

**CERTAIN CONVEXITY THEOREMS FOR UNIVALENT
ANALYTIC FUNCTIONS**

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1. Let m and M be arbitrary fixed real number which satisfy the relations $(m, M) \in R$ where $R = \{(m, M) \mid m > \frac{1}{2}, (m-1) < M < m\}$. Also, let P denote the class of functions $F(z) = 1 + c_0z + c_1z^2 + \dots$ which are regular and satisfy $\Re\{F(z)\} > \alpha$; $0 \leq \alpha < 1$ and $|F(z) - m| < M$. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in the unit disc $D = \{z \mid |z| < 1\}$ and write $G(z) = zf'(z)/f(z)$ and $H(z) = 1 + zf'(z)/f(z)$. Then we denote the class of functions $G(z)$ for which $G(z) \in P$ by $S(m, M)$ while the class of functions $H(z) \in P$ by $K(m, M)$. We further assume that

$$a = \frac{M^2 - m^2 + m}{M} \quad \text{and} \quad b = \frac{m-1}{M}.$$

Then, it follows that

$$f \in S(m, M) \Leftrightarrow \frac{zf'(z)}{f(z)} = \frac{1 + aw(z)}{1 - bw(z)}$$

where $w(z)$ is regular in D and satisfy $w(0) = 0$, $|w(z)| < 1$. Similarly,

$$f \in K(m, M) \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + aW(z)}{1 - bW(z)}$$

where $W(0) = 0$, $|W(z)| < 1$ and is regular in D . If we write $a = \{\alpha - 2N\alpha + N\}/N$ and $b = (N-1)/N$ and make $N \rightarrow \infty$ then, it is equivalent to say that $b \rightarrow 1$ and $a \rightarrow 1 - 2\alpha$. In this case define

$$S^*(\alpha) = \lim_{\substack{a \rightarrow 1-2\alpha \\ b \rightarrow 1}} S(m, M); \quad 0 \leq \alpha < 1$$

and

$$K(\alpha) = \lim_{\substack{a \rightarrow 1-2\alpha \\ b \rightarrow 1}} K(m, M); \quad 0 \leq \alpha < 1.$$

The functions in $S^*(\alpha)$ and $K(\alpha)$ are usual functions of starlike univalent functions of order α and convex functions of order α .

Also, if we let S_0 be the class of regular functions $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$ in $D_0 = \{z \mid |0 < |z| < 1\}$ and Q denote the class of functions F regular and satisfy $|F(z) + m| < M$, then define the class of functions:

$$\Gamma(m, M) = \left\{ f \mid f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \in S_0 \text{ and } z f'(z)/f(z) \in Q \right\}$$

and

$$\Sigma(m, M) = \left\{ f \mid f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \in S_0 \text{ and } 1 + z f'(z)/f(z) \in Q \right\}.$$

As before, we have

$$f \in \Gamma(m, M) \Leftrightarrow \frac{z f'(z)}{f(z)} = -\frac{1 + a w_1(z)}{1 - b w_1(z)}$$

and

$$f \in \Sigma(m, M) \Leftrightarrow 1 + \frac{z f''(z)}{f'(z)} = -\frac{1 + a w_2(z)}{1 - b w_2(z)}$$

where w_i ; $i = 1, 2$ are regular in D and satisfy $w_i(0) = 0$, $|w_i(z)| < 1$. Also it follows that

$$\Gamma^*(\alpha) = \lim_{\substack{a \rightarrow 1-2\alpha \\ b \rightarrow 0}} \Gamma(m, M), \quad 0 \leq \alpha < 1,$$

and

$$\Sigma(\alpha) = \lim_{\substack{a \rightarrow 1-2\alpha \\ b \rightarrow 0}} \Sigma(m, M); \quad 0 \leq \alpha < 1.$$

Then $\Gamma^*(\alpha)$ and $\Sigma(\alpha)$ denote the usual class of starlike and convex functions in D_0 . In this paper, we shall prove the following theorems which in particular include the results proved in [1-3] or else obtained as a limit as $a \rightarrow 1 - 2\alpha$ and $b \rightarrow 0$.

2. We have:

THEOREM 1: *Let $f \in S(m, M)$ and*

$$(2.1) \quad F(z) = \left(\frac{c+1}{z^c} \right) \int_0^z t^{c-1} f(t) dt, \quad c > -\frac{1-a}{1+b}$$

where a, b are defined by

$$(2.2) \quad a = \frac{M^2 - m^2 + m}{M} \text{ and } b = \frac{m-1}{b}; \quad (m, M) \in \mathbb{R}$$

and $r(a, b)$ be the unique positive root of the equation

$$(2.3) \quad \begin{aligned} & (a + 2b + d) - 2(ad + bd + b + d)r - \{2(b^2 - d^2) + (a + d) + 2b(1 - d^2) \\ & - d(ad + b^2)\}r^2 - 2d\{(a + b) + b(b + d)r^3\} \\ & - d(ad + 2bd + b^2)r^4 = 0. \end{aligned}$$

then, $f(z)$ is starlike of order β for $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$(2.4) \quad (1 - \beta) - \{\beta(b - d) + a + b + 2d\}r + d(a + b\beta)r^2 = 0$$

if $r_0 \leq r(a - b)$, otherwise r_0 is the smallest positive root of the equation

$$(2.5) \quad \begin{aligned} 0 = & \sqrt{(1 - d)\{(1 - d) + (1 + d)x\}\{(1 + 2a + 4b + b^2) + (1 - b^2)x\} +} \\ & + (E - 1 + bd) - (1 + bd)x \end{aligned}$$

where

$$(2.6) \quad x = \frac{1 + r^2}{1 - r^2}, \quad E = -\beta(b + d) + 2d - (a + b) \quad \text{and} \quad d = \frac{a - bc}{c + 1}.$$

This result is sharp.

PROOF. Since $F \in S(m, M)$ there exists a regular function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ and

$$(2.7) \quad \frac{zF'(z)}{F(z)} = \frac{1 + aw(z)}{1 + bw(z)}.$$

From (2.7) and (2.1) we get

$$(2.8) \quad \frac{f(z)}{F(z)} = \frac{1 + \frac{a-bc}{c+1}w(z)}{1 - bw(z)} = \frac{1 + dw(z)}{1 - bw(z)}.$$

Differentiating (2.8) logarithmically with respect to z and using (2.7), we get,

$$(2.9) \quad \begin{aligned} & \Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq -\beta + \Re\left\{\frac{1 + aw(z)}{1 - bw(z)}\right\} + \\ & + (b + d)\Re\left\{\frac{w(z)}{(1 - bw(z))(1 + dw(z))}\right\} - \frac{(b + d)(r^2 - |w(z)|^2)}{(1 - r^2)|1 - bw(z)||1 + dw(z)|}. \end{aligned}$$

Here we have used the well known inequality

$$|zw'(z) - w(z)| \leq \frac{r^2 - |w(z)|^2}{1 - r^2}.$$

If we take

$$(2.10) \quad p(z) = \frac{1 + dw(z)}{1 - bw(z)}$$

then

$$(2.11) \quad |p(z) - A| \leq B$$

where

$$(2.12) \quad A = \frac{1 + bdr^2}{1 - b^2r^2}$$

and

$$(2.14) \quad B = \frac{(b + d)r}{1 - b^2r^2}.$$

Substituting value of $w(z)$ from (2.10) in (2.9) we get

$$(2.14) \quad \Re \left\{ \frac{zf'(z)}{f(z)} - \beta \right\} \geq \frac{1}{b + d} \left[E - d \Re \left\{ \frac{1}{p(z)} \right\} + (a - 2b) \Re \{p(z)\} - \frac{r^2 |bp(z) + d|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|} \right].$$

If we take $p(z) = A + u + iv$, $|p(z)| = R$ and use (2.12) and (2.13) in (2.14) we get

$$(2.15) \quad \Re \left\{ \frac{zf'(z)}{f(z)} - \beta \right\} \geq \frac{1}{b + d} \left[E - \frac{d(A + u)}{R^2} + (a + 2b)(A + u) - \frac{B^2 - u^2 - v^2}{R} \left(\frac{1 - b^2r^2}{1 - r^2} \right) \right] \equiv \frac{1}{b + d} \cdot P(u, v).$$

Differentiating $P(u, v)$ partially with respect to v we get

$$(2.16) \quad \frac{\delta P(u, v)}{\delta v} = \frac{v}{R} \left[\frac{2d(A + u)}{R^3} + \left\{ 2 + \frac{B^2 - u^2 - v^2}{R} \right\} \left(\frac{1 - b^2r^2}{1 - r^2} \right) \right].$$

If $d \geq 0$, quantity in the square brackets is positive. If $d < 0$ we see that

$$\frac{1 - b^2r^2}{1 - r^2} + \frac{d(A + u)}{R^3} \geq 1 + \frac{d(1 + br)^2}{(1 - dr)^2} \geq 0$$

and therefore the quantity in the square brackets in (2.16) is positive.

So $\frac{\delta P(u, v)}{\delta v} \geq 0$ if $v \geq 0$ and $\frac{\delta P(u, v)}{\delta v} < 0$ if $v < 0$ therefore

$$(2.17) \quad \begin{aligned} \min_v P(u, v) &= P(u, 0) = \\ &= E - \frac{d}{R} + (a + 2b)R - \frac{B^2 - (R - A)^2}{R} \left(\frac{1 - b^2 r^2}{1 - r^2} \right) \equiv P(R) \end{aligned}$$

where $R = A + u$.

$P'(R)$ is an increasing function of R and $P'(R_0) = 0$ where

$$(2.18) \quad R_0 = \left[\frac{(1 - d)(1 + dr^2)}{(a + 2b + 1) - (a + 2b + b^2)r^2} \right]^{1/2}.$$

Again we see that $P'(A + B) \geq 0$ therefore $R_0 \leq A + B$. Since $P'(R)$ is increasing function of R and $A - B \leq R \leq A + B$ we have

$$(2.19) \quad \min_R P(R) = \begin{cases} P(A - B) & \text{if } 0 \leq R_0 \leq A - B \\ P(R_0) & \text{if } A - B \leq R_0 \leq A + B. \end{cases}$$

$$= \begin{cases} \frac{(b + d)[(1 - \beta) - \{\beta(b - d) + a + b + 2d\}r + d(a + b\beta)r^2]}{(1 - dr)(1 + br)} & \text{if } R_0 \leq A - B \\ (E - 1 + bd) - (1 + bd)x + \\ + \sqrt{(1 - d)\{(1 - d) + (1 + d)x\}\{(1 + 2a + 4b + b^2) + (1 - b^2)x\}} & \text{if } R_0 \geq A - B \end{cases}$$

where $x = \frac{1 + r^2}{1 - r^2}$.

Let us take

$$(2.20) \quad Q(r) = (A - B)^2 - R_0^2 = \left(\frac{1 - dr}{1 + br} \right)^2 - \frac{(1 - d)(1 + dr^2)}{(a + 2b + 1) - (a + 2b + b^2)r^2}.$$

Therefore $Q(r)$ is a decreasing function of r and

$$Q(0) = \frac{(a + b) + (b + d)}{(a + b) + (1 + b)} \geq 0 \quad \text{and} \quad Q(1) = -\frac{2(1 - d)(b + d)}{(1 + b)(1 - b^2)} \leq 0.$$

Therefore $Q(r)$ has unique root in $(0, 1)$.

Let it be $r(a, b)$. Hence if $r \leq r(a, b)$, $Q(r) \geq 0$ i.e. $A - B \geq R_0$ and if $r \geq r(a, b)$, $Q(r) \leq 0$ i.e. $A - B \leq R_0$. So from (2.19) and (2.20) the result follows.

The equality in (2.4) is attained for the function $F(z) = z(1 - bz)^{-\frac{a+b}{b}}$ and that in (2.5) for the function

$$F(z) = z(1 - 2kbz + b^2 z^2)^{-\frac{a+b}{2b}}$$

where k is given by

$$\frac{1 + k(a-b)r - br^2}{1 - 2kbr + b^2r^2} = \left\{ \frac{(1-d)(1+dr^2)}{(a+2b+1) - (a+2b+b^2)r^2} \right\}^{1/2}.$$

Similarly by using the method of theorem 1 following theorems follow.

THEOREM 2. *If $f(z)$ is regular in D and satisfy*

$$F(z) = \left(\frac{c+2}{z^{c+1}} \right) \int_0^z t^{c-1} f(t)g(t)dt, \quad c \geq 0$$

where $F \in S^*(\beta)$ and $g \in S(m, M)$ then $f(z)$ is univalent and starlike of order β in $|z| < r_0$ where r_0 is the smallest positive root of the equation

$$(1-\beta)(c+2) - \{(c+2)(a+2b-b\beta) + 2(1-\beta)(2-\beta)\}r + \{2b(1-\beta)(2-\beta) - (1-\beta)(c+2\beta) - 2(c+1+\beta)(a+b)\}r^2 - (c+2\beta)(a+b\beta)r^3 = 0.$$

This result is sharp.

THEOREM 3. *If $f(z)$ is regular in D and satisfies*

$$F(z) = \left(\frac{c+2}{z^{c+1}} \right) \int_0^z t^{c-1} f(t)g(t)dt, \quad c \geq 0$$

where $F \in S^*(\beta)$ and $g \in K(\alpha)$, then $f(z)$ is starlike of order β for $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$(c+2)(2-\beta) + 2\{(c+\beta+1) - (1-\beta)(2-\beta)\}r + \beta(c+2\beta)r^2 - (1+r)\{(c+2) + (c+2\beta)r\}B(\alpha, r) = 0$$

where

$$B(\alpha, r) = \begin{cases} \frac{(2\alpha-1)r}{(1-r)^{2(1-\alpha)}\{1-(1-r)^{2\alpha-1}\}}, & \alpha \neq \frac{1}{2} \\ -\frac{r}{(1-r)\log(1-r)}, & \alpha = \frac{1}{2} \end{cases}$$

This result is sharp.

THEOREM 4. *If $f(z)$ is regular in D and satisfy*

$$F(z) = \left(\frac{c+2}{z^{c+1}} \right) \int_0^z t^{c-1} f(t)g(t)dt, \quad c \geq 0$$

where $F \in S^*(\beta)$ and $g(z)/z \in P(\alpha)$ then $f(z)$ is univalent and starlike of order β in $|z| < r_0$ where r_0 is the smallest positive root of the equation

$$(c+2)(1-\beta) - 2\{(c+2)(1-\alpha\beta) + (1-\beta)(2-\beta)\}r - 2\{c(3-4\alpha-\beta+\alpha\beta) + (3+2\beta-8\alpha+6\alpha\beta-\beta^2-2\alpha\beta^2)\}r^2 + 2\{(c+2\beta)(2\alpha-\alpha\beta-1) - (2\alpha-1)(1-\beta)(2-\beta)\}r^3 - (2\alpha-1)(1-\beta)(c+2\beta)r^4 = 0.$$

The result is sharp.

THEOREM 5. Let $F \in \Gamma(m, M)$ and $f(z)$ be defined by

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) dt, \quad c \geq 1$$

and $r(a, b)$ be the unique positive root of the equation

$$(a+d) + 2\{d(a+b) - (d-b)\}r + \{2(b^2-d^2) - (a+d) + d(ad+b^2)\}r^2 - 2d\{(a+b) + b(d-b)\}r^3 - d(ad+b^2)r^4 = 0$$

and $d \leq 0$ then $f(z)$ is meromorphic starlike of order β for $|z| < r_0$, where r_0 is the smallest positive root of the equation

$$(1-\beta) + \{(a+b+2d) - (b+d)\beta\}r + (ab+bd+d^2-bd\beta)r^2 = 0$$

if $0 < r_0 \leq r(a, b)$, and that of the equation

$$(E-1+bd) - (1+bd)x + \sqrt{(1+d)\{(1+d) + (1-d)x\}\{(1-2a+b^2) + (1-b^2)x\}}$$

if $r(a, b) \leq r_0$ where

$$x = \frac{1+r^2}{1-r^2}, \quad E = (a-b) - (d-b)\beta \quad \text{and} \quad d = \frac{a+b+c}{c}.$$

Equality is attained for the functions

$$F(z) = \frac{(1+bz)^{\frac{a+b}{b}}}{z}$$

$$F(z) = \frac{[(1-bz)^{1+k}(1+bz)^{1-k}]^{\frac{a+b}{2b}}}{z}$$

where k is determined from

$$\frac{1-k(a+b)z+abz^2}{1-b^2z^2} = \left\{ \frac{(1+d)(1-dr^2)}{(1-a)+(a-b^2)r^2} \right\}^{1/2}.$$

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