Y)Z+ \\ +{}_4^{''} \mathcal{R}(FX,Y)FZ]\end{aligned}$$

is independent on 1-form $\mathcal{A}$, which is covariantly constant with respect to the connection ${}^0\nabla$ and it is equal to the tensor

$$\begin{aligned}K(X,Y)Z - \frac{1}{2(n-1)}[{}''\mathcal{K}(X,Z)Y -{}'' \mathcal{K}(FX,Z)FY -{}'' \mathcal{K}(X,Y)Z \\ +{}'' \mathcal{K}(FX,Y)FZ].\end{aligned}$$

Proof. Substituting (2.1) and (2.2) into (1.6) and using (1.1), (1.11), after a long calculation we find

$$\begin{aligned}{}_4R(X,Y)Z &= K(X,Y)Z + [\mathcal{A}F({}^\circ\nabla_Y Z)X - Y\mathcal{A}F(Z)]X + [\mathcal{A}({}^\circ\nabla_Y Z)- \\ &- Y\mathcal{A}(Z)]F(X) - [\mathcal{A}F({}^\circ\nabla_X Y) - X\mathcal{A}F(Y)]Z - [\mathcal{A}({}^\circ\nabla_X Y) - X\mathcal{A}(Y)]F(Z) \\ &+ [\mathcal{A}F(X)\mathcal{A}F(Z) - \mathcal{A}(X)\mathcal{A}(Z)]Y + [\mathcal{A}(X)\mathcal{A}F(Z) + \mathcal{A}F(X)\mathcal{A}(Z)]F(Y) \\ &- [\mathcal{A}F(X)\mathcal{A}F(Y) - \mathcal{A}(X)\mathcal{A}(Y)]Z - [\mathcal{A}(X)\mathcal{A}F(Y) + \mathcal{A}F(X)\mathcal{A}(Y)]F(Z),\end{aligned}$$

and from here, using the relation (2.5) for covariantly constant 1-form $\mathcal{A}$, we have

$$\begin{aligned}{}_4R(X,Y)Z = K(X,Y)Z + [\mathcal{A}F(X)\mathcal{A}F(Z) - \mathcal{A}(X)\mathcal{A}(Z)Y+ \\ + [\mathcal{A}F(X)\mathcal{A}(Z) + \mathcal{A}(X)\mathcal{A}F(Z)]F(Y) - [\mathcal{A}F(X)\mathcal{A}F(Y)- \\ + \mathcal{A}(X)\mathcal{A}(Y)]Z - [\mathcal{A}F(X)\mathcal{A}(Y) + \mathcal{A}(X)\mathcal{A}F(Y)]F(Z).\end{aligned} \tag{2.8}$$

If we define the corresponding linear map

$$h : \mathfrak{X}(\mathfrak{M}^{2n}) \to \mathfrak{X}(\mathfrak{M}^{2n})$$

by

$$\begin{aligned}h(Y) =_4 R(X,Y)Z - K(X,Y)Z + [\mathcal{A}F(X)\mathcal{A}F(Z) - \mathcal{A}(X)\mathcal{A}(Z)]Y-\\ - [\mathcal{A}F(X)\mathcal{A}(Z) + \mathcal{A}(X)\mathcal{A}F(Z)]F(Y) - [\mathcal{A}F(X)\mathcal{A}F(Y)-\\ - \mathcal{A}(X)\mathcal{A}(Y)]Z - [\mathcal{A}F(X)\mathcal{A}(Y) + \mathcal{A}(X)\mathcal{A}F(Y)]F(Z),\end{aligned}$$

and calculate its trace we get as in the previous case

$$_4''\mathcal{R}(X,Z) -'' \mathcal{K}(X,Z) - 2(n-1)[\mathcal{A}F(X)\mathcal{A}F(Z) - \mathcal{A}(X)\mathcal{A}(Z)] = 0.$$

Since $n > 1$, from here we have

$$\mathcal{A}F(X)\mathcal{A}F(Z) - \mathcal{A}(X)\mathcal{A}(Z) = \frac{1}{2(n-1)}[_4''\mathcal{R}(X,Z) -'' \mathcal{K}(X,Z)].$$

Substituting FX for X in the last relation and using (1.10) we get

$$\mathcal{A}(X)\mathcal{A}F(Z) - \mathcal{A}F(X)\mathcal{A}(Z) = \frac{1}{2(n-1)}[_4''\mathcal{R}(FX,Z) -'' \mathcal{K}(FX,Z)].$$

Let us substitute the corresponding expression from the last two relations into (2.8). Then it yields

$$\begin{aligned}{}_4R(X,Y)Z - \frac{1}{2(n-1)}[_4''\mathcal{R}(X,Z)Y -_4'' \mathcal{R}(FX,Z)FY -_4'' \mathcal{R}(X,Y)Z+\\ +_4'' \mathcal{R}(FX,Y)FZ] = K(X,Y)Z - \frac{1}{2(n-1)}[''\mathcal{K}(X,Y)Z-\\ -'' \mathcal{K}(FX,Z)FY -'' \mathcal{K}(X,Y)Z +'' \mathcal{K}(FX,Y)FZ],\end{aligned}$$

as required.

THEOREM 2.4. *If 1-form* $\mathcal{A}$ *is covariantly constant with respect to the Riemann-Christoffel connection* $^0\nabla$ *of Kählar spece* $\mathfrak{M}^{2n}(n > 1)$ *and the Ricci tensor* $_4''\mathcal{R}(X,Y)$ *vanishes, then the curvature tensor* $_4R(X,Y)Z$ *is equal to the curvature tensor* $K(X,Y)Z$.

Proof. This theorem can be proved analogously to Theorem 2.2 using the linear map $k\colon \mathfrak{X}(\mathfrak{M}^{2n}) \to \mathfrak{X}(\mathfrak{M}^{2n})$

$$\begin{aligned}k(Z) =_4 R(X,Y)Z - K(X,Y)Z-\\ - [\mathcal{A}F(X)\mathcal{A}F(Z) - \mathcal{A}(X)\mathcal{A}(Z)]Y - [\mathcal{A}F(X)\mathcal{A}(Z) + \mathcal{A}(X)\mathcal{A}F(Z)]F(Y)\\ + [\mathcal{A}F(X)\mathcal{A}F(Y) - \mathcal{A}(X)\mathcal{A}(Y)]Z + [\mathcal{A}F(X)\mathcal{A}(Y) + \mathcal{A}(X)\mathcal{A}F(Y)]F(Z),\end{aligned}$$

for any $X, Y \in \mathfrak{X}(\mathfrak{M}^{2n})$.

We emphasize that, unlike Theorems 2.1 and 2.2, Theorems 2.3 and 2.4 are not valid under the assumption

$$(^0\nabla_Y \mathcal{A})(X) =' \mathcal{K}(FX,Y).$$

3. Locally decomposable Riemannian space. Let us consider the same problem for locally decomposable Riemannian space $\mathfrak{M}^n$, $(n > 2,\, p > 1,\, q > 1)$, as we did for complex analytic space. In this section we use the notation

$$\alpha = -\frac{n-2}{(n-2)^2 - \varphi^2}, \quad \beta = \frac{\varphi}{(n-2)^2 - \varphi^2}. \tag{3.1}$$

This time, the results we obtain have various geometric interpretations.

The $({}^1F, {}^2F)$-connection on locally decomposable Riemannian space $\mathfrak{M}^n$ given by (1.17) in the case

$$A^i_{jk} = \frac{1}{2}\mathcal{A}_j F^i_k,$$

where $\mathcal{A}_j$ is a field of 1-forms, and F^i_k tensor of almost product structure has the form

$$\Gamma^i_{jk} = {}^0\Gamma^i_{jk} + \mathcal{A}_j F^i_k + \mathcal{A}_a F^a_j \delta^i_k.$$

Using the notion of connection in the sense of Koszul given by (1.1) we can write this relation in the following way

$${}^1\nabla_X Y = {}^0\nabla_X Y + \mathcal{A}(X)F(Y) + \mathcal{A}F(X)Y. \tag{3.2}$$

In addition, we also define the second connection

$${}^2\nabla_X Y = {}^0\nabla_X Y + \mathcal{A}(Y)F(X) + \mathcal{A}F(Y)X. \tag{3.3}$$

Keeping in mind (1.1). (1.13), (1.14), (2.4), (3.2) and (3.3), the curvature tensor ${}_3R(X,Y)Z$ given by the relation (1.5) after a long calculation, can be represented in the following form

$$\begin{aligned}
{}_3R(X,Y)Z &= K(X,Y)Z + \\
&+ [\mathcal{A}({}^0\nabla_Y FZ) - Y\mathcal{A}F(Z) + \mathcal{A}F(Y)\mathcal{A}F(Z) + \mathcal{A}(Y)\mathcal{A}(Z)]X \\
&+ [\mathcal{A}({}^0\nabla_Y Z) - Y\mathcal{A}(Z) + \mathcal{A}F(Y)\mathcal{A}(Z) + \mathcal{A}(Y)\mathcal{A}F(Z)]F(X) \\
&- [\mathcal{A}({}^0\nabla_X FY) - X\mathcal{A}F(Y) + \mathcal{A}F(Y)\mathcal{A}F(X) + \mathcal{A}(Y)\mathcal{A}(X)]Z \\
&- [\mathcal{A}({}^0\nabla_X Y) - X\mathcal{A}(Y) + \mathcal{A}(Y)\mathcal{A}F(X) + \mathcal{A}(X)\mathcal{A}F(Y)]F(Z).
\end{aligned} \tag{3.4}$$

Using this formula for the curvature tensor ${}_3R(X,Y)Z$, under different assumptions for 1-form $\mathcal{A}$ we prove the following theorems.

THEOREM 3.1. *Let $\mathfrak{M}^n$ be a locally decomposable Riemannian space with a field of 1-forms $\mathcal{A}$ which is either*

a) *covariantly constant with respect to the Riemann-Christoffel connection ${}^0\nabla$, that is*

$$({}^0\nabla_Y \mathcal{A})(X) = 0$$

or

b) $$({}^0\nabla_Y \mathcal{A})(X) = g(X,Y).$$

Then the tensor field

$$
{}_3R(X,Y)Z+\alpha[{}'_3\mathcal{R}(Y,Z)X+'_3(FY,Z)FX-'_3\mathcal{R}(X,Y)Z-'_3\mathcal{R}(FX,Y)FZ]
+\beta({}'_3\mathcal{R}(FY,Z)X+'_3\mathcal{R}(Y,Z)FX-'_3\mathcal{R}(FX,Y)Z-'_3\mathcal{R}(X,Y)FZ]
$$

is independent on the field of 1-forms $\mathcal{A}$ and it is equal to the field of product projective curvature tensors [**1**]. *Here α,β are constants given by* (3.1).

Proof. Let us prove the theorem in case a). Due to the assumption that 1-form $\mathcal{A}$ is covariantly constant with respect to the connection ${}^0\nabla$, (2.5) is valid. Keping in mind this fact the relation (3.4) can be written in the simpler form

$$
\begin{aligned}
(3.5)\qquad {}_3R(X,Y)Z =&K(X,Y)Z+\\
&+[\mathcal{A}F(Y)\mathcal{A}F(Z)+\mathcal{A}(Y)\mathcal{A}(Z)]X+[\mathcal{A}F(Y)\mathcal{A}(Z)+\mathcal{A}(Y)\mathcal{A}F(Z)]FX\\
&-[\mathcal{A}F(Y)\mathcal{A}F(X)+\mathcal{A}(Y)\mathcal{A}(X)]Z-[\mathcal{A}(Y)\mathcal{A}F(X)+\mathcal{A}(X)\mathcal{A}F(Y)]FZ.
\end{aligned}
$$

Let us define a linear map

$$f:\mathfrak{X}(\mathfrak{M}^n)\to\mathfrak{X}(\mathfrak{M}^n)$$

by the following relation

$$
\begin{aligned}
f(X)=&{}_3R(X,Y)Z-K(X,Y)Z-\\
&-[\mathcal{A}F(Y)\mathcal{A}F(Z)+\mathcal{A}(Y)\mathcal{A}(Z)]X-[\mathcal{A}F(Y)\mathcal{A}(Z)+\mathcal{A}(Y)\mathcal{A}F(Z)]FX\\
&+[\mathcal{A}F(Y)\mathcal{A}F(X)=\mathcal{A}(Y)\mathcal{A}(X)]Z+[\mathcal{A}(Y)\mathcal{A}F(X)+\mathcal{A}(X)\mathcal{A}F(Y)]FZ.
\end{aligned}
$$

If we determine the trace of this linear map, using (1.7), (1.13)(1.16), (3.5) we obtain

$$
\begin{aligned}
(3.6)\qquad {}'_3\mathcal{R}(Y,Z)=&'\mathcal{K}(Y,Z)+\\
&+(n-2)[\mathcal{A}(Y)\mathcal{A}(Z)+\mathcal{A}F(Y)\mathcal{A}F(Z)]+\\
&+\varphi[\mathcal{A}F(Y)\mathcal{A}(Z)+\mathcal{A}(Y)\mathcal{A}F(Z)].
\end{aligned}
$$

Substituting FY for Y in (3.6) we get

$$
\begin{aligned}
{}'_3\mathcal{R}(FY,Z)=&'\mathcal{K}(FY,Z)+\\
&+(n-2)[\mathcal{A}F(Y)\mathcal{A}(Z)+\mathcal{A}(Y)\mathcal{A}F(Z)]\\
&+\varphi[\mathcal{A}(Y)\mathcal{A}(Z)+\mathcal{A}F(Y)\mathcal{A}F(Z)],
\end{aligned}
$$

where we used (1.13). Let us multiply (3.6) by φ and (3.7) by $(n-2)$. Subtracting the relations so obtained we find

$$
\begin{aligned}
(3.8)\qquad &\mathcal{A}F(Y)\mathcal{A}(Z)+\mathcal{A}(Y)F(Z)=\\
&=\frac{1}{\varphi^2-(n-2)^2}[\varphi{}'_3\mathcal{R}(Y,Z)-(n-2){}'_3\mathcal{R}(FY,Z)-\varphi'\mathcal{K}(Y,Z)+\\
&+(n-2)'\mathcal{K}(FY,Z)].
\end{aligned}
$$

Let us substitute FY for Y in the relation (3.8). By (1.13) we have

$$\mathcal{A}F(Y)\mathcal{A}F(Z)+\mathcal{A}(Y)\mathcal{A}(Z)=$$
$$=\frac{1}{\varphi^2-(n-2)^2}[\varphi{}_3'\mathcal{R}(FY,Z)-(n-2){}_3'\mathcal{R}(Y,Z)-\varphi'\mathcal{K}(FY,Z)+$$
$$+(n-2)'\mathcal{K}(Y,Z)].$$

Using (3.8), (3.9) and the notation in (3.1), the relation (3.5) can be transformed, so that we get

$$\begin{aligned}{}_3R(X,Y)Z&+\alpha[{}_3'\mathcal{R}(Y,Z)Z+{}_3'\mathcal{R}(FY,Z)FX-{}_3'\mathcal{R}(X,Y)Z-{}_3'\mathcal{R}(FX,Y)FZ]+\\&+\beta[{}_3'\mathcal{R}(FY,Z)X+{}_3'\mathcal{R}(Y,Z)FX-{}_3'\mathcal{R}(FX,Y)Z-{}_3'\mathcal{R}(X,Y)FZ]=\\&=K(X,Y)Z+\alpha['\mathcal{K}(Y,Z)X+'\mathcal{K}(FY,Z)FX,-'\mathcal{K}(X,Y)Z-\\&-'\mathcal{K}(FX,Y)FZ+\beta['\mathcal{K}(FY,Z)X+'\mathcal{K}(Y,Z)FX-'\mathcal{K}(FX,Y)Z-\\&-'\mathcal{K}(X,Y)FZ,\end{aligned}$$

and taking account of [**11**] the theorem is proved in case a).

The theorem can be proved similarly i case b).

From the preceding theorem we immediately obtain the following.

COROLLARY 3.1. *Let $\mathfrak{M}^n$ be a locally decomposable Riemannian space, with a field of 1-forms $\mathcal{A}$ which is either:*

a) *covariantly constant with respect to the Riemann-Christoffel connection, or*

b) $$({}^0\nabla_Y\mathcal{A})(X)=g(X,Y).$$

Then we have

$$\begin{aligned}{}_3R(X,Y)Z=\alpha&[{}_3'\mathcal{R}(X,Y)Z+{}_3'\mathcal{R}(FX,Y)FZ-{}_3'\mathcal{R}(Y,Z)X-{}_3'\mathcal{R}(FY,Z)FX]\\&+\beta[{}_3'\mathcal{R}(FX,Y)Z+{}_3'\mathcal{R}(X,Y)FZ-{}_3'\mathcal{R}(FY,Z)X-{}_3'\mathcal{R}(Y,Z)FX]\end{aligned}$$

if and only if $\mathfrak{M}^n$ is a space of separately constant curvature.

THEOREM 3.2. *Let $\mathfrak{M}^n$ be a locally decomposamble Riemannian space, with a filed of 1-forms $\mathcal{A}$ which is either:*

a) *covariantly constant with respect to the Riemann-Christoffel connection $^\circ\nabla$, or*

b) $$({}^0\nabla_X\mathcal{A})(X)=g(X,Y)$$

and with curvature tensor ${}_3R(X,Y)Z$ which is equal to zero. Then the space $\mathfrak{M}^n$ is flat.

Proof. Let us prove the theorem in case a). It follows directly from theorem 3.1, that a space $\mathfrak{M}^n$ is of s separately constant curvature if ${}_3R(X,Y)Z=0$, for any $X,Y,Z\in\mathfrak{X}\mathfrak{M}^n)$. We want to prove that the space is not only of separately constant curvature but also flat. For that reason we define the linear map.

$$g:\mathfrak{X}(\mathfrak{M}^n)\to\mathfrak{X}(\mathfrak{M}^n)$$

by the following relation;

$$f(Z) =_3 R(X,Y)Z - K(X,Y)Z-$$
$$-[\mathcal{A}F(Y)\mathcal{A}F(Z) + \mathcal{A}(Y)\mathcal{A}(Z)]X - [\mathcal{A}F(Y)\mathcal{A}(Z) + \mathcal{A}(Y)\mathcal{A}F(Z)]F(X)$$
$$+[\mathcal{A}F(Y)\mathcal{A}F(X) + \mathcal{A}(Y)(\mathcal{A}(X)]Z + [\mathcal{A}(Y)\mathcal{A}F(X) + \mathcal{A}(X)\mathcal{A}F(Y)]F(Z).$$

Since ${}^0\nabla$ is a Riemann-Christoffel connection, taking into account (1.9) and (2.4), we obtain

$$'''\mathcal{K}(X,Y) = 0,$$

and determining the trace of the previously defined linear map, we prove as in previous cases that the following equality

$${}_3R(X,Y)Z + \alpha[{}_3'''\mathcal{R}(Z,Y)X +_3''' \mathcal{R}(FZ,Y)FX -_3''' \mathcal{R}(X,Y)Z -_3''' \mathcal{R}(FX,Y)FZ]+$$
$$+\beta[{}_3'''\mathcal{R}(FZ,Y)X +_3''' \mathcal{R}(Z,Y)FX -_3''' \mathcal{R}(FX,Y)Z -_3''' \mathcal{R}(X,Y)FZ] = K(X,Y)Z$$

is valid in both cases a) and b), and from that the conclusion of the theorem follows directly.

Let us now consider the curvature tensor ${}_4R(X,Y)Z$ given by the formula (1.6). Similarly as in the previous case we prove that two following theorems are valid:

THEOREM 3.3. *Let $\mathfrak{M}^n$ be a locally decomposable Riemannian space, with a field of 1-forms $\mathcal{A}$, which is covariantly constant with respect to Rieman-Christoffel connection ${}^0\nabla$. Then the tensor field*

$${}_4R(X,Y)Z + \alpha[{}_4''\mathcal{R}(X,Z)Y +_4'' \mathcal{R}(FX,Z)FY -_4'' \mathcal{R}(X,Y)Z-$$
$$-_4'' \mathcal{R}(FX,Y)FZ] + \beta[{}_4''\mathcal{R}(FX,Z)Y +_4'' \mathcal{R}(X,Z)FY -_4'' (FX,Y)Z$$
$$-_4'' \mathcal{R}(X,Y)FZ]$$

where α, β are constants given by (3.1), is independent on the field of 1-forms $\mathcal{A}$ and has opposite value of tensor field of product-projective curvature $P(Y,X)Z$.

Proof. Substituting expressions (3.2) and (3.3) for corresponding connection in the formula (1.6) for curvature tensor ${}_4R(X,Y)Z$, and using the notations (2.4), (3.1) and the relations (1.1), (1.13) we get

$${}_4R(X,Y)Z = K(X,Y)Z+$$
$$+ [\mathcal{A}(^\circ\nabla_Y Z) - Y\mathcal{A}(Z)]F(X) + [\mathcal{A}F(^\circ\nabla_Y Z) - Y\mathcal{A}F(Z)]X$$
$$- [\mathcal{A}(^\circ\nabla_X Y) - X\mathcal{A}(Y)]F(Z) - [\mathcal{A}F(^\circ\nabla_X Y) - X\mathcal{A}F(Y)]Z$$
$$+ [\mathcal{A}(X)\mathcal{A}(Z) + \mathcal{A}F(X)\mathcal{A}F(Z)]Y + [\mathcal{A}(X)\mathcal{A}F(Z) + \mathcal{A}F(X)\mathcal{A}(Z)]F(Y)$$
$$- [\mathcal{A}(X)\mathcal{A}(Y) + \mathcal{A}F(X)\mathcal{A}F(Y)]Z - [\mathcal{A}(X)\mathcal{A}F(Y) + \mathcal{A}F(X)\mathcal{A}(Y)]F(Z).$$

Since a field of 1-forms $\mathcal{A}$ is covariantly constant, is satisfies the formula (1.2), and therefore the foregoing relation can be written in a simpler form;

(3.10)

$${}_4R(X,Y)Z = K(X,Y)Z+$$
$$+ [\mathcal{A}(X)\mathcal{A}(Z) + \mathcal{A}F(X)\mathcal{A}F(Z)]Y + [\mathcal{A}(X)\mathcal{A}F(Z) + \mathcal{A}F(X)\mathcal{A}(Z)]F(Y)$$
$$- [\mathcal{A}(X)\mathcal{A}(Y) + \mathcal{A}F(X)\mathcal{A}F(Y)]Z - [\mathcal{A}(X)\mathcal{A}F(Y) + \mathcal{A}F(X)\mathcal{A}(Y)]F(Z).$$

If we define a linear map $f:\mathfrak{X}(\mathfrak{M})\to\mathfrak{X}(\mathfrak{M})$ by

$$\begin{aligned} f(Y) ={}_4 R(X,Y)Z - K(X,Y)Z-\\ -[\mathcal{A}(X)\mathcal{A}(Z)+\mathcal{A}F(X)\mathcal{A}F(Z)]Y-[\mathcal{A}(X)\mathcal{A}F(Z)+\mathcal{A}F(X)\mathcal{A}(Z)]F(Y)\\ +[\mathcal{A}(X)\mathcal{A}(Y)+\mathcal{A}F(X)\mathcal{A}F(Y)]Z+[\mathcal{A}(X)\mathcal{A}F(Y)+\mathcal{A}F(X)\mathcal{A}(Y)]F(Z), \end{aligned}$$

for any $X,Y,Z\in\mathfrak{X}(\mathfrak{M})$ and then find the trace of this linear map, bearing in mind (1.8), (1.13), (1.16) and (3.10), we obtain

$$\begin{aligned} {}_4''\mathcal{R}(X,Z) ={}'' \mathcal{K}(X,Z)+\\ +(n-2)[\mathcal{A}(X)\mathcal{A}(Z)+\mathcal{A}F(X)\mathcal{A}F(Z)]+\\ +\varphi[\mathcal{A}(X)\mathcal{A}F(Z)+\mathcal{A}F(X)\mathcal{A}(Z)], \end{aligned}$$

and from that, if we substitute FX for X, we have

$$\begin{aligned} {}_4''\mathcal{R}(FX,Z) ={}'' \mathcal{K}(FX,Z)+(n-2)[\mathcal{A}F(X)\mathcal{A}(Z)+\mathcal{A}(X)\mathcal{A}F(Z)]\\ +\varphi[\mathcal{A}F(X)\mathcal{A}F(Z)+\mathcal{A}(X)\mathcal{A}(Z)]. \end{aligned}$$

Solving the system of linear equations given by the two last relations we find

$$\begin{aligned} \mathcal{A}F(X)\mathcal{A}(Z)+\mathcal{A}(X)\mathcal{A}F(Z) = \alpha[''\mathcal{K}(FX,Z) -{}_4'' \mathcal{R}(FX,Z)]+\\ +\beta[''\mathcal{K}(X,Z) -{}_4'' \mathcal{R}(X,Z)]\\ -\mathcal{A}(X)\mathcal{A}(Z)+\mathcal{A}F(X)\mathcal{A}F(Z) = \alpha[''\mathcal{K}(X,Z) -{}_4'' \mathcal{R}(X,Z)]+\\ +\beta[''\mathcal{K}(FX,Z) -{}_4'' \mathcal{R}(FX,Z)]. \end{aligned}$$

If we substitute the corresponding expressions from the last two relations into (3.10) we have

$$\begin{aligned} {}_4R(X,Y)Z+\alpha[{}_4''\mathcal{R}(X,Z)Y +{}_4'' \mathcal{R}(FX,Z)F(Y) -{}_4'' \mathcal{R}(X,Y)Z -{}_4'' \mathcal{R}(FX,Y)F(Z)]\\ +\beta[{}_4''\mathcal{R}(FX,Z)Y +{}_4'' \mathcal{R}(X,Z)F(Y) -{}_4'' \mathcal{R}(FX,Y)Z -{}_4'' \mathcal{R}(X,Y)F(Z)] =\\ = K(X,Y)Z+\alpha[''\mathcal{K}(X,Z) +'' \mathcal{K}(FX,Z)F(Y) -'' \mathcal{K}(X,Y)Z -'' \mathcal{K}(FX,Y)F(Z)]\\ +\beta[''\mathcal{K}(FX,Z)Y +'' \mathcal{K}(X,Z)F(Y) -'' \mathcal{K}(FX,Y)Z -'' \mathcal{K}(X,Y)F(Z)]. \end{aligned}$$

Since $K(X,Y)Z = -K(Y,X)Z$ we have $''\mathcal{K}(X,Y) = -'\mathcal{K}(X,Y)$ and hence (3.11) can be written in the following way

$$\begin{aligned} {}_4R(X,Y)Z+\alpha[{}_4''\mathcal{R}(X,Z)Y +{}_4'' \mathcal{R}(FX,Z)F(Y) -{}_4'' \mathcal{R}(X,Y)Z -{}_4'' \mathcal{R}(FX,Y)F(Z)\\ +\beta[{}_4''\mathcal{R}(FX,Z)Y +{}_4'' \mathcal{R}(X,Z)F(Y) -{}_4'' \mathcal{R}(FX,Y)Z -{}_4'' \mathcal{R}(X,Y)F(Z)]\\ = -K(Y,X)Z-\alpha['\mathcal{K}(X,Z)Y +' \mathcal{K}(FX,Z)F(Y) -' \mathcal{K}(X,Y)Z\\ -' \mathcal{K}(FX,Y)F(Z)-\beta['\mathcal{K}(FX,Z)Y +' \mathcal{K}(X,Z)F(Y) -' \mathcal{K}(FX,Y)Z\\ -' \mathcal{K}(X,Y)F(Z)]. \end{aligned}$$

This concludes the proof of theorem 3.3.

From the previous theorem we immediately obtain the following.

COROLLARY 3.2. *Let $\mathfrak{M}^n$ be a locally decomposable Riemannian space, with a field of 1-forms $\mathcal{A}$, which is covariantly constant with respect to the Riemann-Chritoffel connection $^\circ\nabla$. Then we have*

$$\begin{aligned}{}_4R(X,Y)Z = \alpha[{}_4''\mathcal{R}(X,Y)Z +_4'' \mathcal{R}(FX,Y)F(Z) -_4'' \mathcal{R}(X,Z)Y -_4'' \mathcal{R}(FX,Z)F(Y)]\\ + \beta[{}_4''\mathcal{R}(FX,Y)Z +_4'' \mathcal{R}(X,Y)F(Z) -_4'' \mathcal{R}(FX,Z)Y -_4'' \mathcal{R}(X,Z)F(Y)]\end{aligned}$$

if and only if a space $\mathfrak{M}^n$ is of separately constant curvature.

THEOREM 3.4. *Let $\mathfrak{M}^n$ be a locally decomposable Riemannian space, with a field of 1-forms $\mathcal{A}$, which is covariantly constant with respect to the Riemann-Christoffel connection $^0\nabla$ and with a field of curvature tensors ${}_4R(X,Y)Z$ which is equal to zero. Then the space $\mathfrak{M}^n$ is flat.*

Proof. From Theorem 3.3 it follows directly that if ${}_4R(X,Y)Z = 0$, then a space $\mathfrak{M}^n$ is a space of separately constant curvature. To prove that a space is also a flat space, we define a linear map $h:\mathfrak{X}(\mathfrak{M}^n) \to \mathfrak{X}(\mathfrak{M}^n)$ by the following relation

$$\begin{aligned}h(Z) =_4 R(X,Y)Z - K(X,Y)Z-\\ - [\mathcal{A}(X)\mathcal{A}(Z) + \mathcal{A}F(X)\mathcal{A}F(Z)]Y - [\mathcal{A}(X)\mathcal{A}F(Z) + \mathcal{A}F(X)\mathcal{A}(Z)]F(Y)\\ + [\mathcal{A}(X)\mathcal{A}(Y) + \mathcal{A}F(X)\mathcal{A}F(Y)]Z + [\mathcal{A}(X)\mathcal{A}F(Y) + \mathcal{A}F(X)\mathcal{A}(Y)]F(Z)\end{aligned}$$

Since $^0\nabla$ is a Riemann-Christoffel connection, we have

$$'''\mathcal{K}(X,Y) = 0$$

and therefore, determining the trace of the previously defined linear map, we can prove as before that

$$\begin{aligned}{}_4R(X,Y)Z + \alpha[{}_4'''\mathcal{R}(X,Y)Z +_4''' \mathcal{R}(FX,Y)F(Z) -_4''' \mathcal{R}(X,Z)Y-\\ {}_4'''\mathcal{R}(FX,Z)FY)] + \beta[{}_4'''\mathcal{R}(FX,Y)Z +_4''' \mathcal{R}(X,Y)F(Z) -_4''' \mathcal{R}(FX,Z)Y-\\ -_4''' \mathcal{R}(X,Z)F(Y)] = K(X,Y)Z\end{aligned}$$

and from here the conclusion of Theorem 3.4 follows directly.

Note that for locally decomposable Riemannian spaces and for complex analytic spaces the theorems related to the curvature tensor ${}_4R(X,Y)Z$ are valid only under the assumptions that the field of 1-forms $\mathcal{A}$ is covariantly constant. Besides that assumptions for a field of 1-forms $\mathcal{A}$, which are related to the curvature tensor ${}_3R(X,Y)Z$, are different for locally decomposable Riemannian space and complex analytic space.

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