

A semi-equivariant Dixmier-Douady invariant

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ABSTRACT. A generalisation of the equivariant Dixmier-Douady invariant is constructed as a second-degree cohomology class within a new semi-equivariant Čech cohomology theory. This invariant obstructs liftings of semi-equivariant principal bundles that are associated to central exact sequences of structure groups in which each structure group is acted on by the equivariance group. The results and methods described can be applied to the study of complex vector bundles equipped with linear/anti-linear actions, such as Atiyah’s Real vector bundles.

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1. Introduction

A *Real vector bundle* (E, τ) is a complex vector bundle equipped with an anti-linear involution that covers an involution on its base space [1]. The $U(n)$ -frame bundle $\text{Fr}(E)$ of a Real vector bundle is equipped with two actions: a left action of \mathbb{Z}_2 induced by τ , and a right action of $U(n)$. Due to the anti-linearity of τ , these actions do not commute. Rather, they combine into an action of $\mathbb{Z}_2 \rtimes U(n)$, where \mathbb{Z}_2 acts on $U(n)$ by elementwise conjugation.

More generally, if G is a Γ -group¹ and P is a principal G -bundle equipped with a left action of Γ that maps fibres to fibres and satisfies $\gamma(pg) = (\gamma p)(\gamma g)$ for all $\gamma \in \Gamma$, $p \in P$ and $g \in G$, then the actions on P combine into an action

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¹a group equipped with an action of Γ by group automorphisms.

of $\Gamma \ltimes G$. In this situation, P is described as a Γ -semi-equivariant principal G -bundle. When the Γ -action on G is trivial, P is an equivariant principal bundle in the usual sense.

This paper solves the following lifting problem for semi-equivariant principal bundles (see Theorem 5.5):

Lifting Problem. *Given a central short exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of Γ -groups and a Γ -semi-equivariant principal C -bundle P , classify the liftings of P by β to a Γ -semi-equivariant principal B -bundle.*

In particular, the obstruction to such liftings is identified as a *semi-equivariant Dixmier-Douady invariant*. This new invariant lies in a *semi-equivariant Čech cohomology theory*, which will be constructed in §4. The semi-equivariant Dixmier-Douady invariant generalises the equivariant Dixmier-Douady invariant, which lies in equivariant cohomology. These constructions and results are motivated by their application to the classification of Spin^c -structures on Real spaces and orientifolds. They originally appeared in the authors thesis [11].

2. Semi-equivariant principal bundles

Before examining semi-equivariant principal bundles, the notion of a semi-direct product is briefly reviewed.

Definition 2.1. Let Γ be a Lie group. A (smooth) Γ -group (G, θ) is a Lie group equipped with a smooth action

$$\theta : \Gamma \rightarrow \text{Aut}(G).$$

If (G, θ) is a Γ -group, the action of an element $\gamma \in \Gamma$ on $g \in G$ will often be denoted by apposition, $\gamma g := \theta_\gamma(g)$.

Definition 2.2. A homomorphism $\varphi : G \rightarrow H$ of Γ -groups is a homomorphism of Lie groups such that, for $\gamma \in \Gamma$ and $g \in G$,

$$\varphi(\gamma g) = \gamma \varphi(g).$$

Definition 2.3. Let (G, θ) be a Γ -group. The (outer) semi-direct product $\Gamma \ltimes_\theta G$ is the Lie group consisting of elements $(\gamma, g) \in \Gamma \times G$ with multiplication defined, for $\gamma_i \in \Gamma$ and $g_i \in G$, by

$$(\gamma_1, g_1)(\gamma_2, g_2) := (\gamma_1 \gamma_2, g_1(\gamma_1 g_2)).$$

One situation in which semi-direct product groups arise is when G and Γ both act on an object X and satisfy the relation $\gamma(gx) = (\gamma g)(\gamma x)$, for some action θ of Γ on G . In this case, the two actions combine to form a single action of the group $\Gamma \ltimes_\theta G$ by $(\gamma, g)x := g(\gamma x)$.

Example 2.4. The standard $U(1)$ -action on \mathbb{C} and the \mathbb{Z}_2 -action on \mathbb{C} by conjugation, combine into a $\mathbb{Z}_2 \ltimes_\kappa U(1)$ -action on \mathbb{C} , where κ is the \mathbb{Z}_2 -action on $U(1)$ by conjugation.

Semi-equivariant principal bundles generalise equivariant principal bundles by using a Γ -group (G, θ) as the structure group. The action θ determines the commutation relation between the left action of Γ and right action of G on the total space of the principal bundle. These actions combine into an action of the semi-direct product $\Gamma \rtimes_{\theta} G$. In the following definitions, let (G, θ) be a smooth Γ -group and X be a manifold equipped with a smooth Γ -action.

Definition 2.5. A (smooth) Γ -semi-equivariant principal (G, θ) -bundle over X is a smooth principal G -bundle $\pi : P \rightarrow X$ equipped with a smooth left action of Γ such that, for $\gamma \in \Gamma, p \in P$ and $g \in G$,

$$\pi(\gamma p) = \gamma \pi(p) \quad \gamma(pg) = (\gamma p)(\gamma g).$$

Definition 2.6. An isomorphism $\varphi : P \rightarrow Q$ of Γ -semi-equivariant principal (G, θ) -bundles is a diffeomorphism such that, for $\gamma \in \Gamma, p \in P$ and $g \in G$,

$$\pi_P = \pi_Q \circ \varphi \quad \varphi(pg) = \varphi(p)g \quad \varphi(\gamma p) = \gamma \varphi(p).$$

Next, let $\lambda : (G, \theta) \rightarrow (H, \vartheta)$ be a homomorphism of Γ -groups, and Q be a Γ -semi-equivariant principal (H, ϑ) -bundle.

Definition 2.7. A lifting of Q by λ is a pair (P, φ) , where P is a Γ -semi-equivariant principal (G, θ) -bundle and $\varphi : P \rightarrow Q$ is a smooth surjective map such that, for $\gamma \in \Gamma, p \in P$ and $g \in G$,

$$\pi_P = \pi_Q \circ \varphi \quad \varphi(pg) = \varphi(p)\lambda(g) \quad \varphi(\gamma p) = \gamma \varphi(p).$$

Definition 2.8. Two liftings (P_1, φ_1) and (P_2, φ_2) of Q by λ are equivalent if there is an isomorphism $\psi : P_1 \rightarrow P_2$ such that $\varphi_2 \circ \psi = \varphi_1$.

The set of all smooth Γ -semi-equivariant principal (G, θ) -bundles will be denoted by $\text{PB}_{\Gamma}(X, (G, \theta))$, and the isomorphisms classes will be denoted by $\text{PB}_{\Gamma}^{\sim}(X, (G, \theta))$.

3. Semi-equivariant transition cocycles

Transition cocycles are used to extract global topological information from a principal bundle into a form which is more easily analysed. A transition cocycle over an open cover $\mathcal{U} := \{U_a\}$ with values in a Lie group G is a collection of smooth maps $\phi_a : U_a \rightarrow G$. Maps on overlapping open sets are required to satisfy a *cocycle condition*. This condition ensures that the cocycle can be used to glue together the patches $U_a \times G$ into a principal G -bundle.

In the equivariant setting, a transition cocycle consists of maps $\phi_a(\gamma, \cdot) : U_a \rightarrow G$ for each $U_a \in \mathcal{U}$ and $\gamma \in \Gamma$. The equivariant cocycle condition then ensures that the elements $\phi_a(1, \cdot)$ can be used to construct the total space of a principal G -bundle, and that the elements $\phi_a(\gamma, \cdot)$ can be used to construct a Γ -action. The derivation of the equivariant cocycle condition uses the fact that the actions of G and Γ commute.

Semi-equivariant transition cocycles can be defined in a similar fashion to equivariant transition cocycles. However, the left and right actions on a Γ -semi-equivariant principal (G, θ) -bundle form an action of $\Gamma \rtimes_{\theta} G$. Thus, the commutation relation between the left and right actions is controlled by θ , and the action θ appears in the semi-equivariant cocycle condition. When this cocycle condition is satisfied, the elements $\phi_a(\gamma, \cdot)$ can be used to construct a semi-equivariant Γ -action.

Throughout this section, let X be a Γ -space, (G, θ) be a Γ -group and $\mathcal{U} := \{U_a\}$ be an open cover of X . The cover \mathcal{U} is not required to be invariant.

Definition 3.1. A (smooth) Γ -semi-equivariant (G, θ) -valued transition cocycle over \mathcal{U} is a collection of smooth maps

$$\phi := \{\phi_{ba}(\gamma, \cdot) : U_a \cap \gamma^{-1}U_b \rightarrow G \mid U_a \cap \gamma^{-1}U_b \neq \emptyset\},$$

satisfying

$$\phi_{aa}(1, x_0) = 1 \quad \phi_{ca}(\gamma'\gamma, x) = \phi_{cb}(\gamma', \gamma x)(\gamma'\phi_{ba}(\gamma, x)), \quad (3.1)$$

for $x_0 \in U_a$, $\gamma', \gamma \in \Gamma$ and $x \in U_a \cap \gamma^{-1}U_b \cap (\gamma'\gamma)^{-1}U_c$.

Note that the conditions (3.1) define a non-equivariant cocycle when restricted to $\gamma = \gamma' = 1$, and an equivariant cocycle when $\theta = \text{id}$.

Definition 3.2. An equivalence of Γ -semi-equivariant (G, θ) -valued transition cocycles ϕ^1 and ϕ^2 with cover \mathcal{U} is a collection of smooth maps

$$\mu := \{\mu_a : U_a \rightarrow G\}$$

such that

$$\mu_b(\gamma x)\phi_{ba}^1(\gamma, x) = \phi_{ba}^2(\gamma, x)(\gamma\mu_a(x)),$$

for $\gamma \in \Gamma$ and $x \in U_a \cap \gamma^{-1}U_b$.

Next, let $\lambda : (G, \theta) \rightarrow (H, \vartheta)$ be a homomorphism of Γ -groups, and ϕ be a Γ -semi-equivariant (H, ϑ) -valued transition cocycle over \mathcal{U} .

Definition 3.3. A lifting of ϕ by λ is a Γ -semi-equivariant (G, θ) -valued transition cocycle ψ such that $\lambda \circ \psi_{ba} = \phi_{ba}$.

Definition 3.4. Two liftings ψ^1 and ψ^2 of ϕ by λ are equivalent if there exists an equivalence μ between ψ^1 and ψ^2 .

The set of smooth Γ -semi-equivariant (G, θ) -valued transition cocycles over \mathcal{U} will be denoted $\text{TC}_{\Gamma}(\mathcal{U}, X, (G, \theta))$. The set of equivalence classes of smooth Γ -semi-equivariant (G, θ) -valued transition cocycles over \mathcal{U} will be denoted by $\text{TC}_{\Gamma}^{\cong}(\mathcal{U}, X, (G, \theta))$.

The first step toward a correspondence between principal bundle and cocycles is to show how a semi-equivariant transition cocycle can be constructed from a semi-equivariant principal bundle. Implicit in the proof of this result is the derivation of the semi-equivariant cocycle property.

Proposition 3.5. *Let $P \in \text{PB}_\Gamma(X, (G, \theta))$ and $s := \{s_a : U_a \rightarrow P|_{U_a}\}$ be a choice of smooth local sections over the cover \mathcal{U} . The collection of maps*

$$\phi^s := \{\phi_{ba}(\gamma, \cdot) : U_a \cap \gamma^{-1}U_b \rightarrow G \mid U_a \cap \gamma^{-1}U_b \neq \emptyset\}$$

defined by

$$\gamma s_a(x) = s_b(\gamma x) \phi_{ba}(\gamma, x). \quad (3.2)$$

is a smooth Γ -semi-equivariant (G, θ) -valued transition cocycle.

Proof. The given condition implies the following three identities

$$\gamma' \gamma s_a(x) = s_c(\gamma' \gamma x) \phi_{ca}^s(\gamma' \gamma, x)$$

$$\gamma' s_b(\gamma x) = s_c(\gamma' \gamma x) \phi_{cb}^s(\gamma', \gamma x)$$

$$\gamma s_a(x) = s_b(\gamma x) \phi_{ba}^s(\gamma, x),$$

which, together, imply

$$\begin{aligned} s_c(\gamma' \gamma x) \phi_{ca}^s(\gamma' \gamma, x) &= \gamma' \gamma s_a(x) \\ &= \gamma' (s_b(\gamma x) \phi_{ba}^s(\gamma, x)) \\ &= (\gamma' s_b(\gamma x)) (\gamma' \phi_{ba}^s(\gamma, x)) \\ &= s_c(\gamma' \gamma x) \phi_{cb}^s(\gamma', \gamma x) (\gamma' \phi_{ba}^s(\gamma, x)). \end{aligned}$$

Thus, ϕ^s satisfies the cocycle property

$$\phi_{ca}^s(\gamma' \gamma, x) = \phi_{cb}^s(\gamma', \gamma x) (\gamma' \phi_{ba}^s(\gamma, x)).$$

□

The map from semi-equivariant principal bundles to semi-equivariant transition cocycles, defined by Proposition 3.5, depends on a choice of local sections. However, if one passes to isomorphism classes of principal bundles and equivalence classes of transition cocycles, this dependence disappears. The next proposition shows that cocycles associated to isomorphic principal bundles by Proposition 3.5 are always equivalent, regardless of which sections are chosen.

Proposition 3.6. *Let $P_i \in \text{PB}_\Gamma(X, (G, \theta))$, and $\phi^i \in \text{TC}_\Gamma(\mathcal{U}, X, (G, \theta))$ be the cocycles associated to local sections $s^i := \{s_a^i : U_a \rightarrow P_i|_{U_a}\}$ as in Proposition 3.5. If $\varphi : P_1 \rightarrow P_2$ is an isomorphism, then the collection of maps*

$$\mu := \{\mu_a : U_a \rightarrow G\}$$

defined by

$$\varphi(s_a^1(x)) := s_a^2(x) \mu_a(x) \quad (3.3)$$

is an equivalence between ϕ^1 and ϕ^2 .

Proof. The properties of semi-equivariant principal bundle isomorphisms and the defining property (3.3) imply that

$$\begin{aligned} \varphi(\gamma s_a^1(x)) &= \gamma \varphi(s_a^1(x)) \\ \varphi(s_b^1(\gamma x) \phi_{ba}^1(\gamma, x)) &= \gamma (s_a^2(x) \mu_a(x)) \end{aligned}$$

$$\begin{aligned}\varphi(s_b^1(\gamma x))\phi_{ba}^1(\gamma, x) &= (\gamma s_a^2(x))(\gamma\mu_a(x)) \\ s_b^2(\gamma x)\mu_b(\gamma x)\phi_{ba}^1(\gamma, x) &= s_b^2(\gamma x)\phi_{ba}^2(\gamma, x)(\gamma\mu_a(x)).\end{aligned}$$

Thus,

$$\mu_b(\gamma x)\phi_{ba}^1(\gamma, x) = \phi_{ba}^2(\gamma, x)(\gamma\mu_a(x)),$$

and μ is an equivalence between ϕ^1 and ϕ^2 for any choice of sections s^i . \square

Corollary 3.7. *The map of Proposition 3.5 induces a well-defined map*

$$\begin{aligned}\mathrm{PB}_{\Gamma}^{\simeq}(X, (G, \theta)) &\rightarrow \mathrm{TC}_{\Gamma}^{\simeq}(\mathcal{U}, X, (G, \theta)) \\ [P] &\mapsto [\phi^s],\end{aligned}$$

where s is any collection of smooth local sections of P .

The correspondence between semi-equivariant cocycles and principal bundles has now been shown in one direction. Next, an inverse map reconstructing a semi-equivariant principal bundle from a semi-equivariant transition cocycle is defined.

Proposition 3.8. *Let $\phi \in \mathrm{TC}_{\Gamma}(\mathcal{U}, X, (G, \theta))$. The bundle P^{ϕ} defined by*

$$\pi : \left(\bigsqcup_{a \in A} U_a \times G / \sim \right) \rightarrow X,$$

where

- (a) $(a, x, g) \sim (b, x, \phi_{ba}(1, x)g)$ defines the equivalence relation \sim
- (b) $\pi[a, x, g] := x$ is the projection map
- (c) $[a, x, g]g' := [a, x, gg']$ defines the right-action of G
- (d) $\gamma[a, x, g] := [b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)]$ defines the left action of Γ ,

is a smooth Γ -semi-equivariant principal (G, θ) -bundle.

Proof. The elements $\{\phi_{ba}(1, \cdot)\}$ satisfy

$$\phi_{ca}(1, x) = \phi_{cb}(1, x)\phi_{ba}(1, x)$$

and so form a G -valued cocycle in the usual sense. Therefore, the usual proof that P^{ϕ} is a principal G -bundle applies. The Γ -action is well-defined on equivalence classes as

$$\begin{aligned}\gamma[b, x, \phi_{ba}(1, x)g] &= [c, \gamma x, \phi_{cb}(\gamma, x)\gamma(\phi_{ba}(1, x)g)] \\ &= [c, \gamma x, \phi_{cb}(\gamma, x)(\gamma\phi_{ba}(1, x))(\gamma g)] \\ &= [c, \gamma x, \phi_{ca}(\gamma, x)(\gamma g)] \\ &= \eta_{\gamma}[a, x, g].\end{aligned}$$

The semi-equivariance property $\gamma(pg) = (\gamma p)(\gamma g)$ is satisfied as

$$\begin{aligned}\gamma([a, x, g]g') &= \gamma([a, x, gg']) \\ &= [b, \gamma x, \phi_{ba}(\gamma, x)(\gamma gg')] \\ &= [b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)(\gamma g')] \\ &= (\gamma[a, x, g])(\gamma g')\end{aligned}$$

Thus, P^ϕ is a Γ -semi-equivariant principal (G, θ) -bundle. \square

This reconstruction map is also well-defined at the level of isomorphism and equivalence classes.

Proposition 3.9. *Let $\phi^i \in \text{TC}_\Gamma(\mathcal{U}, X, (G, \theta))$ and $P_i \in \text{PB}_\Gamma(X, (G, \theta))$ be the associated principal bundles, constructed using Proposition 3.8. If*

$$\mu := \{\mu_a : U_a \rightarrow G\}$$

is an equivalence between ϕ^1 and ϕ^2 then

$$\begin{aligned} \varphi : P_1 &\rightarrow P_2 \\ [a, x, g] &\mapsto [a, x, \mu_a(x)g]. \end{aligned}$$

is an isomorphism.

Proof. That φ is a well-defined isomorphism of principal G -bundles follows immediately from the proof in the non-equivariant case. Compatibility with the Γ -action is satisfied as

$$\begin{aligned} \gamma\varphi([a, x, g]) &= \gamma[a, x, \mu_a(x)g] \\ &= [b, \gamma x, \phi'_{ba}(\gamma, x)\gamma(\mu_a(x)g)] \\ &= [b, \gamma x, \phi'_{ba}(\gamma, x)(\gamma\mu_a(x))(\gamma g)] \\ &= [b, \gamma x, \mu_b(\gamma x)\phi_{ba}(\gamma, x)(\gamma g)] \\ &= \varphi([b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)]) \\ &= \varphi(\gamma[a, x, g]). \end{aligned}$$

Thus, φ is an isomorphism of Γ -semi-equivariant principal (G, θ) -bundles. \square

Corollary 3.10. *The map of Proposition 3.8 induces a well-defined map*

$$\begin{aligned} \text{TC}_\Gamma^\approx(\mathcal{U}, X, (G, \theta)) &\rightarrow \text{PB}_\Gamma^\approx(X, (G, \theta)) \\ [\phi] &\mapsto [P^\phi]. \end{aligned}$$

Finally, one shows that the two maps defined above are inverse to one another.

Proposition 3.11. *The maps*

$$\begin{aligned} \text{TC}_\Gamma^\approx(\mathcal{U}, X, (G, \theta)) &\rightarrow \text{PB}_\Gamma^\approx(X, (G, \theta)) \\ [\phi] &\mapsto [P^\phi] \end{aligned}$$

and

$$\begin{aligned} \text{PB}_\Gamma^\approx(X, (G, \theta)) &\rightarrow \text{TC}_\Gamma^\approx(\mathcal{U}, X, (G, \theta)) \\ [P] &\mapsto [\phi^s] \end{aligned}$$

are inverse to one another.

Proof. Let $P \in \text{PB}_\Gamma(X, (G, \theta))$, $\phi := \phi^s$ and $P' := P^\phi$ for some collection of local sections $s := \{s_a : U_a \rightarrow P|_{U_a}\}$. The sections $\{s_a\}$ define a trivialization $\{t_a\}$ of P by

$$\begin{aligned} t_a : P|_{U_a} &\rightarrow U_a \times G \\ s_a(x) &\mapsto (a, x, 1) \end{aligned}$$

and a collection of maps $\{T_a : P|_{U_a} \rightarrow G\}$ by $t_a(p) =: (a, x, T_a(p))$ where $x = \pi_P(p)$. Note that $T_a(pg) = T_a(p)g$. Define

$$\begin{aligned} \varphi : P &\rightarrow P' \\ p &\mapsto [t_a(p)]. \end{aligned}$$

That φ is a well-defined isomorphism of principal G -bundles follows from the proof in the non-equivariant case. To check that φ is compatible with the Γ -actions, first note that

$$\begin{aligned} t_b \circ \eta_\gamma \circ t_a^{-1}(a, x, g) &= t_b(\gamma(s_a(x)g)) \\ &= t_b((\gamma s_a(x))(\gamma g)) \\ &= t_b(s_b(\gamma x)\phi_{ba}(\gamma, x)(\gamma g)) \\ &= (b, \gamma x, \phi_{ba}(\gamma, x)(\gamma g)) \end{aligned}$$

where η is the Γ -action on P . Thus,

$$\begin{aligned} \gamma\varphi(p) &= \gamma[t_a(p)] \\ &= \gamma[a, x, T_a(p)] \\ &= [b, \gamma x, \phi_{ba}(\gamma, x)T_a(p)] \\ &= [t_b \circ \eta_\gamma \circ t_a^{-1}(a, x, T_a(p))] \\ &= [t_b(\gamma p)] \\ &= \varphi(\gamma p). \end{aligned}$$

Therefore, φ is an isomorphism of Γ -semi-equivariant principal (G, θ) -bundles and $P \mapsto \phi^s \mapsto P^{\phi^s}$ is the identity map at the level of isomorphism classes. \square

The main theorem of this section has now been proved.

Theorem 3.12. *There is a bijective correspondence*

$$\text{PB}_\Gamma^\cong(X, (G, \theta)) \leftrightarrow \text{TC}_\Gamma^\cong(\mathcal{U}, X, (G, \theta))$$

between semi-equivariant cocycles and principal bundles.

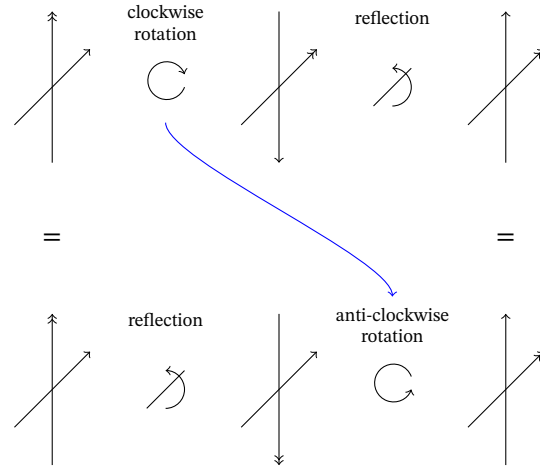


FIGURE 1. This figure corresponds to \mathbb{C} equipped with conjugation as a \mathbb{Z}_2 -action and $U(1)$ acting by rotations, as in Example 2.4. The blue line represents the conjugation automorphism on $U(1)$. This conjugation is required in order to obtain the same final result when the two actions are applied in reversed order.

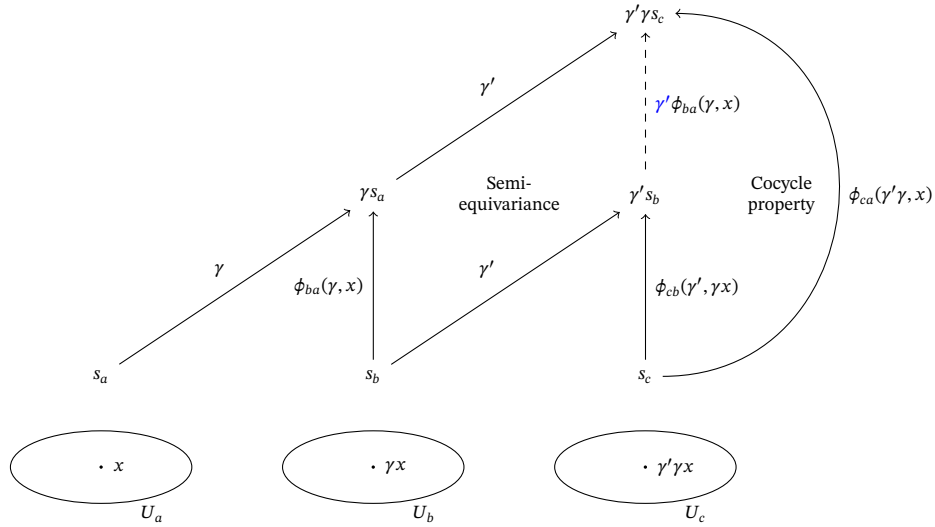


FIGURE 2. This diagram represents the derivation of the semi-equivariant cocycle property, as in Proposition 3.5. Each node of the diagram represents a local section of a principal bundle. The diagonal arrows represent applications of the Γ -action, while the vertical arrows represent the action of a cocycle ϕ via the right action of the structure group. With the exception of the dashed line, all of the arrows follow from the definitions. The dashed line follows by the semi-equivariance property of the principal bundle, the blue γ' is acting on the element $\phi_{ba}(\gamma, x)$ of the structure group.

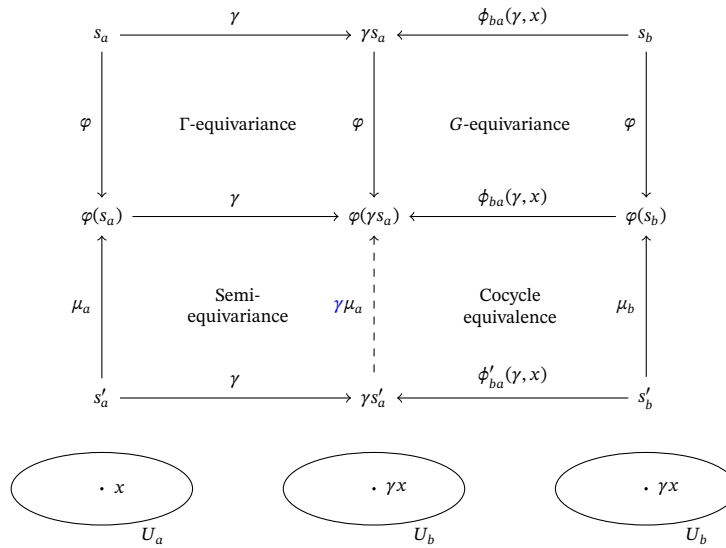


FIGURE 3. This diagram depicts the derivation of the equivalence property for semi-equivariant cocycles, see Definition 3.2. Here, φ is a semi-equivariant principal bundle isomorphism. Each node of the diagram represents a local section of a principal bundle. The arrows running downward are applications of a principal bundle isomorphism φ . The arrows running left to right are applications of the Γ -action. The arrows running right to left are right actions by the cocycle ϕ . Those running upward are right actions of the cocycle equivalence μ . With the exception of the dashed arrow, all of the arrows follow from definitions. The commutation of the top two squares follows from the properties of principal bundle isomorphisms. The dashed arrow follows from the semi-equivariance property of the principal bundle. This twists the equivalence μ_a by the action of Γ on the structure group, which is marked in blue. The lower right square is the semi-equivariant cocycle equivalence condition.

4. Semi-equivariant cohomology

In order to study liftings of semi-equivariant principal bundles, a cohomology theory is needed. The existing notions of equivariant cohomology are inappropriate for this task, and a new cohomology theory must be constructed. In this section, a Γ -semi-equivariant Čech cohomology theory is developed with an abelian Γ -group (G, θ) as its coefficient group. The theory makes use of a simplicial space which encodes the group structure of Γ , and the action of Γ on the manifold X . In addition to these actions, the effect of the action θ must be incorporated. This is achieved by twisting the coboundary map using θ . There are a few details to check, but everything works as one would wish. This semi-equivariant cohomology theory generalises an equivariant cohomology theory outlined by Brylinski [3, §A]. Another helpful reference is [7, §3.3]. One feature of the presentation here is that it avoids the use of hypercohomology. The second dimension of the bicomplex appearing in [3, §A] is an artifact of the choice to separate the cocycle into two parts, one encoding the transition functions for the total space and one encoding the action. Although this is ultimately a notational matter, the reduced book-keeping is helpful when checking higher cocycle conditions.

The construction of semi-equivariant Čech cohomology begins with the definition of a simplicial space. The coboundary map on the underlying chain complex of the cohomology theory will be constructed using the face maps of this space.

Definition 4.1. Let X be a manifold equipped with a smooth action of Γ . The *simplicial space* associated to X is defined by

$$X^\bullet := \{\Gamma^p \times X\}_{p \geq 0}.$$

The simplicial space carries *face* and *degeneracy* maps

$$d_i^p : X^p \rightarrow X^{p-1} \qquad e_i^p : X^p \rightarrow X^{p+1}$$

defined by

$$d_i^p(\gamma_1, \dots, \gamma_p, x) := \begin{cases} (\gamma_2, \dots, \gamma_p, x) & \text{for } i = 0 \\ (\gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_p, x) & \text{for } 1 \leq i \leq p-1 \\ (\gamma_1, \dots, \gamma_{p-1}, \gamma_p x) & \text{for } i = p \end{cases} \quad (4.1)$$

$$e_i^p(\gamma_1, \dots, \gamma_p, x) := (\gamma_1, \dots, \gamma_i, 1, \gamma_{i+1}, \dots, \gamma_p, x) \quad \text{for } 0 \leq i \leq p+1$$

Notice that in (4.1) the face map d_0^p discards the element γ_1 , this element will be used to define the simplicial twisting maps, in Definition 4.3.

Proposition 4.2. *The face and degeneracy maps satisfy the simplicial identities*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i & \text{for } i < j \\ e_i \circ e_j &= e_{j+1} \circ e_i & \text{for } i \leq j \end{aligned} \qquad d_i \circ e_j = \begin{cases} e_{j-1} \circ d_i & \text{for } i < j \\ \text{id} & \text{for } i = j, j+1 \\ e_j \circ d_{i-1} & \text{for } i > j+1 \end{cases}$$

Corresponding to the face maps d_i^p , twisting maps $\theta_i : X^p \times G \rightarrow G$ can be defined. These maps encode the action θ of Γ on G and will be used to twist the coboundary map. They are the basic ingredient needed for generalisation to the semi-equivariant setting. Note that it is only the twisting map θ_0 that has any effect. The rest of the twisting maps are included for notational convenience when dealing with simplicial identities.

Definition 4.3. The simplicial twisting maps $\theta_i : X^p \times G \rightarrow G$ are given by

$$\theta_i^{(\gamma_1, \dots, \gamma_p, x)} := \begin{cases} \theta_{\gamma_1} & \text{for } i = 0 \\ \text{id} & \text{for } 1 \leq i \leq p-1 \\ \text{id} & \text{for } i = p \end{cases}$$

The twisting maps also satisfy simplicial identities which help to ensure that the coboundary map in semi-equivariant cohomology squares to zero.

Proposition 4.4. The simplicial twisting maps satisfy the identities

$$\begin{aligned} \theta_j^{x^{p+1}} \circ \theta_i^{d_j(x^{p+1})} &= \theta_i^{x^{p+1}} \circ \theta_{j-1}^{d_i(x^{p+1})} \quad \text{for } i < j \\ \theta_i^{e_j(x^p)} &= \begin{cases} \theta_i^{x^p} & \text{for } i < j \\ \text{id} & \text{for } i = j, j+1 \\ \theta_{i-1}^{x^p} & \text{for } i > j+1, \end{cases} \end{aligned}$$

where $x^p \in X^p$.

Proof. The identities are trivial for most combinations of i and j . The remaining cases can be checked individually. In particular, the first identity reduces to

$$\begin{aligned} \text{id} \circ \theta_{\gamma_1 \gamma_2} &= \theta_{\gamma_1} \circ \theta_{\gamma_2} && \text{for } i = 0, j = 1 \\ \text{id} \circ \theta_{\gamma_1} &= \theta_{\gamma_1} \circ \text{id} && \text{for } i = 0, j \geq 2 \\ \text{id} &= \text{id} && \text{otherwise.} \end{aligned}$$

□

To construct a Čech-type theory, a simplicial cover \mathcal{U}^\bullet of X^\bullet is needed. Such a cover can be constructed from an appropriate cover $\mathcal{U} := \{U_a \mid a \in A\}$ of X . First, the indexing set of the simplicial cover is defined. This indexing set has a simplicial structure defined by face and degeneracy maps, which will again be denoted by d_i^p and e_i^p .

Definition 4.5. Define the indexing set for \mathcal{U}^\bullet by

$$A^\bullet := \{A^p\}_{p \geq 0}$$

where $A^p := \{(a_0, \dots, a_p) \mid a_i \in A\}$. Elements of A^p will be denoted by a^p . This set carries face and degeneracy maps

$$d_i^p : A^p \rightarrow A^{p-1} \qquad e_i^p : A^p \rightarrow A^{p+1}$$

defined by

$$d_i^p(a_0, \dots, a_p) := (a_0, \dots, \hat{a}_i, \dots, a_p)$$

$$e_i^p(a_0, \dots, a_p) := (a_0, \dots, a_i, a_i, a_{i+1}, \dots, a_p),$$

where \hat{a}_i denotes the removal of the element a_i .

Proposition 4.6. *The face and degeneracy maps of the indexing set A^\bullet satisfy*

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for } i < j$$

$$e_i \circ e_j = e_{j+1} \circ e_i \quad \text{for } i \leq j$$

$$d_i \circ e_j = \begin{cases} e_{j-1} \circ d_i & \text{for } i < j \\ \text{id} & \text{for } i = j, j + 1 \\ e_j \circ d_{i-1} & \text{for } i > j + 1. \end{cases}$$

Before defining the simplicial cover itself, observe that the elements of the simplicial space define sequences of points in X .

Definition 4.7. Let $x^p = (\gamma_1, \dots, \gamma_p, x) \in X^p$. The associated sequence $\{\underline{x}_i^p\}$ is defined by

$$\underline{x}_i^p := \gamma_{p-i} \cdots \gamma_p x \in X.$$

Simplicial covers generalise the nerves of covers. The definition will be made using the definitions of the sequences \underline{x}_i^p and indexing set A^\bullet .

Definition 4.8. The *simplicial cover*

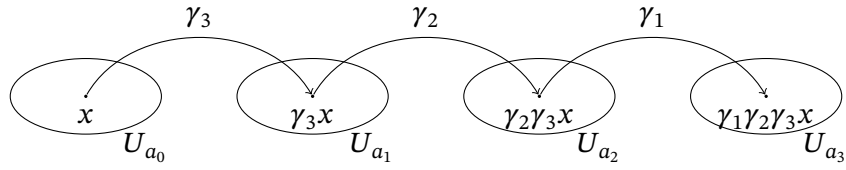
$$\mathcal{U}^\bullet := \{\mathcal{U}^p\}_{p \geq 0}$$

associated to \mathcal{U} is a sequence of covers \mathcal{U}^p of X^p each indexed by A^p . A set

$$U_{(a_0, \dots, a_p)} \in \mathcal{U}^p$$

consists of all points in X^p such that $\underline{x}_i^p \in U_{a_i}$ for $0 \leq i \leq p$.

For example, $(\gamma_1, \gamma_2, \gamma_3, x) \in U_{(a_0, a_1, a_2, a_3)}$ can be visualised as a path



Note that a refinement of \mathcal{U} induces a refinement of \mathcal{U}^\bullet . Also, the face maps of the simplicial cover are compatible with those of the simplicial space. This is necessary to ensure that the coboundary map is well-defined.

Proposition 4.9. *The pullback maps of the simplicial space are compatible with those on the indexing set of the cover in the sense that $d_i(U_{a^p}) \subseteq U_{d_i(a^p)}$.*

Semi-equivariant Čech cohomology is based on a single cochain complex. A p -cochain for this cohomology theory consists of a smooth function on each set in the p th level of the simplicial cover.

Definition 4.10. The group of p -cochains is defined by

$$K_{\Gamma}^p(\mathcal{U}, X, (G, \theta)) := \prod_{a^p \in A^p} C^{\infty}(U_{a^p}, G),$$

with the group operation $(\phi' \phi)_{a^p} := \phi'_{a^p} \phi_{a^p}$.

These cochains can be pulled back by the face maps. In the semi-equivariant setting, the pullback maps are composed with the twisting maps. This modifies the pullback by d_0 .

Definition 4.11. The *twisted pullback maps*

$$\partial_i^p : K_{\Gamma}^p(\mathcal{U}, X, (G, \theta)) \rightarrow K_{\Gamma}^{p+1}(\mathcal{U}, X, (G, \theta))$$

are defined by

$$(\partial_i^p \phi)_{a^{p+1}}(x^{p+1}) := \theta_i^{x^{p+1}} \circ \phi_{d_i^p(a^{p+1})} \circ d_i^p(x^{p+1})$$

Note that the property $d_i(U_{a^p}) \subseteq U_{d_i(a^p)}$ of the cover ensures that $\partial_i(\phi)$ is a well-defined element of $K_{\Gamma}^{p+1}(\mathcal{U}, X, (G, \theta))$.

Proposition 4.12. *The twisted pullback maps are group homomorphisms.*

Proof. Using the fact that θ_{γ} is an automorphism for all $\gamma \in \Gamma$,

$$\begin{aligned} & (\partial_i(\phi' \phi))_{a^{p+1}}(x^{p+1}) \\ &= \theta_i^{x^{p+1}} \circ (\phi' \phi)_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\ &= \theta_i^{x^{p+1}} \left((\phi'_{d_i(a^{p+1})} \circ d_i(x^{p+1})) (\phi_{d_i(a^{p+1})} \circ d_i(x^{p+1})) \right) \\ &= \left(\theta_i^{x^{p+1}} \circ \phi'_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \right) \left(\theta_i^{x^{p+1}} \circ \phi_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \right) \\ &= \left((\partial_i \phi')_{a^{p+1}}(x^{p+1}) \right) \left((\partial_i \phi)_{a^{p+1}}(x^{p+1}) \right) \end{aligned}$$

□

The simplicial identities of the face maps for the simplicial space, the simplicial cover and the twisting maps combine to produce a simplicial identity for the twisted pullback maps.

Proposition 4.13. *For $i < j$ the twisted pullback maps satisfy the identity*

$$\partial_j \circ \partial_i = \partial_i \circ \partial_{j-1}.$$

Proof. Using the corresponding simplicial identities between face maps on the simplicial complex, those on the simplicial cover, and those between the simplicial twisting maps one can directly compute

$$\begin{aligned} (\partial_j(\partial_i \phi))_{a^{p+2}}(x^{p+2}) &= \theta_j^{x^{p+2}} \circ (\partial_i \phi)_{d_j(a^{p+2})} \circ d_j(x^{p+2}) \\ &= \theta_j^{x^{p+2}} \circ \theta_i^{d_j(x^{p+2})} \circ \phi_{d_i \circ d_j(a^{p+2})} \circ d_i \circ d_j(x^{p+2}) \\ &= \theta_i^{x^{p+2}} \circ \theta_{j-1}^{d_i(x^{p+2})} \circ \phi_{d_{j-1} \circ d_i(a^{p+2})} \circ d_{j-1} \circ d_i(x^{p+2}) \end{aligned}$$

$$\begin{aligned}
&= \theta_i^{x^{p+2}} \circ (\partial_{j-1} \phi)_{d_i(a^{p+2})} \circ d_i(x^{p+2}) \\
&= (\partial_i(\partial_{j-1} \phi))_{a^{p+2}}(x^{p+2}).
\end{aligned}$$

□

Finally, the coboundary maps are defined.

Definition 4.14. The *coboundary maps*

$$\partial^p : K_\Gamma^p(\mathcal{U}, X, (G, \theta)) \rightarrow K_\Gamma^{p+1}(\mathcal{U}, X, (G, \theta))$$

are defined by

$$\partial^p := \sum_{0 \leq i \leq p} (-1)^i \partial_i^p.$$

Using the simplicial identity for the twisted pullback maps, the square of the coboundary map is shown to be zero.

Proposition 4.15. *The coboundary map satisfies $\partial \partial = 0$.*

Proof. First note, using Proposition 4.13, that

$$\begin{aligned}
\sum_{i < j, j \leq p+2} (-1)^{i+j} \partial_j \partial_i &= \sum_{i < j, j \leq p+2} (-1)^{i+j} \partial_i \partial_{j-1} \\
&= \sum_{i \leq j, j \leq p+1} (-1)^{i+j} \partial_i \partial_j = \sum_{j \leq i, i \leq p+1} (-1)^{i+j} \partial_j \partial_i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\partial \partial &= \sum_{0 \leq j \leq p+2} (-1)^j \partial_j \left(\sum_{0 \leq i \leq p+1} (-1)^i \partial_i \right) = \sum_{0 \leq j \leq p+2} \sum_{0 \leq i \leq p+1} (-1)^{i+j} \partial_j \partial_i \\
&= \sum_{j \leq i, i \leq p+1} (-1)^{i+j} \partial_j \partial_i + \sum_{i < j, j \leq p+2} (-1)^{i+j} \partial_j \partial_i = 0.
\end{aligned}$$

□

When (G, θ) is abelian, Proposition 4.15 allows the cohomology groups

$$H_\Gamma^p(\mathcal{U}, X, (G, \theta))$$

of the complex $(K_\Gamma^*(\mathcal{U}, X, (G, \theta)), \partial)$ to be defined. The restriction to abelian Γ -groups is necessary to ensure that the coboundary maps ∂^p are group homomorphisms. In order to obtain a cohomology theory which is independent of the cover \mathcal{U} , the direct limit of these cohomology groups will be taken with respect to refinements of the cover. A refinement of \mathcal{U} consists of another cover \mathcal{V} indexed by some set B , and a refining map $r : B \rightarrow A$ such that $V_b \subset U_{r(b)}$ for all $b \in B$. Such a refinement induces a refinement of the associated simplicial covers, and restriction homomorphisms $r_* : K_\Gamma^p(\mathcal{U}, X, (G, \theta)) \rightarrow K_\Gamma^p(\mathcal{V}, X, (G, \theta))$ defined by

$$(r_* \phi)_{(b_0, \dots, b_p)} := \phi_{(r(b_0), \dots, r(b_p))} |_{V_{(b_0, \dots, b_p)}}.$$

These restriction homomorphisms, in turn, induce maps

$$H_\Gamma^p(\mathcal{U}, X, (G, \theta)) \rightarrow H_\Gamma^p(\mathcal{V}, X, (G, \theta))$$

on the cohomology of the complexes. In order for the direct limit of cohomology groups to be well-defined, the maps induced on cohomology by two different refining maps need to be equal. This is true in the equivariant setting, and in the semi-equivariant setting it just needs to be checked that the twisting of the coboundary map using θ doesn't cause any problems.

Lemma 4.16. *Let (\mathcal{V}, r) and (\mathcal{V}, s) be refinements of \mathcal{U} with refining maps $r, s : B \rightarrow A$. The maps induced on semi-equivariant cohomology by r and s are identical.*

Proof. By analogy with the proof in the non-equivariant case (see for example [16, pp. 78-79]), a cochain homotopy

$$\begin{array}{ccc} & K_{\Gamma}^p(\mathcal{U}, X, (G, \theta)) & \xrightarrow{\partial^p} K_{\Gamma}^{p+1}(\mathcal{U}, X, (G, \theta)) \\ & \swarrow h^p & \searrow h^{p+1} \\ K_{\Gamma}^{p-1}(\mathcal{V}, X, (G, \theta)) & \xrightarrow{\partial^{p-1}} & K_{\Gamma}^p(\mathcal{V}, X, (G, \theta)) \end{array}$$

$r_* \downarrow \downarrow s_*$

is defined by

$$(h^p \phi)_{(b_0, \dots, b_{p-1})} = \sum_{k=0}^{p-1} (-1)^k \phi_{(r(b_0), \dots, r(b_k), s(b_k), \dots, s(b_{p-1}))} \circ e_k,$$

where e_k is the k th degeneracy map. Just as in the non-equivariant case, expanding the expression

$$(h^{p+1} \partial^p \phi)_{(b_0, \dots, b_p)} - (\partial^{p-1} h^p \phi)_{(b_0, \dots, b_p)} \in K_{\Gamma}^p(\mathcal{V}, X, (G, \theta))$$

results in a large amount of cancellation. The remaining expression is

$$(\partial_0^p \phi)_{(r(b_0), s(b_0), \dots, s(b_p))} \circ e_0 - (\partial_{p+1}^p \phi)_{(r(b_0), \dots, r(b_p), s(b_p))} \circ e_p.$$

The twisted coboundary maps ∂_0^0 and ∂_{p+1}^p involve the Γ -actions θ on G and σ on X , respectively. However, in the above expression, the degeneracy maps e_0 and e_p ensure that θ and σ only ever act via the identity element of Γ . Thus, the above expression simplifies to

$$\phi_{(s(b_0), \dots, s(b_p))} - \phi_{(r(b_0), \dots, r(b_p))} = (s_* \phi)_{(b_0, \dots, b_p)} - (r_* \phi)_{(b_0, \dots, b_p)}.$$

Therefore, if $\phi \in H_{\Gamma}^p(\mathcal{V}, X, (G, \theta))$ is a cocycle, then

$$(s_* \phi) - (r_* \phi) = h^{p+1} \circ \partial^p(\phi) - \partial^{p-1} \circ h^p(\phi) = \partial^{p-1} \circ h^p(\phi),$$

which is a coboundary. Thus, r_* and s_* induce the same cohomology groups. \square

It is now possible to define the semi-equivariant cohomology groups.

Definition 4.17. The (smooth) Γ -semi-equivariant Čech cohomology groups with coefficients in an abelian Γ -group (G, θ) are defined by

$$H_{\Gamma}^p(X, (G, \theta)) := \lim_{\rightarrow} H_{\Gamma}^p(\mathcal{U}, X, (G, \theta)),$$

where $H_{\Gamma}^p(\mathcal{U}, X, (G, \theta))$ are the cohomology groups of the complex

$$(K_{\Gamma}^{\bullet}(\mathcal{U}, X, (G, \theta)), \partial),$$

and the direct limit is taken with respect to refinements of \mathcal{U} .

Semi-equivariant cohomology is functorial with respect to homomorphisms of abelian Γ -groups.

Proposition 4.18. *A homomorphism $\alpha : A \rightarrow B$ of abelian Γ -groups induces a morphism of complexes*

$$\alpha^{\bullet} : (K_{\Gamma}^{\bullet}(\mathcal{U}, X, A), \partial) \rightarrow (K_{\Gamma}^{\bullet}(\mathcal{U}, X, B), \partial)$$

defined by $(\alpha^p \phi)_{a^p} := \alpha \circ \phi_{a^p}$.

Proof. Let θ be the Γ -action on A and ϑ be the Γ -action on B . As α is a homomorphism of Γ -groups $\alpha^p \circ \vartheta_i^{x^p} = \vartheta_i^{x^p} \circ \alpha^p$ for all $x^p \in X^p$ and $0 \leq i \leq p$. Thus,

$$\begin{aligned} (\alpha^{p+1}(\partial_i \phi))_{a^{p+1}}(x^{p+1}) &= \alpha \circ (\partial_i \phi)_{a^{p+1}}(x^{p+1}) \\ &= \alpha \circ \vartheta_i^{x^{p+1}} \circ \phi_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\ &= \vartheta_i^{x^{p+1}} \circ \alpha \circ \phi_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\ &= \vartheta_i^{x^{p+1}} \circ (\alpha^p \phi)_{d_i(a^{p+1})} \circ d_i(x^{p+1}) \\ &= (\partial_i(\alpha^p \phi))_{a^{p+1}}(x^{p+1}). \end{aligned}$$

Therefore, $\alpha^{p+1} \circ \partial = \partial \circ \alpha^p$ and α^p defines a morphism of complexes. \square

Given a short exact sequence of abelian Γ -groups, connecting maps for a long exact sequence can be constructed.

Theorem 4.19. *A short exact sequence of abelian Γ -groups*

$$1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$$

induces a long exact sequence

$$\dots \xrightarrow{\Delta^{p-1}} H_{\Gamma}^p(X, A) \xrightarrow{\alpha^p} H_{\Gamma}^p(X, B) \xrightarrow{\beta^p} H_{\Gamma}^p(X, C) \xrightarrow{\Delta^p} H_{\Gamma}^{p+1}(X, A) \xrightarrow{\alpha^{p+1}} \dots,$$

where $\Delta^p(\phi) := [\partial(\psi)]$ for any element $\psi \in K_{\Gamma}^p(B)$ such that $\beta^p(\psi) = \phi$.

Proof. The proposition follows by standard diagram chasing arguments applied to the exact sequence of complexes

$$1 \rightarrow (K_{\Gamma}^{\bullet}(X, A), \partial) \xrightarrow{\alpha^{\bullet}} (K_{\Gamma}^{\bullet}(X, B), \partial) \xrightarrow{\beta^{\bullet}} (K_{\Gamma}^{\bullet}(X, C), \partial) \rightarrow 1.$$

For an example, see the proof of [16, Theorem 4.30]. \square

5. Semi-equivariant Dixmier-Douady invariants

In order to apply semi-equivariant cohomology to the classification of semi-equivariant liftings, its relationship with semi-equivariant principal bundles must be clarified. By Theorem 3.12, this reduces to the problem of relating semi-equivariant transition cocycles and semi-equivariant cohomology classes. In this section, semi-equivariant transition cocycles will be interpreted as degree-1 cocycles which can take values in a non-abelian coefficient group. An analogue of Theorem 4.19 will be proved that constructs a connecting map from the transition cocycles into degree-2 cohomology. The theorem can be used to classify certain liftings of semi-equivariant principal bundles between non-abelian structure groups. This method has its origins in the work of Dixmier-Douady on continuous trace C^* -algebras [5]. See also [4, §4] and [16, §4.3].

To begin, note that the p -cochains of Definition 4.10 and the twisted pullback maps of Definition 4.11 are well-defined for non-abelian Γ -groups. Thus, it is possible to make the following definitions.

Definition 5.1.

$$\mathrm{TC}_\Gamma^0(\mathcal{U}, X, (G, \theta)) := \{\mu \in K_\Gamma^0(\mathcal{U}, X, (G, \theta)) \mid (\partial_1\mu)^{-1}(\partial_0\mu) = 1\} \quad (5.1)$$

$$\mathrm{TC}_\Gamma^1(\mathcal{U}, X, (G, \theta)) := \{\phi \in K_\Gamma^1(\mathcal{U}, X, (G, \theta)) \mid (\partial_1\phi)^{-1}(\partial_2\phi)(\partial_0\phi) = 1\} / \sim \quad (5.2)$$

where $\phi^1 \sim \phi^2$ if and only if there exists a $\mu \in K_\Gamma^0(\mathcal{U}, X, (G, \theta))$ such that $(\partial_1\mu)\phi^1 = \phi^2(\partial_0\mu)$.

The set $\mathrm{TC}_\Gamma^1(\mathcal{U}, X, (G, \theta))$ is just $\mathrm{TC}_\Gamma^{\simeq}(\mathcal{U}, X, (G, \theta))$ with the transition cocycle condition and equivalence condition expressed in terms of twisted pullback maps. Note that the particular order of the terms $\partial_i\mu$ in (5.1) and $\partial_i\phi$ in (5.2) is important as the elements μ and ϕ take values in G , which is not necessarily abelian. When G is abelian, these terms may be rearranged to give the corresponding cocycle properties in semi-equivariant cohomology. An abelian structure group also ensures that pointwise multiplication is a well-defined group structure on TC_Γ^0 and TC_Γ^1 , which, in general, are only pointed sets.

Theorem 5.2. When G is abelian

$$\mathrm{TC}_\Gamma^0(\mathcal{U}, X, (G, \theta)) \simeq H_\Gamma^0(\mathcal{U}, X, (G, \theta)) \quad (5.3)$$

$$\mathrm{TC}_\Gamma^1(\mathcal{U}, X, (G, \theta)) \simeq H_\Gamma^1(\mathcal{U}, X, (G, \theta)). \quad (5.4)$$

Proof. When G is abelian, the defining condition on $\mathrm{TC}_\Gamma^0(\mathcal{U}, X, (G, \theta))$ and the 0-cocycle condition on cohomology are equivalent as

$$0 = -(\partial_1\mu) + (\partial_0\mu) = (\partial_0\mu) - (\partial_1\mu) = \partial\mu.$$

This proves (5.3). Similarly, the defining condition on $\mathrm{TC}_\Gamma^1(\mathcal{U}, X, (G, \theta))$ and the 1-cocycle condition on cohomology are equivalent as

$$0 = -(\partial_1\phi) + (\partial_2\phi) + (\partial_0\phi) = (\partial_0\phi) - (\partial_1\phi) + (\partial_2\phi) = \partial\phi,$$

and the equivalence relations on $\mathrm{TC}_\Gamma^1(\mathcal{U}, X, (G, \theta))$ and $H_\Gamma^0(\mathcal{U}, X, (G, \theta))$ are the same as

$$\begin{aligned}(\partial_1\mu) + \phi^1 &= \phi^2 + (\partial_0\mu) \\ \phi^1 - \phi^2 &= (\partial_0\mu) - (\partial_1\mu) \\ \phi^1 - \phi^2 &= \partial\mu.\end{aligned}$$

These two facts imply (5.4). \square

Together, Theorem 4.19 and Theorem 5.2 enable liftings of semi-equivariant principal bundles between abelian structure groups to be classified. However, the construction of a Dirac operator involves the construction of liftings between non-abelian groups. The next theorem is a generalisation of Theorem 4.19 that can be used to classify certain liftings between non-abelian structure groups.

Theorem 5.3. *A short exact sequence of Γ -groups*

$$1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1,$$

where $\alpha(A)$ is central in B , induces an exact sequence of pointed sets

$$\begin{aligned}0 \rightarrow H_\Gamma^0(X, A) &\xrightarrow{\alpha^0} \mathrm{TC}_\Gamma^0(X, B) \xrightarrow{\beta^0} \mathrm{TC}_\Gamma^0(X, C) \xrightarrow{\Delta^0} \dots \\ \dots \xrightarrow{\Delta^0} H_\Gamma^1(X, A) &\xrightarrow{\alpha^1} \mathrm{TC}_\Gamma^1(X, B) \xrightarrow{\beta^1} \mathrm{TC}_\Gamma^1(X, C) \xrightarrow{\Delta^1} H_\Gamma^2(X, A),\end{aligned}$$

where the connecting maps Δ^0 and Δ^1 are defined by

$$\Delta^0([\mu]) := [(\partial_1\eta)^{-1}(\partial_0\eta)] \quad \Delta^1([\phi]) := [(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi)],$$

for any $\eta \in K_\Gamma^0(X, B)$, $\psi \in K_\Gamma^1(X, B)$ satisfying $\beta^0(\eta) = \mu$, $\beta^1(\psi) = \phi$.

Proof. The diagram chasing arguments used in the proof of Theorem 4.19 do not apply directly. However, they can be imitated by carefully working around any lack of commutativity in the groups B and C . Note that Proposition 4.18 and Proposition 4.13 continue to hold when the structure groups involved are non-abelian. Thus, the twisted pullback maps ∂_i commute with the maps α^i and β^i induced by α and β , and also satisfy the simplicial identity $\partial_j \circ \partial_i = \partial_i \circ \partial_{j-1}$ for $i < j$.

First, the map Δ^0 will be considered. Let $\nu := (\partial_1\eta)^{-1}(\partial_0\eta) \in K_\Gamma^1(X, B)$. The cochain η is a lifting by β of μ so $\beta(\nu) = 1$. Thus, ν takes values in $\ker(\beta) \simeq A$ and defines an element of $K_\Gamma^1(X, A)$. The simplicial identity can be used to show that the cochain ν satisfies the cocycle property,

$$\begin{aligned}(\partial_1\nu)^{-1}(\partial_0\nu) &= \left[(\partial_1\partial_1\nu)^{-1}(\partial_1\partial_0\nu) \right]^{-1} \left[(\partial_0\partial_1\nu)^{-1}(\partial_0\partial_0\nu) \right] \\ &= (\partial_1\partial_0\nu)^{-1}(\partial_1\partial_1\nu)(\partial_0\partial_1\nu)^{-1}(\partial_0\partial_0\nu) \\ &= (\partial_1\partial_0\nu)^{-1}(\partial_1\partial_1\nu)(\partial_0\partial_1\nu)^{-1}(\partial_0\partial_0\nu) \\ &= (\partial_0\partial_0\nu)^{-1}(\partial_1\partial_1\nu)(\partial_1\partial_1\nu)^{-1}(\partial_0\partial_0\nu) = 1.\end{aligned}$$

Therefore, $\Delta^0([\mu]) := [\nu] \in H_\Gamma^1(X, A)$.

Next, it needs to be shown that $\Delta^0([\mu]) := [(\partial_1\eta)^{-1}(\partial_0\eta)]$ is independent of the choice of η . Let $\eta' \in K_\Gamma^0(X, B)$ be another element such that $\beta(\eta') = \mu$. Set $\omega := \eta'\eta^{-1}$ and $\nu' := (\partial_1\eta')^{-1}(\partial_0\eta') \in K_\Gamma^1(X, B)$. Then $\beta(\omega) = \beta(\eta'\eta^{-1}) = \mu\mu^{-1} = 1$. Thus, ω defines an element of $K_\Gamma^0(X, A)$ and $\partial\omega \in K_\Gamma^1(X, A)$ is a coboundary. Using the fact that ν and $\partial\omega$ take values in the abelian group A ,

$$\begin{aligned} (\partial\omega)\nu &= (\partial\omega)(\partial_1\eta)^{-1}(\partial_0\eta) \\ &= (\partial_1\eta)^{-1}(\partial\omega)(\partial_0\eta) \\ &= (\partial_1\eta)^{-1}(\partial_1\eta)(\partial_1\eta')^{-1}(\partial_0\eta')(\partial_0\eta)^{-1}(\partial_0\eta) \\ &= (\partial_1\eta')^{-1}(\partial_0\eta') \\ &= \nu'. \end{aligned}$$

Therefore, $[\nu] = [\nu'] \in H_\Gamma^1(X, A)$.

In order to examine the map Δ^1 , let $\nu := (\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi) \in K_\Gamma^2(X, B)$. The cochain $\psi \in K_\Gamma^1(X, B)$ is a β -lifting of the cocycle $\phi \in \text{TC}_\Gamma^1(X, C)$ so $\beta(\nu) = 1$. Therefore, ν defines an element of $K_\Gamma^2(X, A)$. Using the simplicial identity, and the fact that ν takes values in the centre of B , it can be shown that ν satisfies the 2-cocycle property. First, compute

$$\begin{aligned} (\partial_1\nu)(\partial_3\nu) &= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)(\partial_1\partial_0\psi)(\partial_3\nu) \\ &= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)(\partial_3\nu)(\partial_1\partial_0\psi) \\ &= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)\left[(\partial_3\partial_1\psi)^{-1}(\partial_3\partial_2\psi)(\partial_3\partial_0\psi)\right](\partial_1\partial_0\psi) \\ &= (\partial_1\partial_1\psi)^{-1}(\partial_1\partial_2\psi)\left[(\partial_1\partial_2\psi)^{-1}(\partial_3\partial_2\psi)(\partial_3\partial_0\psi)\right](\partial_1\partial_0\psi) \\ &= (\partial_1\partial_1\psi)^{-1}(\partial_3\partial_2\psi)(\partial_3\partial_0\psi)(\partial_1\partial_0\psi) \\ &= (\partial_1\partial_1\psi)^{-1}(\partial_3\partial_2\psi)\left[(\partial_2\partial_0\psi)(\partial_2\partial_0\psi)^{-1}\right](\partial_3\partial_0\psi)(\partial_1\partial_0\psi) \\ &= (\partial_2\partial_1\psi)^{-1}(\partial_2\partial_2\psi)\left[(\partial_2\partial_0\psi)(\partial_0\partial_1\psi)^{-1}\right](\partial_0\partial_2\psi)(\partial_0\partial_0\psi) \\ &= \left[(\partial_2\partial_1\psi)^{-1}(\partial_2\partial_2\psi)(\partial_2\partial_0\psi)\right]\left[(\partial_0\partial_1\psi)^{-1}(\partial_0\partial_2\psi)(\partial_0\partial_0\psi)\right] \\ &= (\partial_2\nu)(\partial_0\nu). \end{aligned}$$

Then

$$\begin{aligned} (\partial\nu) &= (\partial_0\nu)(\partial_1\nu)^{-1}(\partial_2\nu)(\partial_3\nu)^{-1} \\ &= (\partial_0\nu)(\partial_2\nu)(\partial_3\nu)^{-1}(\partial_1\nu)^{-1} \\ &= (\partial_0\nu)(\partial_2\nu)\left[(\partial_1\nu)(\partial_3\nu)\right]^{-1} \\ &= (\partial_0\nu)(\partial_2\nu)\left[(\partial_0\nu)(\partial_2\nu)\right]^{-1} \\ &= 1, \end{aligned}$$

and so $[\nu] \in H_{\Gamma}^2(X, A)$.

Next, it needs to be shown that Δ^1 is well-defined. Specifically, that

$$\Delta^1([\phi]) := [(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi)]$$

is independent of the choice of ψ , and depends only on the class of ϕ in $\mathrm{TC}_{\Gamma}^1(X, C)$. To prove the first statement, let $\psi' \in K_{\Gamma}^1(X, B)$ be another β -lifting of ϕ and $\nu' := (\partial_1\psi')^{-1}(\partial_2\psi')(\partial_0\psi')$ be the corresponding element of $H_{\Gamma}^2(X, A)$. If $\omega := \psi'\psi^{-1}$ then $\beta(\omega) = \beta(\psi'\psi^{-1}) = \phi\phi^{-1} = 1$. Thus, $\omega \in K_{\Gamma}^1(X, A)$ and $\partial\omega \in K_{\Gamma}^2(X, A)$ is a coboundary. Next, using the fact that ω takes values in the center of B ,

$$\begin{aligned} (\partial\omega)\nu &= (\partial_0\omega)(\partial_1\omega)^{-1}(\partial_2\omega)(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi) \\ &= (\partial_1\psi)^{-1}(\partial_1\omega)^{-1}(\partial_2\omega)(\partial_2\psi)(\partial_0\omega)(\partial_0\psi) \\ &= (\partial_1\psi)^{-1}(\partial_1\psi'\psi^{-1})^{-1}(\partial_2\psi'\psi^{-1})(\partial_2\psi)(\partial_0\psi'\psi^{-1})(\partial_0\psi) \\ &= (\partial_1\psi)^{-1}(\partial_1\psi)(\partial_1\psi')^{-1}(\partial_2\psi')(\partial_2\psi)^{-1}(\partial_2\psi)(\partial_0\psi')(\partial_0\psi)^{-1}(\partial_0\psi) \\ &= (\partial_1\psi')^{-1}(\partial_2\psi')(\partial_0\psi') \\ &= \nu'. \end{aligned}$$

Therefore, $[\nu] = [\nu'] \in H_{\Gamma}^2(X, A)$.

In order to prove that $\Delta^1([\phi])$ depends only on the class of ϕ , suppose that ϕ is a coboundary i.e. that $\phi = (\partial_1\tilde{\phi})^{-1}(\partial_0\tilde{\phi})$ for some $\tilde{\phi} \in K_{\Gamma}^0(X, C)$. By surjectivity of β , there exists an element $\tilde{\psi}$ such that $\beta(\tilde{\psi}) = \tilde{\phi}$. Then $\psi := (\partial_1\tilde{\psi})^{-1}(\partial_0\tilde{\psi})$ is a lifting by β of ϕ as

$$\begin{aligned} \beta(\psi) &= \beta[(\partial_1\tilde{\psi})^{-1}(\partial_0\tilde{\psi})] \\ &= (\beta\partial_1\tilde{\psi})^{-1}(\beta\partial_0\tilde{\psi}) \\ &= (\partial_1\beta\tilde{\psi})^{-1}(\partial_0\beta\tilde{\psi}) \\ &= (\partial_1\tilde{\phi})^{-1}(\partial_0\tilde{\phi}) \\ &= \phi. \end{aligned}$$

So, again applying the simplicial identity,

$$\begin{aligned} \Delta^1([\phi]) &= [(\partial_1\psi)^{-1}(\partial_2\psi)(\partial_0\psi)] \\ &= [(\partial_1\partial_0\tilde{\psi})^{-1}(\partial_1\partial_1\tilde{\psi})(\partial_2\partial_1\tilde{\psi})^{-1}(\partial_2\partial_0\tilde{\psi})(\partial_0\partial_1\tilde{\psi})^{-1}(\partial_0\partial_0\tilde{\psi})] \\ &= [(\partial_0\partial_0\tilde{\psi})^{-1}(\partial_1\partial_1\tilde{\psi})(\partial_1\partial_1\tilde{\psi})^{-1}(\partial_0\partial_1\tilde{\psi})(\partial_0\partial_1\tilde{\psi})^{-1}(\partial_0\partial_0\tilde{\psi})] \\ &= 1. \end{aligned}$$

Thus, $\Delta^1([\phi])$ depends only on the class of ϕ in $\mathrm{TC}_{\Gamma}^1(X, C)$. \square

It is now possible to define the semi-equivariant Dixmier-Douady invariant and resolve the main problem of this paper.

Definition 5.4. The *semi-equivariant Dixmier-Douady invariant* of a Γ -semi-equivariant principal C -bundle P associated to a central exact sequence

$$1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$$

is defined by

$$DD(P) := \Delta^1([\phi]) \in H_{\Gamma}^2(X, A),$$

where Δ^1 is the connecting map provided by Theorem 5.3 and $[\phi]$ is the transition cocycle associated to P by Proposition 3.5.

Theorem 5.5. *The exact sequence produced by Theorem 5.3 implies that*

- (a) P can be lifted by β if and only if $DD(P) = 0$,
- (b) when $DD(P) = 0$, the liftings of P by β correspond non-canonically to the classes of $H_{\Gamma}^1(X, A)$.

6. Related work and applications

Semi-equivariant principal bundles are well-known objects that have been studied by several authors, including tom Dieck [17]. A more general type of bundle also appears in the literature [13] [14, IV §1]. These bundles are associated to a short exact sequence $1 \rightarrow G \rightarrow \Omega \rightarrow \Gamma \rightarrow 1$ of Lie groups. They coincide with Γ -semi-equivariant principal (G, θ) -bundles whenever the sequence splits so that $\Omega = \Gamma \ltimes_{\theta} G$.

Semi-equivariant principal bundles can be related to equivariant principal bundles equipped with an appropriate structure group reduction. In the following proposition, let B^{\simeq} denote the set of isomorphism classes of pairs (Q, s) , where $\pi : Q \rightarrow X$ is a Γ -equivariant principal $\Gamma \ltimes_{\theta} G$ -bundle and $s : X \rightarrow Q/G$ is a section that satisfies $s(\gamma x) = \gamma s(x)\gamma^{-1}$. This proposition is a special case of a more general result described in [13, pp. 266-267].

Proposition 6.1. *There is a bijective correspondence*

$$\begin{aligned} \text{PB}_{\Gamma}^{\simeq}(X, (G, \theta)) &\leftrightarrow B^{\simeq} \\ [P] &\mapsto [P \times_G (\Gamma \ltimes_{\theta} G), s], \end{aligned}$$

where $\gamma \in \Gamma$ and $\omega \in \Gamma \ltimes_{\theta} G$ act on $[p, v] \in P \times_G (\Gamma \ltimes_{\theta} G)$ by

$$\gamma[p, v] := [\gamma p, (\gamma, 1)v] \quad [p, v]\omega := [p, v\omega],$$

and s is defined by

$$\begin{aligned} s : X &\rightarrow P \times_G (\Gamma \ltimes_{\theta} G)/G \\ x &\mapsto [p, 1], \end{aligned}$$

for any p in the fibre over x . In the above, $P \times_G (\Gamma \ltimes_{\theta} G)$ is the quotient with respect to the left G -action $g(p, v) := (pg, (1, g)v)$ and $P \times_G (\Gamma \ltimes_{\theta} G)/G$ is the subsequent quotient with respect to the right G -action $[p, v]g := [p, v(1, g)]$.

Semi-equivariant Čech cohomology $H_{\Gamma}^*(X, (G, \theta))$ is closely related to several other cohomology theories. For example,

- (a) When Γ is the trivial group, $H_\Gamma^*(X, (G, \theta))$ is Čech cohomology.
- (b) When θ is the trivial action, $H_\Gamma^*(X, (G, \theta))$ is equivariant Čech cohomology $\check{H}_\Gamma^*(X, G)$. When X is a compact manifold acted upon by a finite group, the equivariant Čech cohomology can be related to Grothendieck's equivariant sheaf cohomology [8, §5.5] or Borel cohomology [3, §A], [7, §3.3].
- (c) There is a restriction homomorphism

$$H_\Gamma^p(X, (G, \theta)) \rightarrow H_{\Gamma_G}^p(X, (G, \theta)) \simeq \check{H}_{\Gamma_G}^p(X, G),$$

where $\Gamma_G \subseteq \Gamma$ is the stabiliser subgroup that acts trivially on G . In this way, the semi-equivariant cohomology can be regarded as a restriction of equivariant cohomology.

- (d) When X is a point, $H_\Gamma^*(X, (G, \theta))$ is the group cohomology $H^*(\Gamma, G_\theta)$ of Γ with coefficients in the Γ -module G_θ defined by G and θ [2, p. 35]. With this in mind, semi-equivariant cohomology can be viewed as a cross between group cohomology and equivariant cohomology.
- (e) When X is a Real space and κ is the conjugation action on $U(1)$,

$$H_{\text{Gal}(\mathbb{C}/\mathbb{R})}^*(X, (U(1), \kappa))$$

is closely related to the Real Čech cohomology of [15], and the Real sheaf cohomology defined in [9]. Note that, in this case, the semi-equivariant cohomology incorporates aspects of equivariant Čech cohomology and Galois cohomology for the field extension \mathbb{C}/\mathbb{R} .

An important application of Theorem 5.3 arises in the study of Spin^c structures on Real spaces [1] and orientifolds [6]. Such structures correspond to semi-equivariant liftings of equivariant principal $\text{SO}(n)$ -bundles via the central exact sequence

$$1 \rightarrow (U(1), \kappa \circ \epsilon) \rightarrow (\text{Spin}^c(n), \kappa \circ \epsilon) \xrightarrow{\text{Ad}^c} (\text{SO}(n), \text{id} \circ \epsilon) \rightarrow 1.$$

Here $\epsilon : \Gamma \rightarrow \mathbb{Z}_2$ is a homomorphism from a finite group Γ , and κ denotes the conjugation action on $U(1)$ and $\text{Spin}^c(n)$. The topic of Spin^c -structures on orientifolds and their associated Dirac operators is treated in [12].

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