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ON QUASI-FROBENIUSEAN AND ARTINIAN RINGS

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Abstract. Left p-injective rings, which extend left self injective rings, have been considered in several papers (cf. for example, [**10**] – [**14**]). The following generalizations of left p-injective rings are here introduced: (1) A is called a left min-injective ring if, for any minimal left ideal U of A (if it exists), any left A-homomorphism $g : U \to A$, there exists $y \in A$ such that $g(b) = by$ for all $b \in U$; (2) A is left np-injective if, for any non-nilpotent element c of A, any left A-homomorphism $g : Ac \to A$, there exists $y \in A$ such that $g(ac) = acy$ for all $a \in A$. New characteristic properties of quasi-Frobeniusean rings are given. It is proved that A is quasi-Frobeniusean iff A is a left Artinian, left and right min-injective ring. If A is left np-injective, then (a) every left or right A-module is divisible and (b) any reduced principal left ideal of A is generated by an idempotent. Further properties of left CM-rings (introduced in [**14**]) are developed. The following nice result is established : If U is a minimal left ideal of a left CM-ring A, the following are then equivalent: (a) ${}_AU$ is injective; (b) ${}_AU$ is projective; (c) ${}_AU$ is p-injective. Consequently, A is semi-simple Artinian iff A is a left CM-ring with finitely generated projective essential left socle. Divison rings are also characterised. Known results are improved.

Introduction. This note contains new characteristic properties of quasi-Frobeniusean rings in terms of min-injective Iings (defined below). It is proved that A is quasi-Frobeniusean in A is a left Artinian, left and right min-injective ring. Left CM-rings (introduced in [**14**]) are further studied. In particular, it is proved that if U is a minimal left ideal of a left CM-ring A, then ${}_AU$ is injective iff it is projective iff it is p-injective. Certain results on CM-rings (14) are improved. A generalization of left p-injective rings, called np-injective rings, is also considered and various properties are derived. Characteristic properties of division rings and semi-simple Artinian rings are given.

Throughout, A refers to an associative ring with identity and A-modules are unitary. Z, J, S will denote respectively the left singular ideal, the Jacobson radical and the left socle of A. A is called left non-singular (resp. semi-simple) iff $Z = 0$ (resp. $J = 0$). Recall that a left A-module M is p-injective (resp. f-injective) iff for any principal (resp. finitely generated) left ideal I of A, any left A-bomomorphism $g : I \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. Then A is

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von Neumann regular iff every left A-module is p-injective (f-injective). It is well-known that A is regular iff every left A-module is flat. If I is a p-injective left ideal of A, then A/I is a flat left A-module.

As usual, a left (right) ideal of A is called reduced iff it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal. In [**14**], the following generalization of semi-simple Artinian, left uniform and left duo rings is introduced: A is called a left CM-ring iff, for any maximal essential left ideal M of A (if it exists), every complement left subideal is an ideal of M. Left CM-rings also extend left PCI-rings [**4**] and the domains constructed by Cozzens [**3**]. Regular left CM-rings are left and right V-rings but left CM, left or right V-rings need not be regular. Recall that (1) A is left pseudo-coherent iff $l(F)$ is a finitely generated left ideal for any finitely generated right ideal F of A; (2) A is ELT (resp. $MELT$) iff every essential (resp. maximal essential, if it exists) left ideal is an ideal of A. Rings whose ideals are left annihilators are called TLA-rings. We know that if A is a semi-prime TLA-ring, then every ideal of A is generated by a central idempotent and hence A is a biregular, fully left and right idempotent ring. Note that semi-prime rings whose left ideals are left annihilators must be semi-simple Artinian. It is well-known that in a quasi-Frobeniusean ring, every one-sided ideal is an annihilator.

H. Tominaga (Math. Reviews 81i#16014) pointed out that [**12**, Theorem 8] depended on a unproved result of R. P. Kurshan [**8**, Proposition 3.4] (cf. [**6**]). By going through the proof of [**8**, Theorem 3.3 and Proposition 3.4] (keeping in view Ginn's remark [**6**, p. 105]), [**12**, Lemma 5] yields.

LEMMA 1. *Let A be a TLA-ring whose essential left ideals are left annihilators satisfying the maximum condition on ideals rtnd essential left ideals such that $l(Z)$ is a finitely generated left ideal and A/Z is semi-simple Artinian. Then A is left Artinian.*

Applying [**14**, Proposition 2.8], [**2**, p. 69)] and Lemma 1, we get

COROLLARY 1.1. *The following conditions are equivalent for an ELT, TLA-ring satisfying the maximum condition on left anihilators:* (1) *A is left Artinian;* (2) *Both A and A/Z have left finite Goldie dimension;* (3) *A/Z is left or right self-injective and $l(Z)$ is a finitely generated left ideal;* (4) *${}_AZ$ is finitely generated.*

We now introduce the following generalization of left p-injective and semi-prime rings.

Definition. A is called a left min-injective ring if, for any minimal left ideal U of A (if it exists), any left A-homomorphism $g : U \to A$, there exists $y \in A$ such that $g(b) = by$ for all $b \in U$.

[**12**, Theorem 8(iii)] is proved in the next result (this extends a result of C .Faith [**5**, p. 209]).

THEOREM 2. *The following conditions are equivalent:*

(1) *A is quasi-Frobeniusean;*

(2) *A is a right p-injective, left min-injective, left pseudocoherent TLA-ring with maximum condition on right annihilators;*

(3) *A is a left Noetherian, left p-injective, right min-injective ring;*

(4) *A is a left Artinian, Ieft and right min-injective ring;*

(5) *A is a right f injective ring with maximum condition on left annihilators.*

Proof. Obviously, (1) implies (2) and (3).

Assume (2). If I is an ideal of A, $T = l(B)$ for some right ideal B of A. Since A satisfies the maximum condition on right annihilators, then $T = l(F)$, where F is a finitely generated right subideal of B and since A is left pseudo-coherent, AT is finitely generated. In particular, ${}_AJ$ is finitely generated. Since A is right p-injective with minimum condition on left annihilators, then A is right perfect by a theorem of M. Ikeda – T. Nakayama which yields that A is left Artinian [**2**, p. 69]. Thus (2) implies (4).

Assume (3). Since A is left p-injective with the minimum condition on right annihilators, then A is left perfect which, together with A left Noetherian, implies A left Artinian. Therefore (3) implies (4).

Assume (4). Let U be a minimal left ideal of A, $0 \neq u \in U$. Since A is both left and right perfect, uA contains a minimal right ideal $V = vA$, $v \in A$. Then $l(u) \subseteq l(v)$ and if $f : Au \to Av$ is the left A-homomorphism defined by $f(au) = av$ for all $a \in A$, since $U = Au$ is a minimal left ideal, then f is an isomorphism and if $g : Av \to Au$ is the inverse isomorphism, $i : Au \to A$ the natural injection, then there exists $w \in A$ such that $u = ig(v) = vw$. Therefore $uA = vA$ is a minimal right ideal and if $0 \neq b \in l(r(u))$, then $r(u) = r(l(r(u))) \subseteq r(b)$ and if $h : uA \to bA$ is the right A-homomorphism defined by $h(ua) = ba$ for all $a \in A$, $j : bA \to A$ the natural injection, then there exists $c \in A$ such that $b = jh(u) = cu \in Au$. Thus $l(r(Au)) = l(r(u)) = Au = U$. Similarly, any minimal right ideal of A is a right annihilator and by [**9**, Proposition 1], (4) implies (5).

Assume (5). Then A is a left Noetherian ring whose left ideals are left annihilators (cf. [**12**, p. 134]). Since A/J is a semi-prime ring whose left ideals are left annihilators, then A/J is semi-simple Artinian (this is the crucial property mentioned by Ginn [**6**]). By [**12**, Lemma 5] and Lemma 1, A is left Artinian (indeed, the proof of [**12**, Lemma 5] shows that $Z = J = l(S)$ and $S = l(Z)$ is an essential left ideal). Since A satisfies the maximum condition on right annihilators, then (5) implies (1) by [**2**, Theorem 4.1].

In general, for an arbitrary ring A, a simple projective left A-module needs not be injective. However, we prove

PROPOSITION 3. *Let A be a left CM-ring. Then any simple projective left A-module is injective.*

Proof. Let W be a simple projective left A-module. Then $W \approx A/K$, where K is a maximal left ideal of A and since ${}_AA/K$ is projective, then $A = K \oplus U$, where $U = Ae$, $e = e^2 \in A$, is a minimal left ideal of A. If L is a proper essential left ideal of A, $f : L \to U$ a non-zero left A-homomorphism, then $L/N \approx U$, where

$N = \ker f$ is a maximal left subideal of L. Now $L = N \oplus V$, where $V(\approx U)$ ia a minimal left ideal of A. If $g : V \to U$ is an isomorphism, then $g(v) = e$ for some non-zero $v \in V$ and $V = Av$. Since $g(ev) = eg(v) = e^2 = e$, then $ev = v$. If $U \cap V \neq 0$, then $U = V$. If $U \cap V = 0$, let I be a complement left ideal such that $E = (U \oplus V) \oplus I$ is an essential left ideal of A. If $E \neq A$, since E is contained in some maximal (essential) left ideal and A is left CM, $v = ev \in U \cap V = 0$, which is a contradiction. Therefore $E = A$ and $V = Aw$, $w = w^2 \in A$ in any case. If M is a maximal left ideal containing L, by Zorn's Lemma, there exists a maximal left subideal C of M containing N with $C \cap V = 0$. Then C is a complement left subideal of M which implies $CM \subseteq C$, whence $NV \subseteq CM \cap V \subseteq C \cap V = 0$. Now for any $y \in L$, $y = d + aw$, $d \in N$, $a \in A$ and since $dw \in NV = 0$; $f(y) = f(aw) = f(dw) + f(aw) = (d + aw)\ f(w) = yf(w)$ which proves that ${}_AU$ is injective, whence A_W is injective.

Since a finitely generated p-injective left ideal of A is a direct summand of ${}_AA$, the next corollary then follows.

Corollary 3.1. *The following conditions are equivalent for any minimal left ideal U of a left CM ring A:* (a) *${}_AU$ is projective;* (b) *${}_AU$ is injective;* (c) *${}_AU$ is p-injective.*

Applying [**14**, Remark 5(2)], we get

Corollary 3.2. *If A is a MELT, left CM-ring, then a simple left A-module is injective iff it is p-injective.*

Corollary 3.3. *If A is left CM, left Noetherian, then ${}_AS$ is injective iff it is projective.*

Corollary 3.4. *If A is a left CM ring with ${}_AS$ finitely generated projective, then A is the ring direct sum of a semisimple Artinian ring and a ring with zero socle.*

Remark 1. A left CM-ring with projective left socle is left min-injective.

Left CM-rings lead us to consider the following class of rings: A is called a left CAM-ring if, for any maximal essential left ideal M of A (if it exists), for any left subideal I of M which is either a complement left subideal of M or a left annihilator ideal in A, I is an ideal of M.

Left CAM-rings generalize semi-simple Artinian, left duo, left PCI-rings and left Ore domains. Note that left CAM (left and right) V-rings need not be regular.

Proposition 4. *If A is a semi prime Ieft CAM-ring, then A is either semi-simple Artinian or reduced.*

Proof. Suppose there exists $0 \neq z \in Z$ such that $z^2 = 0$. Let M be a maximal left ideal of A containing $l(z)$. Then $l(z)M \subseteq l(z)$ implies $(Mz)^2 \leq (Az)Mz) \leq l(z)Mz = 0$, whence $M = l(z)$. Therefore $Az(\approx A/M)$ is a minimal left ideal which is a direct summand of ${}_AA$ contradicting the fact that Z contains no non-zero idempotent. By [**11**, Lemma 2.1], $Z = 0$. Suppose that A is not Artinian. Then there exists a maximal essential left ideal E of A. E is reduced by [**14**, Lemma

1.6(1)], which implies that A is reduced (being an essential extension of ${}_AE$). This proves the proposition.

The next corollary improves [**14**, Theorem 1.9].

COROLLARY 4.1. *If A is a semi prime right self-injective left CAM-ring, then A is either semi-simple Artinian or left self injective strongly regular.*

COROLLARY 4.2. *A semi prime left CAM-ring with maximum or minimum condition on left annihilators is a left and right Goldie ring.*

The next two corollaries give sufficient conditions for rings to be regular and self injective regular with non-zero socle.

Applying [**10**, Theorem 1] to Propositions 3 and 4, we get

COROLLARY 4.3. *The following conditions are eguivalent:*

(1) *A is either semi-simple Artinian or strongly regular with non-zero socle;*

(2) *A is a semi prime left CAM-ring containing a finitelly generated p-injective maximal left ideal;*

(3) *A is a semi-simple left CM-ring containing a finitely generated p-injective maximal left ideal.*

COROLLARY 4.4. *The following conditions are equivalent:*

(1) *A is either semi-simple Artinian or left and right self injective strongly regular with non-zero socle;*

(2) *A is a semi prime left CAM ring containing an injective maximal left ideal.*

Following [**13**], A is called a right WP-ring (weak p-injectvie) if every right ideal not isomorphic to A_A is p-injective. As usual, A is called left uniform iff every non-zero left ideal is an essential left ideal of A. Since a left uniform right semi-hereditary ring is a left Ore domain, then [**13**, Lemma 1.1] yields the next remark.

Remark 2. The following conditions are equivalent:

(1) A is either simple Artinian or a left Ore right principal ideal domain;

(2) A is a prime left CM, right WP-ring.

Remark 3. If A is a left CAM, right WP-ring, then A is either semi-simple Artinian or strongly regular or a right principal ideal domain. (cf. [**13**, Corollary 1.6].)

Remark 4. If A is a left CAM-ring whose essential left ideals are idempotent, then A is fully left and right idempotent.

We now consider another generalization of left p-injective rings: Call A a left np-injective ring if, for any non-nilpotent element c of A, any left A-homomorphism $g : Ac \to A$, there exists $b \in A$ such $g(ac) = acb$ for all $a \in A$. Following [**5**], an element a of A is called left regular iff $l(a) = 0$. [**10**, Theorem 1] is improved in the next proposition.

Proposition 5. *Let A be a left np-injective ring. Then*

(1) *Any left regular element of A is right invertible;*

(2) $Z \subseteq J$;

(3) *Every left or right A-module is divisible;*

(4) *If P is a reduced principal left ideal of A, then $P = Ae$, where $e = e^2 \in A$ and $A(1-e)$ is an ideal of A.*

Proof. (1) Let $c \in A$ such that $l(c) = 0$. For any $u \in A = r(l(c))$, $l(c) = l(r(l(c))) \subseteq l(u)$ and if $g : Ac \to A$ is the left A-homomorphism defined by $g(ac) = au$ for all $a \in A$, since A is left np-injective, there exists $b \in A$ such that $u = g(c) = cb \in cA$. Therefore $A = cA$ which proves (1).

(2) If $z \in Z$, $a \in A$, then $l(1 - za) = 0$ implies $(1 - za)u = 1$ for some $u \in A$ by (1). This proves that $z \in J$.

(3) If c is a non-zero-divisor in A, then $cd = 1$ for some $d \in A$ by (1). Now $r(c) = 0$ implies $dc = 1$ and for any right A-module M, $M = Mdc \subseteq Mc \subseteq M$ implies $M = Mc$. Similarly, any left A-module is divisible.

(4) Let $P = Ab$, $b \in A$, be a non-zero reduced principal left ideal. Then $r(b) \subseteq l(b) = l(b^2)$ and the proof of (1) shows that $r(l(bA)) = bA$ which yields $bA = r(l(b^2)) = r(l(b^2A)) = b^2A$ (since b^2 is non-nilpotent). Therefore $b = b^2c$, $c \in A$, which implies $b = bcb$ (P being reduced), whence P is generated by the idempotent $e = cb$. Also, for any $a \in A$, $(ae - eae)^2 = 0$ implies $ae = eae$, whence $(1-e)Ae = 0$. Therefore $(1-e)A \subseteq A(1-e)$ which establishes the last part of (4).

Remark 5. [**1**, Theorem 12] holds for right *np*-injective rings whose complement right ideals are finitely generated.

We now characterize division rings in terms of the following: A is called a right F-ring if, for any maximal right ideal M of A, any $b \in M$, A/bM_A is flat.

Theorem 6. *The following conditions are equivalent for a semi prime left uniform ring A:*

(1) *A is a division ring;*

(2) *A is a left self injective Ieft F ring;*

(3) *A is a left np-injective left F ring;*

(4) *A is a right F ring.*

Proof. It is evident that (1) implies (2), which, in turn, implies (3).

Assume (3). If $b \in A$, $b \notin Z$, then $l(b) = 0$ which implies $bc = 1$ for some $c \in A$ (Proposition 5(1)). This shows that every maximal right ideal of A is contained in Z, whence A is a local ring with $Z = J$ as the unique maximal right ideal (which is also the only maxi.mal left ideal). Suppose that $Z \neq 0$. Since A is semi-prime, for any $0 \neq z \in J$, $Jz \neq 0$. If $y \in J$ such that $yz \neq 0$, since ${}_AA/Jz$ is flat, $yz = yzwz$ for some $w \in J$. Since $(1 - wz)u = 1$ for some $u \in A$, then $yz(1 - wz) = 0$ implies $yz = 0$, a contradiction. Thus $Z = J = 0$ which proves that A is a division ring and (3) implies (4).

Assume (4). If $u \in A$, $u \neq Z$, then $l(u) = 0$. Suppose that $uA \neq A$. If R is a maximal right ideal containing uA, since A/uR_A is flat, $u^2 = uvu^2$ for some $v \in R$. Then $(1 - uv)uz = 0$ implies $uv = 1$, which contradicts $uA \neq A$. This proves that A is a local ring with $Z = J$ the only maximal right (and left) ideal of A. The proof of "(3) implies (4)" then shows that $J = 0$ which proves that (4) implies (1).

COROLLARY 6.1. *A is simple Artinian iff A is a prime left CM, right F ring.*

Remark 6. A left Noetherian ring is left Artinian iff each of its prime factor rings is a left CM, right F-ring.

[**7**, Corollary 1.18], [**10**, Theorem 1], [**11**, Theorem 1.4], Propositions 4 and 5 yield the next result. (cf. [**14**, Theorem 2.2 and Proposition 2.4].)

THEOREM 7. *The following conditions are equivalent:*

(1) *A is either semi-simple Artinian or strongly regular;*

(2) *A is a semi prime left CAM-ring whose simple right modules are flat;*

(3) *A is a semi-prime left np-injective, left CAM ring.*

Note that a ring with finitely generated projective essential left socle need not be semi-prime. We conclude with a few characteristic properties of semi-simple Artinian rings.

THEOREM 8. it The following conditions are equivalent:

(1) *A is semi-simple Artinian;*

(2) *A is a semi-prime TLA, left CM-ring containing a finitely generated p-injective maximal left ideal;*

(3) *A is a left CM, left Noetherian ring with projective essential left socle;*

(4) *A is a left CM-ring with finitely generated projective essential left socle;*

(5) *A is a semi prime left np-injective, left or right Goldie ring.*

Proof. Apply Propositions 3 and 4, Corollary 4.4 and Proposition 5.

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