

## INTEGRABILITY THEOREMS FOR TRIGONOMETRIC SERIES WITH POSITIVE COEFFICIENTS

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### 1. Introduction and results

Let  $g$  be an odd function and let  $f$  be an even function defined on  $(0, \pi)$ , periodic with period  $2\pi$ , and let their Fourier series be

$$g(x) = \sum_1^{\infty} b_n \sin nx \quad f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx$$

with  $a_n \geq 0$ ,  $b_n \geq 0$ . In his book [3] Boas proved the following two theorems concerning integrability of  $g$  (similar statements hold for  $f$ ).

THEOREM A. Let  $0 < \gamma < 1$ . Then

$$\int_{\rightarrow 0}^{\pi} x^{-\gamma} g(x) dx < \infty \Leftrightarrow \sum_1^{\infty} n^{\gamma-1} b_n < \infty$$

( where  $\int_{\rightarrow 0}^{\pi} = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\pi}$  is the integral in Cauchy's sense ).

THEOREM B. Let  $1 < \gamma < 2$ . Then

$$x^{-\gamma} g(x) \in L(0, \pi) \Leftrightarrow \sum_1^{\infty} n^{\gamma-1} b_n < \infty.$$

Izumi and Izumi [4] have proved a generalization of Theorem A in which the function  $x^{-\gamma}$  is replaced by a monotone decreasing function  $\xi(x)$ , and Hasegawa [5] has proved a generalization of Theorem B in which  $x^{-\gamma}$  is replaced by a function  $\alpha(x)$  having the following properties:  $x\alpha(x)$  is decreasing and  $t^{-1} \int_t^{\eta} \alpha(x) dx \leq C\alpha(t)$ , for some  $\eta$ ,  $0 < \eta \leq \pi$  and all  $t$ ,  $0 < t \leq \eta$ .

In the present paper we prove another generalization of Theorem B (Theorem 1). We remark that no monotony condition is required for the multiplier function  $K$ . This is a consequence of the following observations: different properties of the function  $x^{-\gamma}$  are essential for Theorems A and B. While the monotony of  $x^{-\gamma}$  plays a central role in the proof of Theorem A, in Theorem B no use is made of monotony, but only of the regular variation of  $x^{-\gamma}$ . Indeed, the main step in the proof of both theorems is the estimation of the integral  $\int_0^{\pi} x^{-\gamma} \sin nx dx = n^{\gamma-1} \int_0^{n\pi} x^{-\gamma} \sin x dx$ . For  $1 < \gamma < 2$  this integral is absolutely convergent, and this is a consequence of the regular variation properties of  $x^{-\gamma}$  only (cf. [1]). On the other hand, for  $0 < \gamma < 1$ , the integral converges nonabsolutely and the monotony of  $x^{-\gamma}$  is essential for this statement.

For the definition and properties of 0-regularly varying ( $0 - RV$ ) functions (in the sense of Karamata) we refer to [2]. The symbol  $\mathcal{K}(\underline{\rho}, \bar{\rho})$  denotes the class of all  $0 - RV$  functions with lower index  $\underline{\rho}$  and upper index  $\bar{\rho}$ . The symbol  $\asymp$  is defined by:  $f(x) \asymp g(x)$  on  $[a, \infty)$  if there are two positive constants  $C_1, C_2$  such that  $0 < C_1 g(x) \leq f(x) \leq C_2 g(x) < \infty$ . By the letter  $C$ , possibly with subscripts, we denote a positive constant, not necessarily the same at each appearance.

**THEOREM 1.** *Let  $b_n \geq 0$  and let  $g$  be defined by  $g(x) = \sum_1^{\infty} b_n \sin nx$ . Let  $K$  be a positive function defined on  $(1/\pi, \infty)$  such that  $xK\left(\frac{1}{x}\right) \in L(0, \pi)$  and*

$$(1) \quad \int_{1/\pi}^u t^{-2} K(t) dt \asymp u^{-1} K(u), \quad \text{for } u > \frac{1}{u}.$$

Then

$$(a) \quad \sum_1^{\infty} n b_n \int_0^{1/n} x K\left(\frac{1}{x}\right) dx < \infty \Rightarrow g(x) K\left(\frac{1}{x}\right) \in L(0, \pi)$$

and conversely

$$(b) \quad g(x) K\left(\frac{1}{x}\right) \in L(0, \pi) \Rightarrow \sum_1^{\infty} b_n \frac{K(n)}{n} < \infty.$$

**THEOREM 2.** *Let  $a_n \geq 0$  and let  $f$  be defined by  $f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx$ .*

*Let  $K$  be a positive function defined on  $(1/\pi, \infty)$  such that  $x^2 K\left(\frac{1}{x}\right) \in L(0, \pi)$  and condition (1) is satisfied. Then*

$$(a) \quad \sum_1^{\infty} n^2 a_n \int_0^{1/n} x^2 K\left(\frac{1}{x}\right) dx < \infty \Rightarrow (f(0) - f(x)) K\left(\frac{1}{x}\right) \in L(0, \pi)$$

and conversely

$$(b) \quad (f(0) - f(x))K\left(\frac{1}{x}\right) \in L(0, \pi) \Rightarrow \sum_1^{\infty} a_n \frac{K(n)}{n} < \infty.$$

COROLLARY 1. Let  $b_n \geq 0$  and let  $g(x) = \sum_1^{\infty} b_n \sin nx$ . Let  $K \in \mathcal{K}(1 < \underline{\rho}, \bar{\rho} < 2)$ . Then

$$g(x)K\left(\frac{1}{x}\right) \in L(0, \pi) \Leftrightarrow \sum_1^{\infty} b_n \frac{K(n)}{n} < \infty.$$

COROLLARY 2. Let  $a_n \geq 0$  and let  $f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx$ . Let  $K \in \mathcal{K}(1 \leq \underline{\rho}, \bar{\rho} < 3)$ . Then

$$(f(0) - f(x))K\left(\frac{1}{x}\right) \in L(0, \pi) \Leftrightarrow \sum_1^{\infty} a_n \frac{K(n)}{n} < \infty.$$

## 2. Some properties of 0-RV functions

Let us remark that condition (1) is a characteristic property for 0-regularly varying functions ([2]), i.e. a function  $K$  satisfying (1) is an 0-RV function with lower index  $\underline{\rho} > 1$ :  $K \in \mathcal{K}(1 < \underline{\rho}, \bar{\rho})$ . In the following lemma we list some properties of 0-RV functions which follow from (1).

LEMMA. Let  $K$  be a positive function defined on  $(1/\pi, \infty)$  such that (1) holds. Then

1°  $u^{-1}K(u)$  is almost increasing ( $u^{-1}K(u) \nearrow \nearrow$ ), i.e. there is a constant  $C \geq 0$  such that  $u^{-1}K(u) \leq Cv^{-1}$ , for  $1/\pi < u < v$ .

2° There is a  $\tau < 1$  such that  $u^{-\tau}K(u)$  is almost decreasing ( $u^{-\tau}K(u) \searrow \searrow$ ), i.e. there is a constant  $C \geq 0$  such that  $u^{-\tau}K(u) \geq Cv^{-\tau}K(v)$ , for  $1/\pi < u < v$ .

3° If  $K\left(\frac{1}{x}\right)g(x) \in L(0, \pi)$ , then  $x^{-1}g(x) \in L(0, \pi)$ .

4° Let  $F(x) = \int_0^x |f(t)| dt$ . Then

$$\int_0^1 K\left(\frac{1}{x}\right)F(x) dx \leq C \int_0^1 K\left(\frac{1}{x}\right)x|f(x)| dx$$

for some positive constant  $C$ .

5° If  $c_n \downarrow 0$ , then the series  $\sum_1^{\infty} c_n \frac{K(n)}{n}$  and  $\sum_1^{\infty} (c_n - c_{n+1}) \frac{K(n)}{n}$  are equiconvergent.

*Proof.*

1° and 2° cf. [2].

3° It is easily seen that

$$\begin{aligned} \int_0^\pi x^{-1}|g(x)| dx &= \int_0^\pi |g(x)|K\left(\frac{1}{x}\right)\left(xK\left(\frac{1}{x}\right)\right)^{-1} dx \\ &\leq \sup_{0 < x < \pi} \left(xK\left(\frac{1}{x}\right)\right)^{-1} \int_0^\pi |g(x)|K\left(\frac{1}{x}\right) dx = C \sup_{1/\pi < 1 < \infty} (t^{-1}K(t))^{-1} \\ &= C(\inf_{1/\pi < t < \infty} (t^{-1}K(t))^{-1}) = C_1 \left(\pi K\left(\frac{1}{\pi}\right)\right)^{-1}, \end{aligned}$$

since  $t^{-1}K(t)$  is almost increasing, by 1°.

4° We by changing the order of integration and by making use of (1)

$$\begin{aligned} \int_0^1 K\left(\frac{1}{x}\right)F(x)dx &= \int_0^1 K\left(\frac{1}{x}\right) \int_0^x |f(t)|dt dx = \int_0^1 |f(t)| \int_t^1 K\left(\frac{1}{x}\right) dx dt = \\ &= \int_0^1 |f(t)| \int_1^{1/t} K(u)u^{-2} du dt \leq C \int_0^1 |f(t)| \left(\frac{1}{t}\right)^{-1} K\left(\frac{1}{t}\right) dt = C \int_0^1 |f(t)|tK\left(\frac{1}{t}\right) dt. \end{aligned}$$

5° this is Lemma 2 of [1].

### 3. Proof of the Theorems

We prove Theorem 1 only, the proof of Theorem 2 being very similar.

a) First, by the definition of  $g$ , we have

$$\begin{aligned} (2) \quad \int_0^\pi K\left(\frac{1}{x}\right)|g(x)| dx &= \int_0^\pi K\left(\frac{1}{x}\right) \left| \sum_1^\infty b_n \sin nx \right| dx \leq \int_0^\pi K\left(\frac{1}{x}\right) \sum_1^\infty |b_n| |\sin nx| dx \\ &= \sum_1^\infty b_n \int_0^\pi K\left(\frac{1}{x}\right) |\sin nx| dx. \end{aligned}$$

Next we prove that

$$(3) \quad \int_0^\pi K\left(\frac{1}{x}\right) |\sin nx| dx \leq Cn \int_0^{1/n} xK\left(\frac{1}{x}\right) dx.$$

Really,

$$\int_0^{\pi} K\left(\frac{1}{x}\right) |\sin nx| dx = \int_0^{1/n} + \int_{1/n}^{\pi} = I_1 + I_2.$$

Now for the first integral we have obviously

$$(4) \quad I_1 = \int_0^{1/n} K\left(\frac{1}{x}\right) |\sin nx| dx \leq n \int_0^{1/n} xK\left(\frac{1}{x}\right) dx$$

and for the second

$$(5) \quad I_2 = \int_{1/n}^{\pi} K\left(\frac{1}{x}\right) |\sin nx| dx \leq \int_{1/n}^{\pi} K\left(\frac{1}{x}\right) dx = \int_{1/n}^n K(t)t^{-2} dt \leq Cn^{-1}K(n),$$

since  $K$  satisfies (1).

Now, by Lemma 1°, the function  $u^{-1}K(u)$  is almost increasing, thus

$$(6) \quad n \int_0^{1/n} xK\left(\frac{1}{x}\right) dx = n \int_n^{\infty} t^{-1}K(t)t^{-2} dt \geq Cn \cdot n^{-1}K(n) \int_n^{\infty} t^{-2} dt = \\ = CK(n) \cdot n^{-1}.$$

From (5) and (6) it follows that

$$I_2 \leq Cn \int_0^{1/n} xK\left(\frac{1}{x}\right) dx$$

which, together with (4), proves (3).

Finally, by substituting (3) into (2), we obtain

$$\int_0^{\pi} K\left(\frac{1}{x}\right) |g(x)| dx \leq C \sum_0^{\infty} |b_n| n \int_0^{1/n} xK\left(\frac{1}{x}\right) dx$$

which proves part a) of the Theorem.

b) Assume that  $g(x)K\left(\frac{1}{x}\right) \in L(0, \pi)$ . By Lemma 3° it follows that  $x^{-1}g(x) \in L(0, \pi)$ ; hence by Theorem 4.1. [3] the series  $\sum_1^{\infty} b_n$  is convergent.

Denote  $B_k = \sum_{j=0}^{\infty} b_{2j+k+1}$ . Then  $B_k$  decreases to zero and we obtain by partial summation (cf. Lemma 2.2. [3])

$$g(x) = \sum_1^{\infty} b_n \sin nx = B_0 \sin x + 2 \sum_1^{\infty} B_n \cos nx \sin x$$

or

$$h(x) = \frac{g(x)}{\sin x} = B_0 + 2 \sum_1^{\infty} B_n \cos nx.$$

Denote  $\varphi(x) = h(x) - B_0 = 2 \sum_1^{\infty} B_n \cos nx$ . It is easily seen that

$$(7) \quad \varphi(x)xK\left(\frac{1}{x}\right) \in L(0, \pi).$$

Indeed,

$$\begin{aligned} \int_0^{\pi} |\varphi(x)xK\left(\frac{1}{x}\right)| dx &\leq \int_0^{\pi} |h(x)xK\left(\frac{1}{x}\right)| dx + B_0 \int_0^{\pi} xK\left(\frac{1}{x}\right) dx = \\ &= \int_0^{\pi} |g(x)| \frac{x}{\sin x} K\left(\frac{1}{x}\right) dx + B_0 \int_0^{\pi} xK\left(\frac{1}{x}\right) dx \end{aligned}$$

and both integrals converge by the assumptions of the theorem. Since  $\varphi$  is integrable and  $B_n$  monotone, it follows that  $B_n$  are the Fourier coefficients of  $\varphi$  and we can put

$$\begin{aligned} \Phi(x) &= \int_0^x \varphi(t) dt = 2 \sum_1^{\infty} \frac{B_k}{k} \sin kx \\ F(x) &= \int_0^x \Phi(t) dt = 2 \sum_1^{\infty} \frac{B_k}{k^2} (1 - \cos kx) = 4 \sum_1^{\infty} \frac{B_k}{k^2} \sin^2 \frac{kx}{2}. \end{aligned}$$

Next we prove that

$$(8) \quad F\left(\frac{1}{n+1}\right) \geq C \frac{B_n}{n}$$

Indeed,

$$\begin{aligned} F\left(\frac{1}{n+1}\right) &= 4 \sum_{k=1}^{\infty} \frac{B_k}{k^2} \sin^2 \frac{k}{2(n+1)} \geq 4 \sum_{k=1}^{\infty} \frac{B_k}{k^2} \sin^2 \frac{k}{2(n+1)} \\ 4 \left(\frac{2}{\pi}\right)^2 \sum_{k=1}^n \left(\frac{k}{2(n+1)}\right)^2 &= \frac{4}{\pi^2} \frac{1}{(n+1)^2} \sum_{k=1}^n B_k \geq \frac{4}{\pi^2} \frac{1}{n} B_n \cdot n = \frac{4}{\pi^2} \frac{B_n}{n} \end{aligned}$$

where we have used that  $\sin \frac{k}{2(n+1)} \geq \frac{2}{\pi} \frac{k}{2(n+1)}$ , for  $\frac{k}{2(n+1)} < \frac{1}{2}$  and that  $B_n$  is a monotone decreasing sequence.

Next, put  $\Phi^+(x) = \int_0^x |\varphi(t)| dt$  and  $F^+(x) = \int_0^x |\Phi(t)| dt$ . The functions  $\Phi^+$  and  $F^+$  are positive and increasing and

$$(9) \quad |F(x)| \leq F^+(x) \leq \int_0^x \Phi^+(t) dt \leq x\Phi^+(x).$$

Therefore, by (8) and (9) it follows that

$$(10) \quad \frac{B_n}{n} \leq C \frac{1}{n+1} \Phi^+ \left( \frac{1}{n+1} \right).$$

Now we are ready to prove

$$(11) \quad \sum_1^{\infty} \frac{B_n K(n)}{n^2} \leq \int_0^1 |\varphi(x)| x K \left( \frac{1}{x} \right) dx$$

The proof runs as follows

$$\begin{aligned} \sum_1^{\infty} \frac{B_n}{n} \frac{K(n)}{n} &\leq C \sum_1^{\infty} \frac{1}{n+1} \Phi^+ \left( \frac{1}{n+1} \right) \frac{K(n)}{n} \quad [\text{by (10)}] \\ &\leq C \sum_1^{\infty} \frac{1}{n+1} \Phi^+ \left( \frac{1}{n+1} \right) \int_n^{n+1} \frac{K(t)}{t} dt \quad [\text{since } u^{-1}K(u) \nearrow] \\ &= C \sum_1^{\infty} \frac{1}{n+1} \Phi^+ \left( \frac{1}{n+1} \right) \int_{1/(n+1)}^{1/n} x K \left( \frac{1}{x} \right) x^{-2} dx \\ &\leq C \sum_1^{\infty} \int_{1/(n+1)}^{1/n} x \Phi^+(x) K \left( \frac{1}{x} \right) x^{-1} dx \quad [\text{since } x \Phi^+(x) \uparrow] \\ &= C \int_0^1 K \left( \frac{1}{x} \right) \Phi^+(x) dx \end{aligned}$$

Applying Lemma 4° to the last integral we obtain (11). Thus, by (7) it follows that the series  $\sum_1^{\infty} B_n \frac{K(n)}{n^2}$  is convergent. To complete the proof of the theorem we have

to prove that this implies that the series  $\sum_1^{\infty} b_n \frac{K(n)}{n}$  is convergent. First, by Lemma 5° it follows that

$$(12) \quad \sum_1^{\infty} (B_n - B_{n+1}) \frac{K(n)}{n} < \infty$$

and since  $u^{-1}K(u)$  is almost increasing

$$\begin{aligned} \sum_1^{\infty} (B_{n+1} - B_{n+2}) \frac{K(n)}{n} &\leq C \sum_1^{\infty} (B_{n+1} - B_{n+2}) \frac{K(n+1)}{n+1} \\ (13) \quad &= C \sum_2^{\infty} (B_n - B_{n+1}) \frac{K(n)}{n}. \end{aligned}$$

Thus from (12) and (13) the following series

$$(14) \quad \sum_1^{\infty} (B_n - B_{n+2}) \frac{K(n)}{n} = \sum_1^{\infty} b_{n+1} \frac{K(n)}{n}$$

is also convergent. Now, since by Lemma 2° there is a  $\tau > 1$  such that  $u^{-\tau}K(u)$  is almost decreasing, we have

$$\frac{K(n+1)}{n+1} = (n+1)^{-\tau} K(n+1)(n+1)^{\tau-1} \leq Cn^{-\tau} K(n) C_1 n^{\tau-1} = C_2 n^{-1} K(n).$$

Whence

$$\sum_1^{\infty} b_{n+1} \frac{K(n+1)}{n+1} \leq C_2 \sum_1^{\infty} b_{n+1} \frac{K(n)}{n}$$

and the last series is convergent, by (14). Thus we have proved that  $\sum_1^{\infty} b_n \frac{K(n)}{n}$  is convergent, which completes the proof of the theorem.

*Proof of Corollary 1.*

Let  $K \in \mathcal{K}(1 < \underline{\varrho}, \bar{\varrho} < 2)$ . The assumption  $\underline{\varrho} > 1$  implies condition (1) of Theorem 1. On the other hand, from the assumption  $\bar{\varrho} < 2$  it follows that

$$\int_u^{\infty} t^{-3} K(t) dt \asymp u^{-2} K(u).$$

Thus

$$n \int_0^{1/n} x K\left(\frac{1}{x}\right) dx = n \int_n^{\infty} t^{-1} K(t) t^{-2} dt \asymp n n^{-2} K(n) = n^{-1} K(n)$$

which means that the series

$$\sum_1^{\infty} n b_n \int_0^{1/n} x K\left(\frac{1}{x}\right) dx \quad \text{and} \quad \sum_1^{\infty} b_n \frac{K(n)}{n}$$

are equiconvergent. Now Corollary 1 follows from Theorem 1.

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